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LATTICE-EMBEDDING SCALES OF L^p SPACES INTO ORLICZ SPACES

BY

FRANCISCO L. HERNÁNDEZ* AND BALTASAR RODRIGUEZ-SALINAS*

Dpto Análisis Matemático, Facultad de Matemáticas
Universidad Complutense, 28040 Madrid, Spain
e-mail: pacoh@eucmax.sim.ucm.es

ABSTRACT

We study the set P_X of scalars p such that L^p is lattice-isomorphically embedded into a given rearrangement invariant (r.i.) function space $X[0,1]$. Given $0 < \alpha \leq \beta < \infty$, we construct a family of Orlicz function spaces $X = L^F[0,1]$, with Boyd indices α and β, whose associated sets P_X are the closed intervals $[\gamma, \beta]$, for every γ with $\alpha \leq \gamma \leq \beta$. In particular for $\alpha > 2$, this proves the existence of separable 2-convex r.i. function spaces on [0,1] containing isomorphically scales of L^p-spaces for different values of p. We also show that, in general, the associated set P_X is not closed. Similar questions in the setting of Banach spaces with uncountable symmetric basis are also considered. Thus, we construct a family of Orlicz spaces $\ell^F(I)$, with symmetric basis and indices fixed in advance, containing $\ell^p(\Gamma)$-subspaces for different p's and uncountable $\Gamma \subset I$. In contrast with the behavior in the countable case (Lindenstrauss and Tzafriri [L-T_1]), we show that the set of scalars p for which $\ell^p(\Gamma)$ is isomorphic to a subspace of a given Orlicz space $\ell^F(I)$ is not in general closed.

I. Introduction

The structure of rearrangement invariant (r.i.) Banach function spaces has been studied in the Memoirs of Johnson, Maurey, Schechtman and Tzafriri [J-M-S-T] and Kalton [K_4] (see also Lindenstrauss and Tzafriri [L-T_3]). The problem of classifying subspaces of certain special classes of r.i. function spaces has been considered by several authors: for instance, for Lorentz spaces in [C_1],[C_2], [C-D] and [D-K] and for Orlicz spaces in [B-D], [L-T_1], [H-R_1], [H-Ru], [Ra] and [S].

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N. Kalton in [K_2] proved that if a separable r.i. Banach function space X on [0,1], having no isomorphic copy of c_0, has a sublattice isomorphic to $L^1[0,1]$ then $X[0,1]$ is precisely $L^1[0,1]$. This result is even valid replacing sublattice by subspace ([K_2]) and holds in general for the class of all separable Banach r.i. spaces as it can be deduced from results in [K_4] (see also [H-K]).

The impossibility of extending the result of Kalton mentioned above to $L^p[0,1] = L^p$ spaces (instead of $L^1[0,1]$) has been shown in [H-R_2]: given $1 < p < \infty$, there exist separable Banach r.i. function spaces on $[0,1]$, with Boyd indices fixed in advance, containing a *sublattice* isomorphic to L^p. In particular, in the case $p > 2$ this also proves the existence of (non-trivial) separable Banach r.i. function spaces on [0,1] containing a *subspace* isomorphic to L^p. Notice that in many cases, under additional hypothesis, the existence of an embedding of L^p into an r.i. function space as subspace implies, in fact, the existence of an L^p-sublattice (see [K_3] Thm. 10.9, [K_4] Thm. 8.7 and [H-K] Coroll. 7.4).

One of the purposes of this paper is to study, for separable r.i. function spaces X on [0,1], the associated set P_X of scalars p such that L^p is lattice isomorphically embedded into $X[0,1]$. It is known that this set P_X is empty for some special classes of r.i. function spaces. For example: if X is a (non-trivial) Lorentz space $L_{p,q}[0,1]$ or $L_{w,p}[0,1]$ defined by a submultiplicative weight (Carothers [C_1], [C_2]), or if X is an Orlicz space $L^F[0,1]$ generated by a submultiplicative function [J-M-S-T]. On the other hand, [H-R_2] gives several classes of Orlicz spaces $X = L^F[0,1]$ whose associated sets P_X are precisely singletons. To prove the existence of separable r.i. function spaces X on [0,1] containing L^p-spaces as *sublattices* for different p's (and also as *subspaces* for the case $p > 2$) has remained open.

We answer in the positive the above question: we obtain a class of Banach (and quasi-Banach) Orlicz function spaces $X = L^F[0,1]$ whose associated sets P_X are closed intervals of positive numbers. It is also proved that, in general, the set P_X is not necessarily closed.

For the statements of the main results, we need to recall some definitions. Given an Orlicz space $L^F[0,1]$ the *inclusion index* γ_F^∞ ([H-R_2]),

$$\gamma_F^\infty = \limsup_{x\to\infty} \frac{\log F(x)}{\log x},$$

satisfies $\alpha_F^\infty \leq \gamma_F^\infty \leq \beta_F^\infty$, where α_F^∞ and β_F^∞ denote the usual Boyd indices of the space (cf. [L-T_3]). If L^p is lattice isomorphic to a sublattice of $L^F[0,1]$ then $\gamma_F^\infty \leq p \leq \beta_F^\infty$. This follows from the fact that $L^p[0,1]$ is included in $L^F[0,1]$ (by [J-M-S-T] Thm. 7.1). As a converse, we will prove the following:

THEOREM 1: *Let $0 < \alpha \le \gamma \le \beta < \infty$. There exists an Orlicz function space $L^F[0,1]$ with indices $\alpha_F^\infty = \alpha, \gamma_F^\infty = \gamma$ and $\beta_F^\infty = \beta$ such that L^p is lattice-isomorphic to a sublattice of $L^F[0,1]$ for every $p \in [\gamma_F^\infty, \beta_F^\infty]$.*

And we also show that the associated set P_X of scalars is not necessarily closed:

THEOREM 2: *Given $0 < \alpha \le \gamma < \beta < \infty$, there exists a β-concave Orlicz function space $L^F[0,1]$ with indices $\alpha_F^\infty = \alpha, \gamma_F^\infty = \gamma$ and $\beta_F^\infty = \beta$, such that $L^F[0,1]$ contains a lattice-isomorphic copy of L^p if and only if $p \in [\gamma_F^\infty, \beta_F^\infty)$.*

In order to prove these function space results we need to deal with similar questions in the uncountable discrete setting. For this, we will consider the question of finding Banach spaces with symmetric basis containing $\ell^p(\Gamma)$-subspaces for different p's and Γ uncountable.

S. Troyanski in [T] proved the impossibility of embedding $\ell^1(\Gamma)$ spaces for uncountable Γ into Banach spaces with symmetric basis different of $\ell^1(I)$. In the case of $p > 1$, the existence of non-reflexive Orlicz spaces $\ell^F(I)$ with symmetric basis containing an isomorphic copy of $\ell^p(\Gamma)$ for uncountable Γ was proved in [H-T], while the reflexive case has recently been obtained in [H-R$_2$]. So far, it has been unknown whether or not there exists a Banach space with symmetric basis containing $\ell^p(\Gamma)$-subspaces for different p's. Here we will fill this gap by constructing suitable Banach (and quasi-Banach) Orlicz spaces $\ell^F(I)$ with this property.

We will consider the *inclusion index* γ_F at 0 ([H-R$_2$]),

$$\gamma_F = \liminf_{x \to o} \frac{\log F(x)}{\log x},$$

that satisfies $\alpha_F \le \gamma_F \le \beta_F$, where α_F and β_F are the usual indices of the function F at 0 ([$L - T_2$]). It follows from ([R] Thm. B or [H-T] Prop. 5) that if $\ell^p(\Gamma)$ is isomorphically embedded into an space $\ell^F(I)$ for uncountable $\Gamma \subset I$, then $\alpha_F \le p \le \gamma_F$. Here, as a converse, we will show the following:

THEOREM 3: *Let $0 < \alpha \le \gamma \le \beta < \infty$. There exists an Orlicz space $\ell^F(I)$ with indices $\alpha_F = \alpha, \gamma_F = \gamma$ and $\beta_F = \beta$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$ for any $p \in [\alpha_F, \gamma_F]$.*

In contrast with well-known results in the countable case ([L-T$_1$], [L-T$_2$], in the non-convex case [K$_1$]), it turns out that the set of scalars p such that $\ell^p(\Gamma)$ is isomorphically embedded into an Orlicz space $\ell^F(I)$ is not necessarily closed:

THEOREM 4: *Let $0 < \alpha < \gamma \le \beta < \infty$. There exists an α-convex Orlicz space $\ell^F(I)$ with indices $\alpha_F = \alpha, \gamma_F = \gamma$ and $\beta_F = \beta$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$, for uncountable $\Gamma \subset I$, if and only if $p \in (\alpha_F, \gamma_F]$.*

The constructed Orlicz functions are of type

$$F(x) = \int_0^x (x-t)t^q f(t)dt \quad (0 < x < 1)$$

where the function f is defined by

$$f = \sum_{n=0}^{\infty} \epsilon_n \chi_{(2^{-n-1}, 2^{-n}]}$$

for a suitable coefficient sequence (ϵ_n). The construction of this coefficient sequence (ϵ_n), depending on α, β and γ, requires several technical Lemmas, which are sharp extensions of previous ones given in [H-T] and [H-R$_2$]. In particular, Lemma 1 extends a crucial result for these methods given in ([H-T] Lemma 8) and Lemma 7 improves Lemma 1.2 of [H-R$_2$]. We also make use of the criteria for the isomorphic embedding into spaces $\ell^F(I)$ given in ([R], [H-T]): An Orlicz space $\ell^F(I)$ contains an isomorphic copy of $\ell^G(\Gamma)$ for some uncountable set Γ contained in I if and only if $G \in \Sigma_{F,1}$, where $\Sigma_{F,1}$ is the set of Orlicz functions equivalent at 0 to a function

$$H(x) = \int_0^1 \frac{F(xs)}{F(s)} d\mu(s) \qquad (0 < x < 1),$$

where μ is a probability measure on (0,1].

