Explicit bounds of the first eigenvalue

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Abstract It is proved that the general formulas, obtained recently for the lower bound of the first eigenvalue, can be further bounded by one or two constants depending on the coefficients of the corresponding operators only. Moreover, the ratio of the upper and lower bounds is no more than four.

Keywords: first eigenvalue, elliptic operator, Riemannian manifold, birth-death process.

Some general formulas of the first eigenvalue are presented in refs. [1—4] for elliptic operators, Laplacian on Riemannian manifolds and Markov chains. The formulas are expressed in terms of some class of functions, that is making variation with respect to test functions. Several explicit bounds are further presented here, avoiding the use of test functions. It is surprising that the bounds not only control all the essential estimates produced by the formulas but also deduce a simple criterion for the positiveness of the eigenvalue in one-dimensional situation. Further improvement of bounds will be presented in a subsequent paper.

1 Special case: Illustration of the results and the proofs

The main results and their proofs are illustrated in this section in a particular situation.

Consider differential operator $L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$ on $(0, D)$, where $a(x)$ is positive everywhere, with Dirichlet and Neumann boundary at 0 and D (if $D < \infty$) respectively. Assume that

$$\int_0^D \mathrm{d}x e^{C(x)}/a(x) < \infty, \tag{1.1}$$

where $C(x) = \int_0^x b/a$. Consider the (generalized) eigenvalue of L: $\lambda_0 = \inf\{D(f): f \in C^1(0,D), f(0) = 0, \|f\| = 1\}$, where $D(f) = \int_0^D a(x)f'(x)^2\pi(\mathrm{d}x)$, $\pi(\mathrm{d}x) = (a(x)Z)^{-1}e^{C(x)}\mathrm{d}x$, here and in what follows, Z denotes the normalizing constant, and $\|\cdot\|$ denotes the L^2-norm with respect to π. The following variational formula was presented by Theorem 2.2 in ref. [4]:

$$\lambda_0 \geqslant \xi_0 := \sup_{f \in \mathscr{F}} \inf_{x \in (0,D)} I(f)(x)^{-1}, \tag{1.2}$$

where $\mathscr{F} = \{f \in C^1(0,D): f(0) = 0, f'|_{(0,D)} > 0\}$ and

$$I(f)(x) = \frac{e^{-C(x)}}{f'(x)}\int_x^D \frac{f(u)e^{C(u)}}{a(u)}\mathrm{d}u, \qquad x \in (0,D). \tag{1.3}$$

Moreover, it was proved in ref. [4] that the equality in (1.2) holds under mild assumption.

The test function f used in (1.2) is a mimic of the eigenfunction of λ_0. Note that there is no explicit solution of the eigenfunction. More seriously, the eigenvalues and eigenfunctions are very sensitive. For instance, let $D=\infty$, $a(x)\equiv 1$ and $b(x)=-(x+c)$. Then, for the specific value of constant c: $0,\ 1,\ \sqrt{3},\ \sqrt{3+\sqrt{6}}$, both the eigenvalue λ_0 and the order of its eigenfunction (polynomial) change from 1 to 4 successively. And for the other values of c between the above ones, the eigenfunctions are even not polynomial. Thus, it is hardly imaginable to get a good estimate without using test functions. However, we do have the following result.

Theorem 1.1. Let (1.1) hold and $Q(x)=\int_0^x e^{-C(y)}\mathrm{d}y\int_x^D a(y)^{-1}e^{C(y)}\mathrm{d}y$, where $\nu^{(x)}$ is a probability measure on $(0,x)$ with density $e^{-C(y)}/Z^{(x)}$ (and $Z^{(x)}$ is the normalizing constant), $\delta=\sup_{x\in(0,D)}Q(x)$, $\delta'=2\sup_{x\in(0,D)}\int_0^x Q\mathrm{d}\nu^{(x)}$. Then

$$\delta'^{-1}\geqslant\lambda_0\geqslant\xi_0\geqslant(4\delta)^{-1}, \tag{1.4}$$

and moreover $\delta\leqslant\delta'\leqslant 2\delta$. In particular, when $D=\infty$, $\lambda_0>0$ iff $\delta<\infty$.

When $D=\infty$, in order to justify $\lambda_0>0$, it suffices to consider the limiting behavior of $Q(x)$ as $x\to\infty$. For this, there are some simpler sufficient conditions. Let the corresponding process be non-explosive on $[0,\infty)$ (with reflecting boundary at 0): $\int_0^\infty e^{-C(s)}\mathrm{d}s\int_0^s a(u)^{-1}e^{C(u)}\mathrm{d}u=\infty$. By using the l'Hospital's rule, from (1.4), it follows that whenever the limit $\kappa:=\lim_{x\to\infty}[e^C/\sqrt{a}](x)\int_0^x e^{-C}(\leqslant\infty)$ exists, then $\lambda_0>0$ iff $\kappa<\infty$. Especially, if $a(x)\in C^1$, $\lim_{x\to\infty}[\sqrt{a}e^{-C}](x)=\infty$ and the limit $\kappa':=\lim_{x\to\infty}[\sqrt{a}/(a'/2-b)](x)$ exists, then $\lambda_0>0$ iff $\kappa'<\infty$. Furthermore, recall the Mean Value Theorem: if $f(0)=g(0)=0$ or $f(D)=g(D)=0$ but $g'|_{(0,D)}\neq 0$, then $\sup_{x\in(0,D)}f(x)/g(x)\leqslant\sup_{x\in(0,D)}f'(x)/g'(x)$. Thus, if $a\in C^1$, then $\lambda_0>0$ once $\sup_{x>0}[\sqrt{a}/(a'/2-b)](x)<\infty$.