The paper is organized as follows. Firstly, we consider the uncountable discrete case. In Section 2 we give some basic Lemmas in order to prove in Section 3 partial statements of Theorem 3. Section 4 collects several Lemmas which are the key to obtain Orlicz spaces $\ell^F(I)$ containing $\ell^p(\Gamma)$-subspaces for different p's. Section 5 contains the proofs of the main results in the discrete case.

Section 6 is devoted to showing the function space results of the paper . They are deduced quite easily from the previous ones in the uncountable discrete case, by using a simple transference argument and the criteria for lattice isomorphic embedding Orlicz spaces into an Orlicz function space given in ([J-M-S-T]): An Orlicz space $L^F[0,1]$ contains a lattice isomorphic copy of $L^G[0,1]$ provided that $G \in \Sigma_{F,1}^{\infty}$, where $\Sigma_{F,1}^{\infty}$ is the set of Orlicz functions equivalent at ∞ to a function

$$H(x) = \int_0^{\infty} \frac{F(xs)}{F(s)} d\mu(s) \quad \text{for } x > 1,$$

where μ is a probability measure on $(0, \infty)$ satisfying

$$\int_0^\infty \frac{d\mu(s)}{F(s)} \leq 1.$$

Our notation is standard and we refer to the monographs [L-T$_3$], [Gu] and [M] for unexplained definitions.

II. Preliminary lemmas

In this Section we give some technical Lemmas. We begin with a crucial Lemma which extends Lemma 8 in [H-T]:

LEMMA 1: *Given a sequence $(h_i)_{i=0}^\infty$ of positive integers ($h_0 = 1$), there exist two integer sequences $(k_i)_{i=0}^\infty$ and $(m_i)_{i=0}^\infty$, strictly increasing, with $m_i > k_i = \sum_{j=0}^{i-1} m_j$ for $i = 1, 2, \ldots$ such that*

$$\lim_{i\to\infty} (m_{i+1} - m_i) = \infty, \tag{1}$$

$$\sum_{i=0}^{\infty} f(n + k_i) = h_n \quad (n = 0, 1, 2, \ldots), \tag{2}$$

$$\sum_{i=0}^{\infty} f(k_i - n) \leq (n+2)^2 \quad (n = 1, 2, \ldots), \tag{3}$$

where f is the function

$$f(x) = \sum_{i=0}^{\infty} \chi_{[m_i, m_i+1)}(x).$$

Proof: We will proceed by induction. Assume that we have built

$$1 = k_0 < k_1 < \cdots < k_j,$$
$$1 = m_0 < m_1 < \cdots < m_j,$$
$$0 = \ell_0 < \ell_1 \leq \cdots \leq \ell_j,$$

where $k_{i+1} = m_i + k_i$; $m_{i+1} = k_{i+1} + \ell_{i+1}$ for $i = 0, \ldots, j-1$.

We will say that the integer $n \geq 1$ is "covered" in the step j if there exist exactly h_n couples $(k_{i'}, m_{j'})$ such that $n = m_{j'} - k_{i'}$ with $i', j' \leq j$.

Let ℓ_{j+1} be the smallest integer which is not covered in this step j. Let us consider $k_{j+1} = m_j + k_j$; $m_{j+1} = k_{j+1} + \ell_{j+1}$. It is clear that

$$m_{j+1} - k_{j+1} = \ell_{j+1},$$
$$m_{j+1} - k_i \geq m_{j+1} - k_j = m_j + \ell_{j+1} > m_j$$

for $i = 0, 1, \ldots, j$, and

$$m_{j'} - k_{i'} < m_{j'} \le m_j$$

for $i', j' \le j$. Hence the new differences introduced in the last step $(j+1)$ are either equal to ℓ_{j+1} (so uncovered) or they appear for the first time. Then, it follows that the sequences (k_i) and (m_i) satisfy the conditions (1) and (2).

Let us now prove (3). By the definition of the function f,

$$\sum_{j=0}^{\infty} f(k_j - n) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \chi_{[m_i, m_i+1)}(k_j - n).$$

If $k_j - m_i \ge n \ge 1$ we have $k_j > m_i$, and hence, $j > i$. And if $k_j - (m_i + 1) < n$, we have

$$n > k_j - (m_i + 1) = \sum_{\substack{\ell=0 \\ \ell \ne i}}^{j-1} m_\ell - 1 \ge j - 2.$$

Hence, we deduce $i < j < n + 2$, and

$$\begin{aligned}\sum_{j=0}^{\infty} f(k_j - n) &= \sum_{j=0}^{n+1}\sum_{i<j} \chi_{[m_i, m_i+1)}(k_j - n) \\ &\le \sum_{j=0}^{n+1} (n+2) \le (n+2)^2\end{aligned}$$

holds, which concludes the proof. ∎

Lemma 2: *Given $\delta \ge 0$, there exist two positive sequences $(\alpha_n)_{n=0}^{\infty}$ and $(\delta_n)_{n=0}^{\infty}$ such that*

$$\sum_{n=0}^{\infty} \alpha_n 2^{-\delta n} = \infty,$$

$$\frac{1}{2} \le \sum_{n=0}^{\infty} \alpha_n \delta_{n+k} 2^{-(n+k)\delta} \le 1,$$

and

$$\sum_{n=k}^{\infty} \alpha_n \delta_{n-k} 2^{-(n-k)\delta} \le (k+2)^2 2^{\delta k}$$

for every $k \in \mathbb{N}$.

Proof: We will consider the sequences (k_i) and (m_i) constructed in Lemma 1 for the case of the sequence (h_n) equal to $([2^{n\delta}])$ ([ ] denotes here the integral part). And we define the sequences (α_n) and (δ_n) by $\alpha_{k_i} = 2^{k_i\delta}; \delta_{m_i} = 1$ for $i = 0, 1, \ldots$, and $\alpha_j = \delta_{j'} = 0$ in the other remaining cases. Then, using Lemma 1, we have

$$\sum_{n=0}^{\infty} \alpha_n 2^{-\delta n} = \infty,$$

$$\sum_{n=0}^{\infty} \alpha_n \delta_{n+k} 2^{-(n+k)\delta} = \sum_{i=0}^{\infty} 2^{k_i\delta} \delta_{k_i+k} 2^{-(k_i+k)\delta} = 2^{-k\delta} h_k \geq \frac{1}{2}$$

for $k = 0, 1, \ldots$. Further

$$\sum_{n=k}^{\infty} \alpha_n \delta_{n-k} 2^{-(n-k)\delta} = \sum_{i=0}^{\infty} 2^{k_i\delta} \delta_{k_i-k} 2^{-(k_i-k)\delta} \leq (k+2)^2 2^{k\delta},$$

which concludes the proof of the Lemma. ∎

LEMMA 3: *Given $\epsilon > \delta \geq 0$, there exist a constant B and two sequences $(\alpha_n)_{n=0}^{\infty}$ and $(\epsilon_n)_{n=0}^{\infty}$ of positive numbers such that*

$$\sum_{n=0}^{\infty} \alpha_n 2^{-\delta n} = \infty, \quad \limsup_n \sqrt[n]{\epsilon_n} = 2^{-\delta}, \quad \text{and} \quad \epsilon_n \leq c\epsilon_{n+1}$$

for $n \in \mathbb{N}$, and $c = 2^{\epsilon} > 1$, verifying

$$A = \frac{1}{2} \leq \sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \leq B$$

for every $k \in \mathbb{N}$.

Proof: We apply Lemmas 1 and 2 in taking (α_n) and (m_i) as given there. Let $M = \{m_i : i = 0, 1, \ldots\}$,

$$M_i = (M+i) \setminus \bigcup_{j=0}^{i-1} (M+j)$$

and

$$a_k = \sum_{n=k}^{\infty} \alpha_n \delta_{n-k} 2^{-(n-k)\delta} \leq (k+2)^2 2^{\delta k}$$

for $k = 1, 2, \ldots$. Let us define the sequence (ϵ_n) by $\epsilon_0 = 0$ and

$$\epsilon_n = \begin{cases} 2^{-n\delta} & \text{if } n \in M = M_0, \\ c^{-k} 2^{-(n-k)\delta} = c^{-k} 2^{-(n-k)\delta} \delta_{n-k} & \text{if } n \in M_k, \end{cases}$$

where (δ_n) is defined as in Lemma 2. It is clear that $\sum_{n=0}^{\infty} \alpha_n 2^{-n\delta} = \infty$, $\limsup_n \sqrt[n]{\epsilon_n} = 2^{-\delta}$, and $\epsilon_n \leq c\epsilon_{n+1} (n \in \mathbb{N})$; in addition, by Lemma 2, we have

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \geq \sum_{n+k \in M} \alpha_n 2^{-(n+k)\delta} = \sum_{n=0}^{\infty} \alpha_n \delta_{n+k} 2^{-(n+k)\delta} \geq \frac{1}{2}$$

for every $k \in \mathbb{N}$.