We point out that the result is meaningful for the three situations mentioned at the beginning of this paper. This is due to the coupling method, which reduces the higher-dimensional case to dimension one. To avoid the use of too much notations at the same time, the results are not listed here but discussed case by case in the subsequent sections.

When $b(x)\equiv 0$, the estimate $\delta^{-1}\geqslant\lambda_0\geqslant(4\delta)^{-1}$ was obtained in ref. [5] and is also true for λ_1(see ref. [1]). But the result is not true for λ_1 in the case of $b(x)\not\equiv 0$ (see Example 3.9).

Proof of Theorem 1.1. The original motivation comes from ref. [6], in which the weighted Hardy's inequality

$$\int_0^\infty f(x)^2\nu(\mathrm{d}x)\leqslant A\int_0^\infty f'(x)^2\lambda(\mathrm{d}x),\quad f\in C^\infty,\ f(0)=0$$

was studied, where the optimal constant A obeys the following estimates:

$$B\leqslant A\leqslant 4B, \tag{1.5}$$

here ν and λ are non-negative Borel measures on $[0,\infty)$, $B=\sup_{x>0}\nu[x,\infty)\int_0^x p_\lambda(u)^{-1}\mathrm{d}u$, and p_λ is the derivative of the absolutely continuous part of λ with respect to the Lebesgue measure. However, (1.4) is more precise than (1.5) and so a different proof is needed. The methods of the proofs adopted here mainly come from refs. [1—4].

(a) The second inequality in (1.4) is just (1.2), proved in ref. [4].

(b) To prove the last inequality in (1.4), we need the following result which is an analog of Lemma 6.1 (2) in ref. [1].

Lemma 1.2. Let m, n be non-negative functions satisfying $\int_0^D m(x)\mathrm{d}x < \infty$ and let $c := \sup_{x\in(0,D)} \int_0^x n(y)\mathrm{d}y \int_x^D m(y)\mathrm{d}y < \infty$. Then for every $\gamma \in (0, 1)$, we have $\int_x^D \varphi(y)^\gamma m(y)\mathrm{d}y \leqslant c(1-\gamma)^{-1}\varphi(x)^{\gamma-1}$ for all $x\in(0,D)$, where $\varphi(x) = \int_0^x n(y)\mathrm{d}y$.

Proof. Let $M(x) = \int_x^D m(y)\mathrm{d}y$ and $\gamma\in(0,1)$. Then, by assumption, $M(x)\leqslant c\,\varphi(x)^{-1}$. By using the integration by parts formula, we get

$$\begin{aligned}\int_x^D \varphi(y)^\gamma m(y)\mathrm{d}y &= -\int_x^D \varphi(y)^\gamma \mathrm{d}M(y) \leqslant [\varphi^\gamma M](x) + \gamma\int_x^D [\varphi^{\gamma-1}\varphi' M](y)\mathrm{d}y \\ &\leqslant c\varphi(x)^{\gamma-1} + c\gamma\int_x^D \varphi^{\gamma-2}\varphi' = c\varphi(x)^{\gamma-1} + \frac{c\gamma}{\gamma-1}\int_x^D \mathrm{d}\varphi(y)^{\gamma-1} \\ &\leqslant \frac{c}{1-\gamma}\varphi(x)^{\gamma-1}, \qquad x\in(0,D).\end{aligned}$$

The first and the last inequalities cannot be replaced by equalities because one may ignore a negative term in the case of $D=\infty$. QED

Now, take $m(x) = e^{C(x)}/a(x)$ and $n(x) = e^{-C(x)}$. Because of (1.1) and $\delta<\infty$, the assumptions of Lemma 1.2 are satisfied. Then $\int_x^D [a^{-1}\varphi^\gamma e^C](u)\mathrm{d}u \leqslant c(1-\gamma)^{-1}\varphi(x)^{\gamma-1}$. Next, take $f(x) = \varphi(x)^\gamma$. Then

$$I(f)(x) = \frac{e^{-C(x)}}{f'(x)}\int_x^D \frac{fe^C}{a}(u)\mathrm{d}u \leqslant \frac{e^{-C}}{\gamma\varphi^{\gamma-1}e^{-C}}(x)\cdot\frac{\delta}{1-\gamma}\varphi(x)^{\gamma-1} = \frac{\delta}{\gamma(1-\gamma)}.$$

Optimizing the right-hand side with respect to γ, we obtain $\gamma = 1/2$ and then the required assertion follows.