On the other hand,

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} = \sum_{i=0}^{\infty} \sum_{n+k \in M_i} \alpha_n \epsilon_{n+k}.$$

Now, as

$$\sum_{i=0}^{\infty} \sum_{n+k \in M_i} \alpha_n \epsilon_{n+k} \leq \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha_n \delta_{n+k-i} 2^{-(n+k-i)\delta}}{2^{i\epsilon}} \leq \sum_{i=0}^{\infty} \frac{1}{2^{i\epsilon}} = \frac{1}{1-2^{-\epsilon}},$$

and

$$\sum_{i=k+1}^{\infty} \sum_{n+k \in M_i} \alpha_n \epsilon_{n+k} \leq \sum_{i=k+1}^{\infty} \sum_{n} \frac{\alpha_n \delta_{n+k-i} 2^{-(n+k-i)\delta}}{2^{i\epsilon}} \leq \sum_{i=k+1}^{\infty} \frac{a_{i-k}}{2^{i\epsilon}} \leq \sum_{i=k+1}^{\infty} \frac{(i-k+2)^2}{2^{i\epsilon}} 2^{(i-k)\delta} \leq \sum_{i=1}^{\infty} \frac{(i+2)^2}{2^{i(\epsilon-\delta)}} < \infty,$$

we deduce

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \leq \frac{1}{1-2^{-\epsilon}} + \sum_{i=1}^{\infty} \frac{(i+2)^2}{2^{i(\epsilon-\delta)}} = B < \infty,$$

which concludes the proof. ∎

LEMMA 4: *Let $\epsilon > 0$ and $c_k = (k+1)^4$ for $k \in \mathbb{N}$. There exist a constant B and two sequences $(\alpha_n)_{n=0}^{\infty}$ and $(\epsilon_n)_{n=0}^{\infty}$ of positive numbers such that*

$$\sum_{n=0}^{\infty} \alpha_n 2^{-n\epsilon} = \infty, \quad \limsup_{n\to\infty} \sqrt[n]{\epsilon_n} = 2^{-\epsilon}, \quad \text{and} \quad \epsilon_n \leq 2^{k\epsilon} c_k \epsilon_{n+k}$$

for $n, k \in \mathbb{N}$, verifying

$$A = \frac{1}{2} \leq \sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \leq B$$

for every $k \in \mathbb{N}$.

Proof: We proceed as in Lemma 3, using now Lemma 2 for $\delta = \epsilon$. We define the sequence $(\epsilon_n)_{n=0}^{\infty}$ by

$$\epsilon_n = \begin{cases} 2^{-n\epsilon} & \text{if } n \in M = M_0, \\ 2^{-n\epsilon} c_k^{-1} = 2^{-n\epsilon} c_k^{-1} \delta_{n-k} & \text{if } n \in M_k. \end{cases}$$

It is clear that $\sum_{n=0}^{\infty} \alpha_n 2^{-n\epsilon} = \infty$, $\limsup_n \sqrt[n]{\epsilon_n} = 2^{-\epsilon}$, and $\epsilon_n \leq 2^{k\epsilon} c_k \epsilon_{n+k}$ for $n, k \in \mathbb{N}$. From Lemma 2 we have

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \geq \sum_{n=0}^{\infty} \alpha_n \delta_{n+k} 2^{-(n+k)\epsilon} \geq \frac{1}{2}$$

for every $k \in \mathbb{N}$. On the other hand,

$$\begin{aligned} \sum_{i=0}^{k} \sum_{n+k \in M_i} \alpha_n \epsilon_{n+k} &\leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \alpha_n \delta_{n+k-i} 2^{-(n+k)\epsilon} \\ &\leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \frac{\alpha_n \delta_{n+k-i} 2^{-(n+k-i)\epsilon}}{2^{i\epsilon}} \\ &\leq \sum_{i=0}^{k} \frac{1}{2^{i\epsilon}} < \frac{1}{1-2^{-\epsilon}} \end{aligned}$$

and

$$\begin{aligned} \sum_{i=k+1}^{\infty} \sum_{n+k \in M_i} \alpha_n \epsilon_{n+k} &\leq \sum_{i=k+1}^{\infty} \sum_{n} \frac{\alpha_n \delta_{n+k-i} 2^{-(n+k-i)\epsilon}}{2^{i\epsilon} c_i} \\ &\leq \sum_{i=k+1}^{\infty} \frac{a_{i-k}}{2^{i\epsilon} c_i} \leq \sum_{i=k+1}^{\infty} \frac{(i-k+2)^2 2^{(i-k)\epsilon}}{2^{i\epsilon} c_i} \\ &\leq \sum_{i=1}^{\infty} \frac{(i+2)^2}{c_i} < \infty. \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \alpha_n \epsilon_{n+k} \leq \frac{1}{1-2^{-\epsilon}} + \sum_{i=1}^{\infty} \frac{(i+2)^2}{(i+1)^4} = B < \infty,$$

which concludes the proof. ∎

III. A basic result

An important step in order to get the main results for discrete spaces is the following result (which extends Theorems B and B′ of [H-R$_2$] given in the particular case of p equal to the *inclusion index* γ_F):

PROPOSITION 5: *Let $0 < \alpha \le p \le \gamma \le \beta < \infty$. Then there exists an Orlicz space $\ell^F(I)$ with indices $\alpha_F = \alpha, \beta_F = \beta$ and $\gamma_F = \gamma$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$ for any set $\Gamma \subset I$.*

Proof: Since the case $p = \gamma$ has been solved in [H-R$_2$], we have to consider the cases (A) $\alpha < p < \gamma \le \beta$ and (B) $\alpha = p < \gamma \le \beta$. Now in the case (A), by using convexification and basic properties of the sets $\Sigma_{F,1}$ (see [H-R$_2$]), we can reduce to distinguish only two subcases:

(A.I) THE CASE $\alpha = 1 < p < \gamma < \beta$: We will built a convex Orlicz function F with indices $\alpha_F = 1, \beta_F = \beta = p + \epsilon$ and $\gamma_F = \gamma = p + \delta$ with $0 < \delta < \epsilon$.

Let (ϵ_n) be the sequence defined in Lemma 3. Let f be the function defined by

$$f(x) = \sum_{n=0}^{\infty} \epsilon_n \chi_{(2^{-(n+1)},2^{-n}]}(x),$$

and F be the convex function given by

$$F(x) = \int_0^x (x-t)t^{p-2} f(t)dt$$

for $0 \le x \le 1$. Using Lemma 3 we have

$$A \le \sum_{n=0}^{\infty} \alpha_n f\left(\frac{x}{2^n}\right) \le B$$

for $0 < x \le 1$. This implies by integration and the Beppo–Levi Theorem that

$$(*) \qquad \frac{Ax^p}{p(p-1)} \le \sum_{n=0}^{\infty} \alpha_n 2^{pn} F\left(\frac{x}{2^n}\right) \le \frac{Bx^p}{p(p-1)}$$

for $0 \le x \le 1$.

It follows now that the space ℓ^F(I) contains a subspace isomorphic to $\ell^p(\Gamma)$ for Γ-uncountable. Indeed, if μ denotes the discrete measure on [0,1] defined by $\mu(2^{-n}) = \alpha_n 2^{pn} F(2^{-n})$, we consider the function

$$G(x) = \int_0^1 \frac{F(xt)}{F(t)} d\mu(t) \qquad (0 \le x \le 1).$$

Then, by $(*)$, the function G is equivalent to x^p at 0, so, by [R] Theorem B or [H-T] Proposition 5, we deduce that $\ell^F(I)$ contains a subspace isomorphic to $\ell^p(\Gamma)$ (we prove below that the function F satisfies the Δ_2^0-condition).

Let us compute the associated indices. Since

$$F\left(\frac{1}{2^n}\right) = \sum_{k=n}^{\infty} \epsilon_k \int_{2^{-k-1}}^{2^{-k}} (2^{-n} - t)t^{p-2}dt$$
$$= \sum_{k=n}^{\infty} \epsilon_k 2^{-(p-1)k}(a2^{-n} - b2^{-k})$$

with

$$a = \frac{1-2^{-(p-1)}}{p-1} \quad \text{and} \quad b = \frac{1-2^{-p}}{p},$$

we deduce

$$(**) \qquad 2^{pn}F(2^{-n}) = \sum_{k=0}^{\infty}(a - b2^{-k})2^{-(p-1)k}\epsilon_{n+k}.$$