(c) We now prove the first inequality in (1.4). Fix $x\in(0,D)$. Take $f(y) = f_x(y) = \int_0^{x\wedge y} e^{-C(s)}\mathrm{d}s$, $y\in(0,D)$. Then $f'(y) = e^{-C(y)}$ if $y<x$ and $f'(y) = 0$ if $y\in(x,D)$. Furthermore, $\|f\|^2 = \int_0^x f(y)^2\pi(\mathrm{d}y) + f(x)^2\pi[x,D)$, $D(f) = \int_0^x e^{-2C(y)}e^{C(y)}\mathrm{d}y/Z = f(x)/Z$, where $\pi[p,q] = \int_p^q \mathrm{d}\pi$. Hence

$$\begin{aligned}\lambda_0^{-1} &\geqslant \|f\|^2/D(f) = Zf(x)^{-1}\int_0^x f(y)^2\pi(\mathrm{d}y) + Zf(x)\pi[x,D) \\ &= -Zf(x)^{-1}\int_0^x f(y)^2\mathrm{d}(\pi(y,D)) + Q(x) \\ &= -Zf(x)^{-1}[f(y)^2\pi(y,D)]\Big|_0^x + Q(x) + 2f(x)^{-1}\int_0^x e^{-C(y)}Q(y)\mathrm{d}y \\ &= 2\int_0^x Q\mathrm{d}\nu^{(x)}. \qquad (1.6)\end{aligned}$$

Making supremum with respect to x, it follows that $\lambda_0\leqslant \delta_0'^{-1}$.

(d) By (1.6), we have $2\int_0^x Q\mathrm{d}\nu^{(x)} = Zf(x)^{-1}\int_0^x f(y)^2\pi(\mathrm{d}y) + Q(x) > Q(x)$. Hence $\delta' \geqslant \delta$. On the other hand, from the definitions, it follows immediately that $\delta' \leqslant 2\delta$. Usually, we have $\delta < \delta'$ unless $\delta = \infty$. QED

2 Higher dimensional case: Euclidean space and compact manifolds

This section applies Theorem 1.1 to the higher-dimensional Euclidean space and compact Riemannian manifolds. First, consider elliptic operator $L = \sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i$, $\partial_i = \partial/\partial x_i$ in $\mathbb{R}^d$, where $a(x) := (a_{ij}(x))$ is positive definite, $a_{ij}\in C^2(\mathbb{R}^d)$, $b_i = \sum_{j=1}^{d}(a_{ij}\partial_j V + \partial_j a_{ij})$, $V \in C^2(\mathbb{R}^d)$. Assume additionally that the corresponding diffusion process is non-explosive, having stationary distribution $\pi(\mathrm{d}x) = Z^{-1}\exp[V(x)]\mathrm{d}x$, where $Z := \int \exp[V(x)]\mathrm{d}x < \infty$, and its Dirichlet form $(D, \mathscr{D}(D))$ is regular: $D(f) = \int\langle a\nabla f, \nabla f\rangle \mathrm{d}\pi$, $\mathscr{D}(D) \supset C_0^\infty(\mathbb{R}^d)$. Since L has trivial maximum eigenvalue 0 in the present situation, we are interested only in the first non-trivial one (i.e., the spectral gap): $\lambda_1 = \inf\{D(f): f \in \mathscr{D}(D), \pi(f) = 0, \pi(f^2) = 1\}$, where $\pi(f) = \int f\mathrm{d}\pi$.

The main steps of the study on λ_1 by couplings are as follows. Take and fix a distance $d(x,y)$ in $\mathbb{R}^d$, it belongs to C^2, out of the diagonal. Set $D = \sup_{x,y} d(x,y)$. For each coupling operator $\tilde{L}$ and $f\in C^2[0,D)$, there always exist two functions A and B in $\mathbb{R}^d\times\mathbb{R}^d$ such that $\tilde{L}f\circ d(x,y) = A(x,y)f''(d(x,y)) + B(x,y)f'(d(x,y))$, $x\neq y$. The key step of the method is finding a coupling operator $\tilde{L}$ and a function $f\in C^2[0,D)$ satisfying $f(0)=0$, $f'|_{(0,D)} > 0$ and $f''\leqslant 0$ so that for some constant $\delta > 0$,

$$\tilde{L}f\circ d(x,y) \leqslant -\delta f\circ d(x,y), \qquad x\neq y. \tag{2.1}$$

We now choose $\alpha, \beta\in C(0,D)$ such that $\alpha(r) \leqslant \inf_{d(x,y)=r} A(x,y)$ and $\beta(r) \geqslant \sup_{d(x,y)=r} B(x,y)$. Then, (2.1) holds provided $\alpha(r)f''(r) + \beta(r)f'(r) \leqslant -\delta f(r)$ for $r\in(0,D)$. Thus, the higher-dimensional case is reduced to dimension one.

Replacing $a(x)$ and $b(x)$ used in the last section by $\alpha(r)$ and $\beta(r)$ respectively, define the correspondent function $C(r)$, operator $I(f)$ and the class $\mathscr{F}$ of test functions. Then, the variational formula given by Theorem 4.1 in ref. [1] is as follows:

$$\lambda_1 \geqslant \xi_1 := \sup_{f\in\mathscr{F}} \inf_{r\in(0,D)} I(f)(r)^{-1}. \tag{2.2}$$

Now, define δ and δ' as in Theorem 1.1, from which, one deduces immediately the following result.

Theorem 2.1. $\delta'^{-1} \geqslant \xi_1 \geqslant (4\delta)^{-1}$.

Comparing this theorem with Theorem 1.1, the difference is that here we have upper bound only for ξ_1 rather than λ_1.

We now turn to manifolds. Let M be a compact, connected Riemannian manifold, without or with convex boundary ∂M. Let $L = \Delta + \nabla V$, $V\in C^2(M)$. When $\partial M \neq \phi$, we adopt Neumann boundary condition. Next, let $\mathrm{Ric}_M \geqslant -K$ for some $K\in\mathbb{R}$. Denote by d, D and ρ respectively the dimension, diameter and the Riemannian distance. Let $K(V) = \inf\{r: \mathrm{Hess}_V -$

$\mathrm{Ric}_M \leqslant r\}$ and denote by $\mathrm{cut}(x)$ the cut locus of x. Define

$$a_1(r) = \sup\{\langle \nabla\rho(x,\cdot)(y), \nabla V(y)\rangle + \langle \nabla\rho(\cdot,y)(x), \nabla V(x)\rangle : \rho(x,y) = r,\ y \notin \mathrm{cut}(x)\}, \qquad r \in (0,D].$$

By convention, $a_1(0) = 0$. Choose $\gamma \in C[0,D]$ so that $\gamma(r) \geqslant \min\{K(V)r,\ a_1(r) + 2\sqrt{|K|(d-1)}\,a_2(r)\}$, where $a_2(r) = \tanh\left[\frac{r}{2}\sqrt{K/(d-1)}\right]$ if $K \geqslant 0$ and $a_2(r) = -\tan\left[\frac{r}{2}\sqrt{-K/(d-1)}\right]$ if $K \leqslant 0$. Redefine $C(r) = \frac{1}{4}\int_0^r \gamma(s)\mathrm{d}s$, $r \in [0,D]$. Then, the variational formula obtained by ref. [2] can be stated as follows.

$$\lambda_1 \geqslant 4\xi_1 := 4 \sup_{f \in \mathscr{F}} \inf_{r \in (0,D)} f(r)\left\{\int_0^r e^{-C(s)}\mathrm{d}s \int_s^D [e^C f](u)\mathrm{d}u\right\}^{-1}, \tag{2.3}$$

where $\mathscr{F} = \{f \in C[0,D] : f|_{(0,D)} > 0\}$. Note that $C(r)$ was used in ref. [2] instead of $e^{C(r)}$ used here. We now have the following result.

Theorem 2.2. Define δ and δ' as in Theorem 1.1 but set $a(x) \equiv 1$ and $b(x) = \gamma(x)/4$. Then $\delta^{-1} \geqslant \delta'^{-1} \geqslant \xi_1 \geqslant (4\delta)^{-1}$.

Proof. The proof is similar to the one of Theorem 1.1, but there are two places needed to be modified. The first one is the proof (b). Let $\varphi(r) = \int_0^r e^{-C(x)}\mathrm{d}s$. By Lemma 1.2 (with $n(s) = e^{-C(s)}$, $m(s) = e^{C(s)}$ and $c = \delta$), we have $\int_r^D \varphi^\gamma e^C \leqslant \delta(1-\gamma)^{-1}\varphi(r)^{\gamma-1}$, $\gamma \in (0,1)$. Hence

$$\int_0^r e^{-C(s)}\mathrm{d}s \int_s^D \varphi^\gamma e^C \leqslant \frac{\delta}{1-\gamma}\int_0^r e^{-C}\varphi^{\gamma-1} = \frac{\delta}{\gamma(1-\gamma)}\int_0^r \mathrm{d}\varphi^\gamma = \frac{\delta}{\gamma(1-\gamma)}\varphi(r)^\gamma,\quad r \in (0,D).$$

In particular, setting $\gamma = 1/2$ and $f(r) = \varphi(r)^\gamma$, we obtain $\xi_1 \geqslant (4\delta)^{-1}$. To complete the proof, one needs to show that ξ_1 is a lower bound of the eigenvalue of operator $L = \mathrm{d}^2/\mathrm{d}r^2 + [\gamma(r)/4]\mathrm{d}/\mathrm{d}r$. Then the upper bound $\xi_1 \leqslant \delta'^{-1}$ follows from Theorem 1.1. The proof for the required assertion is similar to the one of (1.2), but is left to a subsequent paper[1]. QED

Example 2.3. Consider the case of zero curvature. Let $V = 0$. Then $\delta = D^2/4$, $\delta' = 3D^2/8$. The precise solution is $4D^2/\pi^2$, which can be deduced by using the test function $\sin(r\pi/2D)$.

Of course, the above idea is also meaningful for Dirichlet eigenvalue in higher-dimensional situation.

3 The general relation between λ_0 and λ_1 and one-dimensional case

The main purpose of this section is to deal with λ_1, by comparing it with λ_0. We now study a general relation between λ_0 and λ_1.