Let us show that $\alpha_F = 1$. It is enough to check that

$$\sup_{m,n} \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} = \infty$$

for every $q > 1$. Indeed, for $m = m_i - n > m_{i-1}$, using $(**)$ we have

$$2^{p(m+n)}F(2^{-m-n}) \geq (a-b)\epsilon_{m+n} = (a-b)\epsilon_{m_i}$$

and

$$2^{pm}F(2^{-m}) \leq a\left(\frac{\epsilon_m}{1-2^{-(p-1)}} + \sum_{k=0}^{\infty} \epsilon_{m_i+k}2^{-(p-1)(n+k)}\right)$$
$$\leq \frac{a}{1-2^{-(p-1)}}\left(\epsilon_m + 2^{-(p-1)n}\epsilon_{m_i}\right)$$

with

$$\frac{\epsilon_m}{\epsilon_{m_i}} = \frac{2^{-(m-m_{i-1})\epsilon}}{2^{-(m_i-m_{i-1})\delta}} = 2^{\epsilon n}2^{-(m_i-m_{i-1})(\epsilon-\delta)} \longrightarrow 0$$

for $i \to \infty$ and n fixed. Then

$$\sup_m \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} \geq \frac{a-b}{a}(1-2^{-(p-1)})2^{(q-1)n}$$

and

$$\sup_{m,n} \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} = \infty.$$

Let us now see that $\beta_F = (p+\epsilon)$. It follows from $(**)$ that

$$\frac{2^{-pn}F(2^{-m})}{F(2^{-m-n})} \le \frac{a}{a-b}\frac{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon_{m+k}}{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon_{m+n+k}} \le \frac{a}{a-b}2^{\epsilon_n},$$

so we deduce that $\beta_F \le p+\epsilon$. In order to show the converse inequality, let us consider $m = m_i < m_{i+1} - n$. Then

$$2^{pm}F(2^{-m}) \ge (a-b)\epsilon_m = (a-b)\epsilon_{m_i}$$

and

$$\begin{aligned} 2^{p(m+n)}F(2^{-m-n}) &\le \frac{a}{1-2^{-(p-1)}}\left(\epsilon_{m+n} + \epsilon_{m_{i+1}}\right) \\ &\le \frac{a}{1-2^{-(p-1)}}\left(2^{-\epsilon n}\epsilon_{m_i} + \epsilon_{m_{i+1}}\right). \end{aligned}$$

Hence, as

$$\frac{\epsilon_{m_{i+1}}}{\epsilon_{m_i}} = 2^{-\delta(m_{i+1}-m_i)} \longrightarrow 0$$

for $i \longrightarrow \infty$, we have

$$\sup_m \frac{2^{-pn}F(2^{-m})}{F(2^{-m-n})} \ge \frac{a-b}{a}(1-2^{-(p-1)})2^{\epsilon n},$$

which implies that $\beta_F \ge (p+\epsilon)$. Then $\beta_F = p+\epsilon$.

Finally, let us show that $\gamma_F = \gamma$. Since

$$2^{pm_i}F(2^{-m_i}) \le (a-b)\epsilon_{m_i} = (a-b)2^{-m_i\delta}$$

we have

$$\limsup_n \sqrt[n]{F(2^{-n})} \le 2^{-(p+\delta)}.$$

On the other hand, it follows from $(**)$ that

$$\begin{aligned} 2^{pn}F(2^{-n}) &\le a\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon_{n+k} \\ &\le a\sum_{k=0}^{\infty} 2^{-(p-1)k}2^{-(n+k)\delta} \le \frac{a}{1-2^{-(p-1)}}2^{-n\delta} \end{aligned}$$

and, hence,

$$\limsup_n \sqrt[n]{F(2^{-n})} \le 2^{-(p+\delta)}.$$

Thus $\gamma_F = p+\delta = \gamma$, which ends the proof in this case (A.I).

(A.II) THE CASE $\alpha = 1 < p < \gamma = \beta < \infty$: Given $\epsilon > 0$ with $\beta = p + \epsilon$, we proceed as in the above case but now using the sequence (ϵ_n) of Lemma 4 in order to define the functions f and F.

Let us show that $\alpha_F = 1$. Like above we consider $m = m_i - n > m_{i-1}$. Then

$$2^{p(m+n)} F(2^{-m-n}) \geq (a-b)\epsilon_m,$$

and

$$2^{pm} F(2^{-m}) \leq \frac{a}{1-2^{-(p-1)}} (\epsilon_m + 2^{-(p-1)n}\epsilon_{m_i})$$

with

$$\frac{\epsilon_m}{\epsilon_{m_i}} = \frac{2^{-\epsilon m} c_{m-m_{i-1}}^{-1}}{2^{-\epsilon m_i}} = \frac{2^{\epsilon n}}{c_{m-m_{i-1}}} \longrightarrow 0$$

for $i \longrightarrow \infty$ and fixed n. Now

$$\sup_m \frac{2^{qn} F(2^{-m-n})}{F(2^{-m})} \geq \frac{a-b}{a}(1-2^{-(p-1)}) 2^{(q-1)n}$$

for $q > 1$, and

$$\sup_{m,n} \frac{2^{qn} F(2^{-m-n})}{F(2^{-m})} = \infty.$$

Hence $\alpha_F = 1$.

Let us now prove that $\beta_F = (p+\epsilon)$. It follows from $(**)$ that

$$\begin{aligned} \frac{2^{-pn} F(2^{-m})}{F(2^{-m-n})} &\leq \frac{a}{a-b} \frac{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon_{m+k}}{\sum_{k=0}^{\infty} 2^{-(p-1)k}\epsilon_{m+n+k}} \\ &\leq \frac{a}{a-b} 2^{n\epsilon} c_n, \end{aligned}$$

which implies that $\beta_F \leq p + \epsilon$. In order to prove the converse inequality let us consider $m = m_i < m_{i+1} - n$. Now, as above,

$$2^{pm} F(2^{-m}) \geq (a-b)\epsilon_{m_i}$$

and

$$2^{p(m+n)} F(2^{-m-n}) \leq \frac{a}{1-2^{-(p-1)}} (2^{-n\epsilon} c_n^{-1} \epsilon_{m_i} + \epsilon_{m_{i+1}}).$$

Hence, as

$$\frac{\epsilon_{m_{i+1}}}{\epsilon_{m_i}} = 2^{-\epsilon(m_{i+1}-m_i)} \longrightarrow 0$$

for $i \longrightarrow \infty$, we have

$$\sup_m \frac{2^{-pn} F(2^{-m})}{F(2^{-m-n})} \geq \frac{a-b}{a}(1-2^{-(p-1)}) 2^{\epsilon n} c_n,$$

which implies $\beta_F \geq p+\epsilon$. Thus $\beta_F = p+\epsilon = \beta$.

Finally, let us also show that $\gamma_F = p+\epsilon$. Indeed, as

$$2^{pm_i} F(2^{-m_i}) \geq (a-b) 2^{-m_i \epsilon},$$

we have

$$\limsup_n \sqrt[n]{F(2^{-n})} \geq 2^{-(p+\epsilon)}.$$

On the other hand, it follows from (**) that

$$\begin{aligned} 2^{pn} F(2^{-n}) &\leq a \sum_{k=0}^{\infty} 2^{-(p-1)k} \epsilon_{n+k} \leq a \sum_{k=0}^{\infty} 2^{-(p-1)k} 2^{-(n+k)\epsilon} \\ &\leq \frac{a}{1-2^{-(p-1)}} 2^{-n\epsilon} \end{aligned}$$

and, hence,

$$\limsup_n \sqrt[n]{F(2^{-n})} \leq 2^{-(p+\epsilon)}.$$

Thus $\gamma_F = (p+\epsilon) = \beta$. This ends the proof in the cases of type (A). (Notice that the constructed Orlicz function F is always α-convex.)

We pass now to the case (B). As above, by standard tricks, we only need to consider two subcases:

(B.I) THE CASE $1 < \alpha = p < \gamma < \beta < \infty$: Let (α_n) and (ϵ_n) be sequences as in Lemma 3 for $\epsilon = (\beta - p) > \delta = (\gamma - p) > 0$. Let us consider

$$\epsilon'_n = \sum_{k=0}^{\infty} \frac{\epsilon_{n+k}}{c_k}$$

where $c_k = (k+1)^4$. Then $\epsilon'_n \leq c^k \epsilon'_{n+k} = 2^{\epsilon k} \epsilon'_{n+k}$ and there exist positive constants A' and B' such that

$$A' \leq \sum_{n=0}^{\infty} \alpha_n \epsilon'_{n+k} \leq B'$$

for $k \in \mathbb{N}$. Let

$$f(x) = \sum_{n=0}^{\infty} \epsilon'_n \chi_{(2^{-n-1}, 2^{-n}]}(x)$$

for $0 < x \leq 1$, and

$$F(x) = \int_0^x (x-t) t^{p-2} f(t)\, dt.$$

For $0 \leq x < 1$,

$$A' \frac{x^p}{p(p-1)} \leq \sum_{n=0}^{\infty} \alpha_n 2^{pn} F\left(\frac{x}{2^n}\right) \leq B' \frac{x^p}{p(p-1)}$$

holds, which implies, by using [R] Thm B or [H-T] Prop. 5, that the space $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$ for $\Gamma \subset I$.

Let us show that $\alpha_F = p$. We have

$$2^{pn} F(2^{-n}) = \sum_{k=0}^{\infty} (a - b2^{-k}) 2^{-(p-1)k} \epsilon'_{n+k},$$

and

$$\begin{aligned} \frac{2^{pn} F(2^{-m-n})}{F(2^{-m})} &\leq \frac{a}{a-b} \frac{\sum_{k=0}^{\infty} 2^{-(p-1)k} \epsilon'_{m+n+k}}{\sum_{k=0}^{\infty} 2^{-(p-1)k} \epsilon'_{m+k}} \\ &\leq \frac{a}{a-b} c_n = \frac{a}{a-b}(n+1)^4, \end{aligned}$$

since

$$\epsilon'_m \geq \sum_{k=0}^{\infty} \frac{\epsilon_{m+n+k}}{c_{n+k}} \geq \frac{1}{c_n} \sum_{k=0}^{\infty} \frac{\epsilon_{m+n+k}}{c_k} = \frac{1}{c_n} \epsilon'_{m+n}.$$

Hence $\alpha_F = p$.

Let us show now that $\beta_F = p + \epsilon$. Since

$$\frac{2^{-pn} F(2^{-m})}{F(2^{-m-n})} \leq \frac{a}{a-b} 2^{\epsilon n}$$

we have that F is $(p+\epsilon)$-concave, so $\beta_F \leq p + \epsilon$. In order to show the converse inequality, we consider $m = m_i < m_{i+1} - n$, so

$$2^{pm} F(2^{-m}) \geq (a-b)\epsilon'_m \geq (a-b)\epsilon_{m_i}$$

and

$$\begin{aligned} 2^{p(m+n)} F(2^{-m-n}) &\leq a \sum_{k=0}^{\infty} 2^{-(p-1)k} \epsilon'_{m+n+k} \\ &\leq a\epsilon'_{m+n} \sum_{k=0}^{\infty} c_k 2^{-(p-1)k} = a' \epsilon'_{m+n} \end{aligned}$$

and, hence,

$$\begin{aligned} \sup_m \frac{2^{-pn} F(2^{-m})}{F(2^{-m-n})} &\geq \frac{a-b}{a'} \sup_i \frac{\epsilon_{m_i}}{\epsilon'_{m_i+n}} \\ &\geq \frac{a-b}{a''} 2^{\epsilon n} \end{aligned}$$

for some positive constants $a'' > a'$, since

$$\begin{aligned}\frac{\epsilon'_{m_i+n}}{\epsilon_{m_i}} &= \frac{1}{\epsilon_{m_i}} \sum_{k=0}^{\infty} \frac{\epsilon_{m+n+k}}{c_k} \\ &\leq 2^{-\epsilon n} \sum_{k=0}^{m_{i+1}-m-n-1} \frac{1}{c_k} + \frac{\epsilon_{m_{i+1}}}{\epsilon_{m_i}} \sum_{k=m_{i+1}-m-n}^{\infty} \frac{1}{c_k} \\ &\longrightarrow 2^{-\epsilon n} \sum_{k=0}^{\infty} \frac{1}{c_k}\end{aligned}$$

for $i \longrightarrow \infty$. Hence $\beta_F \geq (p+\epsilon)$, so $\beta_F = p + \epsilon = \beta$.

Finally, let us show that $\gamma_F = p + \delta = \gamma$. Since

$$2^{pm_i} F(2^{-m_i}) \geq (a-b)\epsilon_{m_i} = (a-b)2^{-m_i\delta},$$

we have

$$\limsup_n \sqrt[n]{F(2^{-n})} \geq 2^{-(p+\delta)}.$$

On the other hand,

$$\begin{aligned}2^{pn} F(2^{-n}) &\leq a \sum_{k=0}^{\infty} 2^{-(p-1)k} \epsilon'_{n+k} \leq a' \sum_{k=0}^{\infty} 2^{-(p-1)k} 2^{-(n+k)\delta} \\ &\leq \frac{a'}{1-2^{-(p-1)}} 2^{-k\delta},\end{aligned}$$

hence

$$\limsup_n \sqrt[n]{F(2^{-n})} \leq 2^{-(p+\delta)}.$$

Thus $\gamma_F = p + \delta = \gamma$, which ends the proof of this case (B.I).

(B.II) THE CASE $1 < \alpha = p < \gamma = \beta < \infty$: We will proceed as in the above case but considering now (α_n) and (ϵ_n) as defined in Lemma 4. Let

$$\epsilon'_n = \sum_{k=0}^{\infty} \frac{\epsilon_{n+k}}{c_k} \qquad \text{where } c_k = (k+1)^4.$$

Then $\epsilon'_n \leq 2^{k\epsilon} c_k \epsilon'_{n+k}$ for $\epsilon = (\beta - p)$, and there exist positive constants A' and B' such that

$$A' \leq \sum_{n=0}^{\infty} \alpha_n \epsilon'_{n+k} \leq B'$$

for every $k \in \mathbb{N}$. Let

$$f(x) = \sum_{n=0}^{\infty} \epsilon_n' \chi_{(2^{-(n+1)}, 2^{-n}]}(x) \quad \text{and} \quad F(x) = \int_0^x (x-t)t^{p-2} f(t)dt$$

for $0 \le x \le 1$. Reasoning as above we get that the space $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$ for $\Gamma \subset I$, and also that $\alpha_F = p$ and $\beta_F \le p + \epsilon$. In order to see that $\beta_F \ge p + \epsilon$, we consider $m = m_i < m_{i+1} - n$, so

$$2^{pn} F(2^{-m}) \ge (a-b)\epsilon_{m_i}$$

and

$$2^{p(m+m)} F(2^{-m-n}) \le a' \epsilon'_{m+n}.$$

Hence

$$\begin{aligned} \sup_m \frac{2^{-pn} F(2^{-m})}{F(2^{-m-n})} &\ge \frac{a-b}{a'} \sup_i \frac{\epsilon_{m_i}}{\epsilon'_{m_i+n}} \\ &\ge \frac{a-b}{a''} c_n 2^{\epsilon n} \end{aligned}$$

for some positive constants $a'' > a'$, since

$$\begin{aligned} \frac{\epsilon'_{m_i+n}}{\epsilon_{m_i}} &= \frac{1}{\epsilon_{m_i}} \sum_{k=0}^{\infty} \frac{\epsilon_{m_i+n+k}}{c_k} \\ &\le c_n^{-1} 2^{-n\epsilon} \sum_{k=0}^{m_{i+1}-m-n-1} \frac{1}{c_k} + \frac{\epsilon_{m_{i+1}}}{\epsilon_{m_i}} \sum_{k=m_{i+1}-m-n} \frac{1}{c_k} \\ &\longrightarrow c_n^{-1} 2^{-n\epsilon} \sum_{k=0}^{\infty} \frac{1}{c_k} \end{aligned}$$

for $i \longrightarrow \infty$. Thus $\beta_F = p + \epsilon$.

Finally, let us show $\gamma_F = p + \epsilon = \gamma$. As

$$2^{pm_i} F(2^{-m_i}) \ge (a-b)\epsilon_{m_i} = (a-b)2^{-m_i\epsilon}$$

we have

$$\limsup_n \sqrt[n]{F(2^{-n})} \ge 2^{-(p+\epsilon)}.$$

On the other hand,

$$\begin{aligned} 2^{pn} F(2^{-n}) &\le a' \epsilon'_n \le a' \sum_{k=0}^{\infty} 2^{-(p-1)k} 2^{-(n+k)\epsilon} \\ &\le \frac{a'}{1 - 2^{-(p-1)}} 2^{-n\epsilon}, \end{aligned}$$

which implies

$$\limsup_n \sqrt[n]{F(2^{-n})} \le 2^{-(p+\epsilon)}.$$

Hence $\gamma_F = p + \epsilon = \gamma$, which concludes the proof. ∎

Remark: Note that for the case $p > \alpha$ the function F constructed in the above Proposition is α-convex. It holds in general that if an α-convex space $\ell^F(I)$ contains an isomorphic copy of $\ell^p(\Gamma)$ for $p = \alpha$ and Γ uncountable, then $\ell^F(I) = \ell^\alpha(I)$ and the function F is equivalent to x^α at 0.

IV. Key lemmas

In this Section we present some technical Lemmas, which are the key to prove later that the Orlicz spaces $\ell^F(\mathrm{I})$, constructed in Proposition 5, contain $\ell^p(\Gamma)$-subspaces for different values of p.

LEMMA 6: *Given $\delta > 0$, for every $0 \le \delta' \le \delta$ there exists an integer $k_0 \ge 0$ and a positive sequence $(\alpha'_n)_{n=0}^\infty$ such that if $(\delta_n)_{n=0}^\infty$ is the sequence defined in Lemma 2, then*

$$2^{-sk-1} \le \sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} \le 2^{-sk+1}$$

for $s = (\delta - \delta') \ge 0$ and every $k \ge k_0$.

Proof: We can assume, by Lemma 2, that $\delta' < \delta$. Thus there exists an integer $k_0 > 0$ such that $[2^{k\delta'}] < [2^{k\delta}]$ for every $k \ge k_0$.

We now apply Lemma 1 in the case of taking the sequence (h_n) as $([2^{n\delta}])$, hence finding the associated sequences $(k_i)_{i=0}^\infty$ and $(m_i)_{i=0}^\infty$. Thus, for every natural n there exist exactly $[2^{n\delta}]$ couples (k_i, m_j) such that $m_j - k_i = n$. Let us denote by A_n the set of the $[2^{n\delta}]$ natural numbers k_i that appear in this pair (k_i, m_j), and by A'_n we denote the subset of A_n given by the last $[2^{n\delta'}]$ natural members k_i which are in A_n.

We claim that the set

$$B_k = \bigcup_{\substack{n \ge k_0 \\ n \ne k}} A'_n \cap A_k$$

contains at most one element. Indeed, if $k_i \in B_k$ then there exists m_j such that $m_j - k_i = k$, and hence $j \ge i$. Now if $j > i$ it follows from the construction in Lemma 1 that $m_{j-1} < k < m_j$, and this implies that there is at most a natural j verifying those. Let us suppose now $j = i$, which means $k = m_i - k_i = \ell_i$. Since $k_i \in A'_n$ for some $n \ge k_0$ with $n \ne k$, there exists $j' > i$ such that $n = m_{j'} - k_i$.

Hence $n > k$. Let us now show that k_i is the first element of the set A_n, which will contradict that $k_i \in A'_n$. If $n = m_{i'} - k_{i'} = \ell_{i'}$, as $n > k$, we have $i' > i$ and hence $k_{i'} > k_i$. Thus k_i is the first element of A_n.