Let $(D, \mathscr{D}(D))$ be a Dirichlet form on a general probabilistic space $(E, \mathscr{E}, \pi)$, it determines a Markov transition probability $p(t,x,\mathrm{d}y)$. Assume that $p(t,x,E) = 1$ for all $t \geqslant 0$ and $x \in E$. Define $\lambda_1 = \inf\{D(f) : f \in \mathscr{D}(D),\ \pi(f) = 0,\ \pi(f^2) = \|f\|^2 = 1\}$. For each $A \in \mathscr{E}$ with $\pi(A) \in (0,1)$, let $\lambda_0(A) = \inf\{D(f) : f \in \mathscr{D}(D),\ f|_{A^c} = 0,\ \|f\| = 1\}$. Then, we have the following result.

1) Chen, M. F., Variational formulas and approximation theorems for the first eigenvalue in dimension one, Science in China, Ser. A, 2000, in press.

Theorem 3.1. $\inf_{\pi(A)\in(0,1/2]}\lambda_0(A)\leqslant\lambda_1\leqslant\inf_{\pi(A)\in(0,1)}\min\{\lambda_0(A)/\pi(A^c),\ \lambda_0(A^c)/\pi(A)\}\leqslant 2\inf_{\pi(A)\in(0,1/2]}\lambda_0(A)$. In particular, $\lambda_1>0$ iff $\inf_{\pi(A)\in(0,1/2]}\lambda_0(A)>0$.

The theorem also holds for general symmetric forms studied in ref. [7], and improves Theorem 1.4 there.

Proof of Theorem 3.1. First, by spectral representation theorem,

$$D(f)=\lim_{t\downarrow 0}\frac{1}{2t}\int\pi(\mathrm{d}x)\int p(t,x,\mathrm{d}y)[f(y)-f(x)]^2,\qquad f\in L^2(\pi),$$

(cf. ref. [8], §6.7). Replacing $J(\mathrm{d}x,\ \mathrm{d}y)$ by $\frac{1}{2t}\pi(\mathrm{d}x)p(t,x,\mathrm{d}y)$, in proof (b) of Theorem 1.2 in ref. [7], or in the last paragraph of part 3 in ref. [9], then setting $t\downarrow 0$, it follows that $\lambda_1\geqslant\inf_{\pi(A)\in(0,1/2]}\lambda_0(A)$.

Next, by Theorem 3.1 in ref. [7], we know that $\lambda_1\leqslant\lambda_0(A)/\pi(A^c)$ for all $A:\pi(A)\in(0,1)$. Hence $\lambda_1\leqslant\inf_{\pi(A)\in(0,1)}\min\{\lambda_0(A)/\pi(A^c),\ \lambda_0(A^c)/\pi(A)\}=\inf_{\pi(A)\in(0,1/2]}\min\{\lambda_0(A)/\pi(A^c),\ \lambda_0(A^c)/\pi(A)\}\leqslant\inf_{\pi(A)\in(0,1/2]}\lambda_0(A)/\pi(A^c)\leqslant 2\inf_{\pi(A)\in(0,1/2]}\lambda_0(A)$. QED

The simplest case is that A consists of a single point, say $A=\{0\}\subset E$ for instance. Then the proof becomes rather easy. For simplicity, let $\lambda_0=\lambda_0(\{0\}^c)$ (but not $\{0\}$). Then, we have the following result.

Proposition 3.2. $\lambda_1\geqslant\lambda_0$.

Proof. Simply noting that $\mathrm{Var}(f)=\|f-\pi(f)\|^2=\inf_{c\in\mathbb{R}}\|f-c\|^2$, we have $\lambda_1=\inf_{f\neq\text{const.}}D(f)/\mathrm{Var}(f)\geqslant\inf_{f\neq\text{const.}}D(f)/\|f-f(0)\|^2=\lambda_0$. QED

In one-dimensional situation, because of the linear order, Theorem 3.1 takes a much simpler form. For instance, the proof of Theorem 3.1 and the property of linear order give us immediately that $\lambda_1\leqslant\inf_{c\in(p,q)}\{[\lambda_0(p,c)\pi(c,q)^{-1}]\wedge[\lambda_0(c,q)\pi(p,c)^{-1}]\}$. However, we have a much stronger result as follows.

Theorem 3.3. Let $L=a(x)\mathrm{d}^2/\mathrm{d}x^2+b(x)\mathrm{d}/\mathrm{d}x$ be an elliptic operator on the interval (p,q), where $a(x)$ is positive everywhere. When p (resp., q) is finite, we adopt Neumann boundary condition. Assume that the process in non-explosive and (1.1) holds. Then,

$$\sup_{c\in(p,q)}\{\lambda_0(p,c)\wedge\lambda_0(c,q)\}\leqslant\lambda_1\leqslant\inf_{c\in(p,q)}\{\lambda_0(p,c)\vee\lambda_0(c,q)\}.$$

Note that when $c\uparrow$, we have $\lambda_0(p,c)\downarrow$ and $\lambda_0(c,q)\uparrow$. Thus, once the two curves $\lambda_0(p,\cdot)$ and $\lambda_0(\cdot,q)$ intersect, the two inequalities become equalities. The conclusion holds once both $a(x)$ and $b(x)$ are continuous. Actually, denoting by x_0 the unique point at which the eigenfunction of λ_1 vanishes, we have $\lambda_1=\lambda_0(p,x_0)=\lambda_0(x_0,q)$ (the proof needs Theorem 1.1 in the subsequent paper[1)]).

Theorem 3.4. Consider birth-death processes. Let $b_i>0(i\geqslant 0)$, and $a_i>0(i\geqslant 1)$ be the birth and death rates respectively. Define $\pi_i=\mu_i/\mu$, $\mu_0=1$, $\mu_i=b_0\cdots b_{i-1}/a_1\cdots a_i$, $\mu=\sum_i\mu_i$ and $D(f)=\sum_i\pi_ib_i[f_{i+1}-f_i]^2$, $\mathscr{D}(D)=\{f\in L^2(\pi):\ D(f)<\infty\}$. Assume that $\mu<\infty$ and the process is non-explosive (equivalently, $\sum_{k=0}^{\infty}(b_k\mu_k)^{-1}\sum_{i=0}^{k}\mu_i=\infty$). Reset

1) See footnote 1) on page 1055.

$\lambda_0([0,k]) = \lambda'_0(k)$, $\lambda_0([k,\infty)) = \lambda''_0(k)$ and adopt the convention $\lambda'_0(-1) = \infty$, here $\lambda_0(A)$ and λ_1 are defined at the beginning of this section. Then

$$\sup_{k\geqslant 0}\{\lambda'_0(k-1) \wedge \lambda''_0(k+1)\} \leqslant \lambda_1 \leqslant \inf_{k\geqslant 1}\{\lambda'_0(k-1) \vee \lambda''_0(k+1)\}.$$

Proof. Here, we prove Theorem 3.4 only, the proof of Theorem 3.3 is similar and even simpler. Given $f\in \mathscr{D}(D)$ and $k\geqslant 0$, let $\tilde f = f - f_k$. Then

$$\begin{aligned} D(f) &= D(\tilde f) = \sum_{i\leqslant k-1}\pi_i b_i[\tilde f_{i+1} - \tilde f_i]^2 + \sum_{i\geqslant k}\pi_i b_i[\tilde f_{i+1} - \tilde f_i]^2 \\ &\geqslant \lambda'_0(k-1)\sum_{i\leqslant k-1}\pi_i\tilde f_i^2 + \lambda''_0(k+1)\sum_{i\geqslant k+1}\pi_i\tilde f_i^2 \geqslant [\lambda'_0(k-1)\wedge\lambda''_0(k+1)]\sum_i \pi_i\tilde f_i^2 \\ &\geqslant [\lambda'_0(k-1)\wedge\lambda''_0(k+1)]\mathrm{Var}(\tilde f) = [\lambda'_0(k-1)\wedge\lambda''_0(k+1)]\mathrm{Var}(f). \end{aligned}$$

Making supremum with respect to k and infimum with respect to f, the required lower bound follows.

We now prove the upper estimate. Given $\varepsilon > 0$, take $f_1, f_2 \geqslant 0$ such that $f_1|_{[k,\infty)} = 0$, $f_2|_{[0,i]} = 0$, $\| f_1 \| = \| f_2 \| = 1$ and $D(f_1) \leqslant \lambda'_0(k-1) + \varepsilon$, $D(f_2) \leqslant \lambda''_0(k+1) + \varepsilon$. Set $f = -f_1 + \alpha f_2$, where α is the constant so that $\pi(f) = 0$. Then $D(f) = \sum_{i\geqslant 0}\pi_i b_i[f_{i+1} - f_i]^2 = D(f_1) + \alpha^2 D(f_2) \leqslant \lambda'_0(k-1) + \varepsilon + (\lambda''_0(k+1) + \varepsilon)\alpha^2 \leqslant (\lambda'_0(k-1) \vee \lambda''_0(k+1) + \varepsilon)\| f \|^2$. Letting $\varepsilon\to 0$ and then making infimum with respect to $k\geqslant 1$, we obtain the required assertion. QED

For birth-death processes, the following variational formulas were presented in refs. [3, 4].

$$\lambda_0 = \sup_{w\in\mathscr{W}_0}\inf_{i\geqslant 0} I_i(w)^{-1}, \quad \lambda_1 = \sup_{w\in\mathscr{W}_1}\inf_{i\geqslant 0} I_i(w)^{-1},$$

where $I_i(w) = [\mu_i b_i(w_{i+1} - w_i)]^{-1}\sum_{j\geqslant i+1}\mu_j w_j$, $\mathscr{W}_0 = \{w: w_0 = 0,\ w_i \text{ is increasing in } i\}$, $\mathscr{W}_1 = \{w: w_i \text{ is strictly increasing in } i \text{ and } \pi(w) \geqslant 0\}$. Our new result is as follows.