We pass now to define the sequence $(\alpha'_n)_{n=0}^{\infty}$ by

$$\alpha'_n = \begin{cases} 2^{n\delta} & \text{if } n \in \bigcup_{h=k_0}^{\infty} A'_h, \\ 0 & \text{otherwise.} \end{cases}$$

Since $A_k \cap \bigcup_{h=k_0}^{\infty} A'_h = A'_k \cup B_k$ we have

$$\sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} = \sum_{n \in A'_k \cup B_k} 2^{n\delta} \delta_{n+k} 2^{-(n+k)\delta}$$

for $k \geq k_0$. Hence

$$\sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} \leq \frac{1}{2^{k\delta}}(1 + [2^{k\delta'}]) \leq 2^{-ks+1}$$

and

$$\sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} \geq \frac{[2^{k\delta'}]}{2^{k\delta}} \geq 2^{-ks-1}$$

for every $k \geq k_0$ and $s = (\delta - \delta')$, which concludes the proof. ∎

LEMMA 7: *Let $\epsilon > \delta \geq 0, (\epsilon_n)$ be the sequence defined in Lemma 3, and (α'_n) be the sequence defined in Lemma 6 associated with each $0 \leq \delta' \leq \delta$. Then*

$$\sum_{n=0}^{\infty} \alpha'_n 2^{-n\delta} = \infty$$

and

$$A' 2^{-sk} \leq \sum_{n=0}^{\infty} \alpha'_n \epsilon_{n+k} \leq B' 2^{-s(k+1)}$$

for every natural $k \geq 0$, and $s = (\delta - \delta') \geq 0$, and where A' and B' are positive constants.

Proof: We will proceed as in Lemma 3 but using now Lemma 6. We have

$$\sum_{n=0}^{\infty} \alpha'_n \epsilon_{n+k} \geq \sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\delta} \geq \frac{1}{2} 2^{-sk}$$

for $k \geq k_0$, and

$$\sum_{i=0}^{k} \sum_{n+k \in M_i} \alpha'_n \epsilon_{n+k} \leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \frac{\alpha'_n \delta_{n+k-i} 2^{-(n+k-i)\delta}}{2^{i\epsilon}}$$
$$\leq \sum_{i=0}^{\infty} \frac{2^{-s(k-i)+1}}{2^{i\epsilon}} = 2^{-s(k+1)} \sum_{i=0}^{\infty} \frac{2^{1+s}}{2^{i(\epsilon-s)}} < \infty,$$

since $\epsilon - s \geq \epsilon - \delta > 0$.

Using now that $\alpha'_n \leq \alpha_n$ we have

$$\sum_{i=k+1}^{\infty} \sum_{n+k \in M_i} \alpha'_n \epsilon_{n+k} \leq \sum_{i=k+1}^{\infty} \sum_{n} \frac{\alpha'_n \delta_{n+k-i} 2^{-(n+k-i)\delta}}{2^{i\epsilon}}$$
$$\leq \sum_{i=k+1}^{\infty} \frac{(i-k+2)^2}{2^{i\epsilon}} 2^{(i-k)\delta}$$
$$= \frac{1}{2^{(k+1)\epsilon}} \sum_{i=k+1}^{\infty} \frac{2^{\epsilon}(i-k+2)^2}{2^{(i-k)(\epsilon-\delta)}}$$
$$= \frac{1}{2^{(k+1)\epsilon}} \sum_{i=1}^{\infty} \frac{2^{\epsilon}(i+2)^2}{2^{i(\epsilon-\delta)}} < \infty.$$

Hence as $s < \epsilon$, there exist positive constants $A' = \frac{1}{2}$ and B' such that

$$A' 2^{-sk} \leq \sum_{n=0}^{\infty} \alpha'_n \epsilon_{n+k} \leq B' 2^{-s(k+1)}$$

for $k \geq k_0$, which concludes the proof since we can replace the constants in order to get that the above inequality holds for every $k \geq 0$. ∎

In the extreme case of $\delta = \epsilon$ we also have a similar result:

LEMMA 8: *Let $\epsilon > 0$ and (ϵ_n) be the sequence defined in Lemma 4. Let $0 < \delta' \leq \delta = \epsilon$ and (α'_n) be the sequence defined in Lemma 6. Then*

$$\sum_{n=0}^{\infty} \alpha'_n 2^{-n\delta} = \infty$$

and

$$A' 2^{-sk} \leq \sum_{n=0}^{\infty} \alpha'_n \epsilon_{n+k} \leq B' 2^{-s(k+1)}$$

for every $k \geq 0$ and $s = (\epsilon - \delta') \geq 0$, and where A' and B' are positive constants.

Proof: We will proceed in a similar way as in Lemma 4. We have, by Lemma 6, that

$$\sum_{n=0}^{\infty} \alpha'_n \epsilon_{n+k} \geq \sum_{n=0}^{\infty} \alpha'_n \delta_{n+k} 2^{-(n+k)\epsilon} \geq \frac{1}{2} 2^{-sk}$$

for $k \geq k_0$. Now,

$$\sum_{i=0}^{k} \sum_{n+k \in M_i} \alpha'_n \epsilon_{n+k} \leq \sum_{i=0}^{k} \sum_{n=0}^{\infty} \frac{\alpha'_n \delta_{n+k-i} 2^{-(n+k-i)\epsilon}}{2^{i\epsilon} c_i}$$
$$\leq \sum_{i=0}^{\infty} \frac{2^{-s(k-i)+1}}{2^{i\epsilon} c_i} = 2^{-s(k+1)} \sum_{i=0}^{\infty} \frac{2^{1+s}}{2^{i\delta'} c_i} < \infty$$

since $\epsilon - s = \delta' \geq 0$ and $c_i = (i+2)^4$. And, as $\alpha'_n \leq \alpha_n$, we have

$$\sum_{i=k+1}^{\infty} \sum_{n+k \in M_i} \alpha'_n \epsilon_{n+k} \leq \sum_{i=k+1}^{\infty} \frac{(i-k+2)^2 2^{(i-k)\epsilon}}{2^{i\epsilon} c_i}$$
$$= \frac{1}{2^{\epsilon(k+1)}} \sum_{i=k+1}^{\infty} \frac{2^{\epsilon}(i-k+2)^2}{c_i}$$
$$\leq \frac{1}{2^{\epsilon(k+1)}} \sum_{i=1}^{\infty} \frac{2^{\epsilon}(i+2)^2}{c_i} < \infty.$$

Hence, as $s \leq \epsilon$, there exist positive constants $A' = \frac{1}{2}$ and B' such that for $k \geq k_0$ we have

$$A' 2^{-sk} \leq \sum_{n=0}^{\infty} \alpha'_n \epsilon_{n+k} \leq B' 2^{-s(k+1)}$$

for every $k \geq k_0$, which concludes the proof as in Lemma 7. ∎

V. Main results in the discrete case

This section is devoted to prove the main results in the discrete case stated in the Introduction.

THEOREM 3: *Let $0 < \alpha \leq \gamma \leq \beta < \infty$. Then there exists an Orlicz space $\ell^F(I)$ with indices $\alpha_F = \alpha$, $\beta_F = \beta$ and $\gamma_F = \gamma$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^q(\Gamma)$ if and only if $q \in [\alpha_F, \gamma_F]$.*

Proof: The extreme case $\alpha = \gamma$ has been solved in [H-R$_2$] Thm. B′, so we can assume $\alpha < \gamma \leq \beta$. We will distinguish two cases:

(i) The case $\alpha < \gamma < \beta$

Using standard arguments (see [H-R$_2$]) we can suppose w.l.o.g. $1 < \alpha = p < \gamma = p + \delta < \beta = p + \epsilon$. Let f and F be the functions defined in Proposition 5, i.e.

$$f(x) = \sum_{n=0}^{\infty} \epsilon_n \chi_{[2^{-(n+1)}, 2^{-n})}(x) \quad \text{and} \quad F(x) = \int_0^x (x-t) t^{p-2} f(t) dt$$

for $0 \le x \le 1$. Let (α'_n) be the sequence defined in Lemma 7 associated with each $0 \le \delta' \le \delta$, hence there exist positive constants A' and B' such that

$$A' 2^{-sk} \le \sum_{n=0}^{\infty} \alpha'_n \epsilon_{n+k} \le B' 2^{-s(k+1)}$$

for every $k \ge 0$ and $s = \delta - \delta' \ge 0$. This implies that

$$A' x^s \le \sum_{n=0}^{\infty} \alpha'_n f\left(\frac{x}{2^n}\right) \le B' x^s$$

for $0 < x \le 1$. And, by integration and the Beppo–Levi Theorem, we get

$$A' \frac{x^{p+s}}{(p+s)(p+s-1)} \le \sum_{n=0}^{\infty} \alpha'_n 2^{pn} F\left(\frac{x}{2^n}\right) \le B' \frac{x^{p+s}}{(p+s)(p+s-1)}$$

for $0 < x \le 1$. Then, if μ' denotes the discrete measure on (0,1] defined by $\mu'(2^{-n}) = \alpha'_n 2^{pn} F(2^{-n})$ and we consider the function

$$G(x) = \int_0^1 \frac{F(xt)}{F(t)} d\mu'(t) \quad (0 \le x \le 1),$$

we get that G is equivalent to the function $x^{p+s} = x^{p+\delta-\delta'}$ at 0. Hence, by using ([R] Thm B or [H-T] Prop. 5), we conclude that $\ell^F(I)$ contains an isomorphic copy of $\ell^q(\Gamma)$ for every $\alpha \le q = p + s = p + \delta - \delta' \le p + \delta = \gamma < \beta$.