Theorem 3.5. Let $\mu < \infty$, $Q_i = \sum_{j\leqslant i-1}(\mu_j b_j)^{-1}\sum_{j\geqslant i}\mu_i$ and $Q'_i = \left[\sum_{j\leqslant i-1}(\mu_j b_j)^{-1} + (2\mu_i b_i)^{-1}\right]\sum_{j\geqslant i+1}\mu_j$, where $\nu^{(k)}$ is a probability measure on $\{0,1,\cdots,k-1\}$ with density $\nu_j^{(k)} = (\mu_j b_j)^{-1}/Z^{(k)}$ (where $Z^{(k)}$ is the normalizing constant). Next, let $\delta = \sup_{n>0} Q_n$ and let $\delta' = 2\sup_{n>0}\sum_{j=0}^{n-1} Q'_j\nu_j^{(n)}$. Then $\delta'^{-1} \geqslant \lambda_0 \geqslant (4\delta)^{-1}$. Assume additionally that the process is non-explosive, then $\lambda_0/\pi_0 \geqslant \lambda_1 \geqslant \lambda \setminus -0$. In particular, λ_0(resp., λ_1) > 0 iff $\delta < \infty$.

Proof. The lower bound of λ_1 comes from Proposition 3.2 (or Theorem 3.4 with $k = 0$). The proof of Theorem 3.1 shows that $\lambda_1 \leqslant \inf_{k\geqslant 0}\{[\lambda'_0(k-1)\pi[k,\infty)^{-1}] \wedge [\lambda''_0(k+1)\pi[0,k]^{-1}]\}$. Then the upper bound follows by setting $k = 0$. The proof for λ_0 is similar to the one of Theorem 1.1. First, prove the following result, which is the discrete version of Lemma 1.2 and improves Lemma 2.2 (2) in ref. [3].

Lemma 3.6. Let (m_i) and (n_i) be non-negative sequences satisfying $\sum_{i=0}^{\infty} m_i < \infty$, $\sup_{n>0}\sum_{i=0}^{n-1} n_i \sum_{i=n}^{\infty} m_i =: c < \infty$. Then for every $\gamma\in(0,1)$, we have $\sum_{j\geqslant i}\varphi_j^\gamma m_j \leqslant c(1 -$

$\gamma)^{-1}\varphi_i^{\gamma-1}$, where $\varphi_n = \sum_{i=0}^{n-1} n_i$.

Proof. Let $M_n = \sum_{j\geqslant n} m_j$. Fix $N > i$. Then by summation by parts formula and $M_n \leqslant c\varphi_n^{-1}$, we get

$$\sum_{j=i}^{N}\varphi_j^{\gamma}m_j \leqslant \varphi_j^{\gamma}M_i + \sum_{j=i}^{N}[\varphi_{j+1}^{\gamma} - \varphi_j^{\gamma}]M_{j+1} \leqslant c\left\{\varphi_i^{\gamma-1} + \sum_{j=i}^{N}[\varphi_{j+1}^{\gamma} - \varphi_j^{\gamma}]/\varphi_{j+1}\right\}.$$

By using the elementary inequality $\gamma(1-\gamma)^{-1}(x^{\gamma-1}-1) + x^{\gamma} \geqslant 1 (x > 0)$, it is easy to check that $\varphi_{j+1}^{\gamma-1} - \varphi_j^{\gamma}/\varphi_{j+1} \leqslant \gamma(1-\gamma)^{-1}[\varphi_j^{\gamma-1} - \varphi_{j+1}^{\gamma-1}]$. Combining this with the last estimate gives us the required assertion. QED

We now take $\gamma = 1/2$, $m_i = \mu_i$, $n_i = (\mu_i b_i)^{-1}$ and $c = \delta$. Then

$$I_i(\sqrt{\varphi}) = \frac{1}{b_i\mu_i(\sqrt{\varphi_{i+1}} - \sqrt{\varphi_i})}\sum_{j\geqslant i+1}\mu_j\sqrt{\varphi_j} \leqslant \frac{2\delta}{b_i\mu_i(\sqrt{\varphi_{i+1}} - \sqrt{\varphi_i})}\cdot\frac{1}{\sqrt{\varphi_{i+1}}} \leqslant 4\delta.$$

Therefore $\lambda_0 \geqslant (4\delta)^{-1}$.

It remains to show that $\lambda_0 \leqslant \delta'^{-1}$. Fix $k \geqslant 1$ and take $f_i = f_i^{(k)} = \sum_{j=0}^{(i-1)\wedge(k-1)}(\pi_j b_j)^{-1}$. Then $\| f \|^2 = \sum_{i\leqslant k-1}\pi_i f_i^2 + f_k^2\sum_{i\geqslant k}\pi_i$, $D(f) = \sum_{i\leqslant k-1}\pi_i b_i[f_{i+1} - f_i]^2 = \sum_{i\leqslant k-1}(\pi_i b_i)^{-1} = f_k$. By using the summation by parts formula again, we get

$$\frac{1}{\lambda_0} \geqslant \frac{\| f \|^2}{D(f)} = \frac{1}{f_k}\sum_{i=0}^{k-1}\pi_i f_i^2 + f_k\sum_{i=k}^{\infty}\pi_i = \frac{1}{f_k}\sum_{i=0}^{k-1}(f_{i+1}^2 - f_i^2)\sum_{j=i+1}^{\infty}\pi_j$$
$$= \frac{2}{f_k}\sum_{i=0}^{k-1}\frac{\mu Q'_i}{\mu_i b_i} = 2\sum_{i=0}^{k-1}Q'_i\nu_i^{(k)}.$$

Making supremum with respect to $k \geqslant 1$ gives the required assertion. QED

Because λ_1 coincides with exponential convergence rate (cf. Theorem 9.21 in ref. [8]), Theorem 3.5 gives us at the same time (and is indeed for the first time) an explicit criterion for exponential ergodicity. By using comparison method (cf. Theorem 4.58 in ref. [8]), this result can be further applied to a class of multidimensional Markov chains. Finally, we return to the case of half-line discussed at the beginning of the paper.

Theorem 3.7. Consider the operator $L = a(x)\mathrm{d}^2/\mathrm{d}x^2 + b(x)\mathrm{d}/\mathrm{d}x$ on $[0, \infty)$, where $a(x)$ is positive everywhere. Let the process be non-explosive (equivalently, $\int_0^{\infty} e^{-C(s)}\mathrm{d}s\int_0^s a(u)^{-1}e^{C(u)}\mathrm{d}u = \infty$) and let (1.1) hold. Then $(4\delta'(c_0))^{-1} \leqslant \lambda_1 \leqslant (\delta'(c_0))^{-1}$, where

$$\delta'(c) = \sup_{x\in(0,c)}\int_c^x e^{-C}\int_x^0 e^C/a, \quad \delta''(c) = \sup_{x\in(c,\infty)}\int_c^x e^{-C}\int_x^{\infty} e^C/a,$$

and c_0 is the unique solution to the equation $\delta'(c) = \delta''(c)$. In particular, $\lambda_1 > 0$ iff $\delta < \infty$.

Proof. First, when $c\uparrow$, we have $\delta'(c)\uparrow$ and $\delta''(c)\downarrow$. Obviously, $\lim_{c\to 0}\delta'(c) = 0$, $\lim_{c\to 0}\delta''(c) = \delta$ and moreover $\lim_{c\to\infty}\delta''(c) \leqslant \delta$. On the other hand, since the process is non-explosive, when $x\uparrow\infty$, we have $\varphi(x) = \int_0^x e^{-C}\uparrow\infty$. It follows that $\delta'(c) \geqslant \int_1^c e^{-C}\int_0^1 e^C/a \to \infty$ as $c\to\infty$. Next, when $c_1 < c_2$, we have

$$0 < \int_x^{c_2} e^{-C}\int_0^x e^C/a - \int_x^{c_1} e^{-C}\int_0^x e^C/a \leqslant \left[\int_0^{c_2} e^C/a\right]\int_{c_1}^{c_2} e^{-C} \to 0, \quad \text{if } c_2 - c_1 \to 0;$$

$$0 < \int_{c_1}^x e^{-C}\int_x^\infty e^C/a - \int_{c_2}^x e^{-C}\int_x^\infty e^C/a \leqslant \left[\int_{c_1}^\infty e^C/a\right]\int_{c_1}^{c_2} e^{-C} \to 0, \quad \text{if } c_2 - c_1 \to 0.$$

Hence both $\delta'(c)$ and $\delta''(c)$ are continuous in c. Therefore, the equation $\delta'(c) = \delta''(c)$ has a unique solution. Then the first assertion follows from Theorem 3.3. Clearly $\delta < \infty$ if and only if $\delta''(c) < \infty$. Hence we obtain the last assertion. QED

In a similar way, one can deduce a criterion for the existence of spectral gap of diffusion on the full-line (cf. sec. 3 in ref. [1]). One may also study the bounds for the processes on finite intervals.

Example 3.8. Take $b(x) \equiv 0$. Define δ as in Theorem 1.1. Then, by Theorem 1.1 and Corollary 2.5 (5) in ref. [1], we know that $\delta^{-1} \geqslant \lambda_1 \geqslant \lambda_0 \geqslant (4\delta)^{-1}$. In particular, when $a(x) = (1+x)^2$, we have $\delta = 1$ (but $\delta' = 2$) and $\lambda_1 = \lambda_0 = 1/4$. Hence, our lower bound is exact.

Example 3.9. Take $a(x) \equiv 1$ and $b(x) = -x$. Then Example 2.10 in ref. [1] gives us $\lambda_1 = 2$. It is easy to check that $\lambda_0 = 1$ (having eigenfunction $g(x) = x$) and $\delta \approx 0.4788$ (but $\delta' \approx 0.9285$). Hence $\delta^{-1} > \lambda_1 > \lambda_0 > (4\delta)^{-1}$.

Example 3.10. An extreme example is the space with two points $\{0,1\}$ only. Then $\lambda_1 = \lambda_0/\pi_0$. Therefore the upper bound of λ_1 in Theorem 3.5 is exact but $\delta^{-1} = \lambda_0 < \lambda_1$. Thus, δ^{-1} is not an upper bound of λ_1 in general.

Added in proof. In the recent paper[10], the estimate $\delta^{-1} \geqslant \lambda_0 \geqslant (4\delta)^{-1}$ for birth-death processes is also obtained by using the discrete Hardy's inequality. Refer also to refs. [11—13] for related study and further references.

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