(ii) The case $\alpha < \gamma = \beta$

We can assume w.l.o.g. that $1 < \alpha < \gamma = \beta$. Let us denote $\alpha = p < \beta = p + \epsilon$ and define the functions f and F as in the case (B) of Proposition 5. Now we consider the sequence (α'_n) associated with $0 \le \delta' \le \delta = \epsilon$ in Lemma 8. Then there exist two positive constants A' and B' such that

$$A' 2^{-sk} \le \sum_{n=0}^{\infty} \alpha'_n \epsilon'_{n+k} \le B' 2^{-s(k+1)}$$

for every $k \geq 0$ and $s = \delta - \delta' = \epsilon - \delta' \geq 0$. Hence

$$A'x^s \leq \sum_{n=0}^{\infty} \alpha'_n f\left(\frac{x}{2^n}\right) \leq B'x^s$$

for $0 < x \leq 1$, and by integration,

$$A' \frac{x^{p+s}}{(p+s)(p+s-1)} \leq \sum_{n=0}^{\infty} \alpha'_n 2^{pn} F\left(\frac{x}{2^n}\right) \leq B' \frac{x^{p+s}}{(p+s)(p+s-1)}$$

for $0 \leq x \leq 1$. This implies, reasoning as in the above case, that the space $\ell^F(I)$ contains an isomorphic copy of $\ell^q(\Gamma)$ for every $\alpha \leq q = p + s \leq p + \epsilon = \gamma$.

This ends the proof of the Theorem since we have already computed the indices of the spaces $\ell^F(\mathrm{I})$ in Proposition 5. ∎

The following result shows that in general the set of scalar p's such that $\ell^p(\Gamma)$ embeds isomorphically into a given space $\ell^F(I)$ is not closed (compare with the countable case [L-T$_1$], [L-T$_2$], [K$_1$])

THEOREM 4: *Let $0 < \alpha < \gamma \leq \beta < \infty$. There exists an α-convex Orlicz space $\ell^F(I)$ with indices $\alpha_F = \alpha$, $\beta_F = \beta$ and $\gamma_F = \gamma$ such that $\ell^F(I)$ contains an isomorphic copy of $\ell^q(\Gamma)$ if and only if $q \in (\alpha_F, \gamma_F]$.*

Proof: We shall differentiate two cases:

(i) The case $\alpha < \gamma < \beta$

Using standard tricks we can assume w.l.o.g. $1 = \alpha < \gamma = 1 + \delta < \beta = 1 + \epsilon$. Let (ϵ_n) be the sequence defined in Lemma 3, and f and F be the functions

$$f(x) = \sum_{n=0}^{\infty} \epsilon_n \chi_{(2^{-(n+1)}, 2^{-n}]}(x) \quad \text{and} \quad F(x) = \int_0^x (x-t)t^{-1} f(t) dt$$

for $0 \leq x \leq 1$. It holds that F is a convex function (since $F'(x) \geq 0$ and increasing) and

$$(*) \qquad F(2^{-n}) = \sum_{k=n}^{\infty} \epsilon_k (a 2^{-n} - b 2^{-k}) < \infty$$

where $a = \log 2$ and $b = \frac{1}{2}$ (since $\limsup_n \sqrt[n]{\epsilon_n} = 2^{-\delta} = 2^{-(\gamma-1)} < 1$).

In order to show that α_F=1 let us prove

$$\sup_{m,n} \frac{2^{qn} F(2^{-m-n})}{F(2^{-m})} = \infty$$

for every $q > 1$. Indeed, for $m = m_i - n > m_{i-1}$ and using $(*)$ we have

$$2^{m+n}F(2^{-m-n}) \geq (a-b)\epsilon_{m+n} = (a-b)\epsilon_{m_i}$$

and

$$\begin{aligned} 2^m F(2^{-m}) &\leq a\sum_{k=m}^{\infty} \epsilon_k = a(\epsilon_m + \cdots + \epsilon_{m_i} + \cdots) \\ &\leq \frac{a}{1-2^{-\delta}}(\epsilon_m + \epsilon_{m_i}) \end{aligned}$$

with

$$\frac{\epsilon_m}{\epsilon_{m_i}} = \frac{2^{-(m-m_{i-1})}}{2^{-(m_i-m_{i-1})\delta}} = 2^{\epsilon n}2^{-(m_i-m_{i-1})(\epsilon-\delta)} \longrightarrow 0$$

for $i \to \infty$ and n fixed since $\epsilon > \delta$. Then

$$\sup_m \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} \geq \frac{a-b}{a}(1-2^{-\delta})2^{(q-1)n}$$

and

$$\sup_{m,n} \frac{2^{qn}F(2^{-m-n})}{F(2^{-m})} = \infty$$

for every $q > 1$. Hence $\alpha_F = 1$.

Let us show now that $\beta_F = 1 + \epsilon$. It follows from $(*)$ that

$$\frac{2^{-n}F(2^{-m})}{F(2^{-m-n})} \leq \frac{a}{a-b}\frac{\sum_{k=0}^{\infty}\epsilon_{m+k}}{\sum_{k=0}^{\infty}\epsilon_{m+n+k}} \leq \frac{a}{a-b}2^{\epsilon n},$$

which implies that $\beta_F \leq 1+\epsilon$. In order to get the converse inequality, let us take $m = m_i < m_{i+1} - n$. Then

$$2^m F(2^{-m}) \geq (a-b)\epsilon_m = (a-b)\epsilon_{m_i}$$

and

$$\begin{aligned} 2^{m+n}F(2^{-m-n}) &\leq \frac{a}{1-2^{-\delta}}(\epsilon_{m+n} + \epsilon_{m_{i+1}}) \\ &\leq \frac{a}{1-2^{-\delta}}(2^{-\epsilon n}\epsilon_{m_i} + \epsilon_{m_{i+1}}). \end{aligned}$$

Hence, as

$$\frac{\epsilon_{m_{i+1}}}{\epsilon_{m_i}} = 2^{-\delta(m_{i+1}-m_i)} \longrightarrow 0$$

for $i \longrightarrow \infty$, we have

$$\sup_m \frac{2^{-n}F(2^{-m})}{F(2^{-m-n})} \geq \frac{a-b}{a}(1-2^{-\delta})2^{\epsilon n},$$

which implies that $\beta_F \geq 1+\epsilon$. Thus $\beta_F = 1+\epsilon$.

It holds also that $\gamma_F = \gamma$. Indeed, as

$$2^{m_i} F(2^{-m_i}) \geq (a-b)\epsilon_{m_i} = (a-b)2^{-m_i\delta},$$

we deduce that

$$\limsup_n \sqrt[n]{F(2^{-n})} \geq 2^{-(1+\delta)}.$$

On the other hand, it follows from $(*)$ that

$$2^n F(2^{-n}) \leq a \sum_{k=0}^{\infty} \epsilon_{n+k} \leq a \sum_{k=0}^{\infty} 2^{-(n+k)\delta} \leq \frac{a}{1-2^{-\delta}} 2^{-\delta n}$$

and, hence,

$$\limsup_n \sqrt[n]{F(2^{-n})} \leq 2^{-(1+\delta)}.$$

Thus $\gamma_F = 1+\delta = \gamma$.

Finally, given $1 = \alpha < \gamma = 1+\delta < \beta = 1+\epsilon$, it follows from Lemma 7 that for every $0 \leq \delta' < \delta$ there exists a sequence (α'_n) verifying

$$A' x^s \leq \sum_{n=0}^{\infty} \alpha'_n f\left(\frac{x}{2^n}\right) \leq B' x^s$$

for $0 < x \leq 1, s = \delta - \delta' \geq 0$ and A' and B' positive constants. This implies, by integration and the Beppo–Levi Theorem, that

$$A' \frac{x^{1+s}}{(1+s)s} \leq \sum_{n=0}^{\infty} \alpha'_n 2^n F\left(\frac{x}{2^n}\right) \leq B' \frac{x^{1+s}}{(1+s)s}$$

for $0 < x \leq 1$ and $s > 0$. Now, using [R] Thm B or [H-T] Prop. 5, we conclude that $\ell^F(I)$ contains a subspace isomorphic to $\ell^q(\Gamma)$ for every $q = 1+s \leq 1+\delta = \gamma$ and hence for every $1 < q \leq \gamma$. Finally, as the function F is convex, the space $\ell^F(I)$ cannot contain a $\ell^1(\Gamma)$-subspace.

(ii) The case $\alpha < \gamma = \beta$

We can assume w.l.o.g. $1 = \alpha < \gamma = \beta = 1+\epsilon$. We proceed as above to build the sequences (ϵ_n) and (α_n), using now Lemma 4. Thus, let f and F be defined by

$$f(x) = \sum_{n=0}^{\infty} \epsilon_n \chi_{(2^{-(n+1)}, 2^{-n}]} \quad \text{and} \quad F(x) = \int_0^x (x-t)t^{-1} f(t)dt$$

for $0 \leq x \leq 1$.

Let us show that $\alpha_F = 1$. If $m = m_i - n > m_{i-1}$ then

$$2^{m+n} F(2^{-m-n}) \geq (a-b)\epsilon_{m_i}$$

and

$$2^m F(2^{-m}) \leq \frac{a}{1-2^{-\epsilon}}(\epsilon_m + \epsilon_{m_i}),$$

with

$$\frac{\epsilon_m}{\epsilon_{m_i}} = \frac{2^{-\epsilon m} c_{m-m_{i-1}}^{-1}}{2^{-\epsilon m_i}} = \frac{2^{\epsilon n}}{c_{m-m_{i-1}}} \longrightarrow 0$$

for $i \longrightarrow \infty$ and fixed n. Thus

$$\sup_m \frac{2^{qn} F(2^{-m-n})}{F(2^{-m})} \geq \frac{a-b}{a}(1-2^{-\epsilon})2^{(q-1)n}$$

and

$$\sup_{m,n} \frac{2^{qn} F(2^{-m-n})}{F(2^{-m})} = \infty$$

for every $q > 1$, and, hence, $\alpha_F = 1$.

Let us show that $\beta_F = 1 + \epsilon$. It follows from $(*)$ that

$$\frac{2^{-n} F(2^{-m})}{F(2^{-m-n})} \leq \frac{a}{a-b} \frac{\sum_{k=0}^{\infty} \epsilon_{m+k}}{\sum_{k=0}^{\infty} \epsilon_{m+n+k}} \leq \frac{a}{a-b} 2^{n\epsilon} c_n,$$

which implies that $\beta_F \leq 1 + \epsilon$. To prove the converse inequality we consider $m = m_i < m_{i+1} - n$. Then

$$2^m F(2^{-m}) \geq (a-b)\epsilon_{m_i}$$

and

$$\begin{aligned} 2^{m+n} F(2^{-m-n}) &\leq \frac{a}{1-2^{-\epsilon}}(\epsilon_{m_{i+n}} + \epsilon_{m_{i+1}}) \\ &\leq \frac{a}{1-2^{-\epsilon}}(2^{-n\epsilon} c_n^{-1} \epsilon_{m_i} + \epsilon_{m_{i+1}}). \end{aligned}$$

Hence, as

$$\frac{\epsilon_{m_{i+1}}}{\epsilon_{m_i}} = 2^{-\epsilon(m_{i+1}-m_i)} \longrightarrow 0$$

for $i \longrightarrow \infty$, we have

$$\sup_m \frac{2^{-n} F(2^{-m})}{F(2^{-m-n})} \geq \frac{a-b}{a}(1-2^{-\epsilon})2^{\epsilon n} c_n,$$

which implies $\beta_F \geq 1 + \epsilon$. Thus $\beta_F = 1 + \epsilon$.

It also holds that $\gamma_F = 1+\epsilon = \gamma$. Indeed, since

$$2^{m_i} F(2^{-m_i}) \geq (a-b) 2^{-m_i \epsilon}$$

we have $\limsup_n \sqrt[n]{F(2^{-n})} \geq 2^{-(1+\epsilon)}$. On the other hand, from (*) it follows that

$$2^n F(2^{-n}) \leq a \sum_{k=0}^{\infty} \epsilon_{n+k} \leq a \sum_{k=0}^{\infty} 2^{-(n+k)\epsilon} \leq \frac{a}{1-2^{-\epsilon}} 2^{\epsilon n}.$$

Hence $\limsup_n \sqrt[n]{F(2^{-n})} \leq 2^{-(1+\epsilon)}$. Thus $\gamma_F = 1+\epsilon$.

Finally, reasoning as in the above case, using now Lemma 8 instead of Lemma 7, we deduce that $\ell^F(I)$ contains an isomorphic copy of $\ell^q(\Gamma)$, for $\Gamma \subset I$ uncountables, if (and only if) $1 < q \leq \gamma = \beta$. ∎

VI. Main function space results

In this Section we prove the main function space Theorems, which follow now quite easily from the above results:

THEOREM 1: *Let $0 < \alpha \leq \gamma \leq \beta < \infty$. There exists an Orlicz function space $L^F[0,1]$ with indices $\alpha_F^\infty = \alpha$, $\beta_F^\infty = \beta$ and $\gamma_F^\infty = \gamma$ such that $L^F[0,1]$ contains a lattice-isomorphic copy of L^p for every $p \in [\gamma_F^\infty, \beta_F^\infty]$.*

Remark: In the case $\alpha = \gamma = \beta$ the above space $L^F[0,1]$ is non-trivial (i.e. $L^F[0,1] \neq L^p[0,1]$).

Proof: Let $0 < \alpha \leq \gamma \leq \beta < r < \infty$. We consider $\alpha_0 = r - \beta, \beta_0 = r - \alpha$ and $\gamma_0 = r - \gamma$. It follows from Theorem 3 that there exists an Orlicz function F_0 with indices at 0, $\alpha_{F_0} = \alpha_0$, $\beta_{F_0} = \beta_0$ and $\gamma_{F_0} = \gamma_0$ such that for every q with $\alpha_{F_0} \leq q \leq \gamma_{F_0}$ there exists a probability measure μ_q on [0,1] for which the function G_q, defined by

$$G_q(x) = \int_0^1 \frac{F_0(xt)}{F_0(t)} d\mu_q(t) \quad (0 \leq x \leq 1),$$

is equivalent to the function x^q at 0. Since $r > \beta_0$ we can assume w.l.o.g. that $F_0(st) \geq s^r F_0(t)$ for $0 \leq s,t \leq 1$.

Let us consider now the non-decreasing function F defined by

$$F(x) = x^r F_0\left(\frac{1}{x}\right), \quad \text{for } x \geq 1.$$

Now $\alpha_F^\infty = r - \beta_0 = \alpha$, $\beta_F^\infty = r - \alpha_0 = \beta$ and $\gamma_F^\infty = r - \gamma_0 = \gamma$ hold. Also, the function G, defined by $G(x) = x^r G_q(x^{-1})$ for $x \geq 1$, satisfies

$$G(x) = \int_1^\infty \frac{F(xt)}{F(t)} d\mu(t) \quad (x \geq 1),$$

where μ is a probability measure on $[1, \infty)$ given by $\mu(t) = \mu_q(1/t)$. Hence G is equivalent to the function $x^{r-q} = x^p$ at ∞. Thus $x^p \in \Sigma_{F,1}^\infty$ for $\gamma \leq p \leq \beta$ and, by [J-M-S-T] Thm. 7.7, [H-R$_2$], we conclude that L^p is lattice-isomorphic to a sublattice of $L^F[0,1]$ for every $\gamma_F^\infty \leq p \leq \beta_F^\infty$. ∎

Remark: When $\alpha < \gamma \leq \beta$, the above result can even be obtained for *α-convex* Orlicz function spaces $L^F[0,1]$.

The proof is the same, only we need to realize that the associated function F_0 defined at 0 is equivalent to a β_0-concave function at 0 in all the possible cases: when $\alpha_0 = \gamma_0$ it is proved in Theorem B$'$ of [H-R$_2$] and when $\alpha_0 < \gamma_0$ it is shown in Proposition 5. Now from the β_0-concavity of F_0 at 0 it follows that the function F, defined at ∞ by $F(x) = x^r F_0(1/x)$ is equivalent to an α-convex function at ∞, since

$$\sup_{1 \leq x,y} \frac{y^\alpha F(x)}{F(xy)} = \sup_{0 \leq u,v \leq 1} \frac{v^{\beta_0} F_0(u)}{F_0(uv)} < \infty.$$

Finally, let us show that in general for r.i. function spaces $X[0,1]$ the set P_X of scalar p's for which L^p is lattice embedded into $X[0,1]$ is not closed:

THEOREM 2: *Given $0 < \alpha \leq \gamma < \beta < \infty$, there exists a β-concave Orlicz function space $L^F[0,1]$ with indices $\alpha_F^\infty = \alpha$, $\beta_F^\infty = \beta$ and $\gamma_F^\infty = \gamma$ such that $L^F[0,1]$ contains a lattice-isomorphic copy of L^p if and only if $p \in [\gamma_F^\infty, \beta_F^\infty)$.*

Proof: Let us consider the case $2 < \alpha < \beta < r < \infty$ (the general case can be deduced from this by convexification). Let us take $\alpha_0 = r - \beta < \gamma_0 = r - \gamma \leq \beta_0 = r - \alpha$. We apply Theorem 4 to find an Orlicz function F_0 with indices α_0, γ_0 and β_0 at 0, which is α_0-convex at 0 and $x^q \in \Sigma_{F_0,1}$ for every $q \in (\alpha_0, \gamma_0]$. Now we define the Orlicz function F at ∞, by $F(x) = x^r F_0(1/x)$ for $x > 1$. Also, $\alpha_F^\infty = \alpha$, $\beta_F^\infty = \beta$ and $\gamma_F^\infty = \gamma$ hold, and $x^p \in \Sigma_{F,1}^\infty$ for every $p \in [\gamma_F^\infty, \beta_F^\infty)$. Using [J-M-S-T] Thm. 7.7, we deduce that $L^F[0,1]$ contains a sublattice lattice-isomorphic to L^p for every $p \in [\gamma_F^\infty, \beta_F^\infty)$.

Finally, since F_0 is an α_0-convex function at 0, and

$$\inf_{1 \leq x,y} \frac{y^\beta F(x)}{F(xy)} = \inf_{0 < u,v \leq 1} \frac{v^{\alpha_0} F_0(u)}{F_0(uv)} > 0,$$

we have that F_0 is equivalent to a β-concave function at ∞. Hence it follows from Proposition 5 in [G-H] (also Theorem 3.6 in [H-K]), that the space L^β cannot be lattice isomorphic to a subspace of $L^F[0,1]$, which concludes the proof. ∎

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