Some dimensional results for homogeneous Moran sets*

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Abstract Let $\mathscr{M}(\{n_k\}_{k\geqslant1},\{c_k\}_{k\geqslant1})$ be the collection of homogeneous Moran sets determined by $\{n_k\}_{k\geqslant1}$ and $\{c_k\}_{k\geqslant1}$, where $\{n_k\}_{k\geqslant1}$ is a sequence of positive integers and $\{c_k\}_{k\geqslant1}$ a sequence of positive numbers. Then the maximal and minimal values of Hausdorff dimensions for elements in $\mathscr{M}$ are determined. The result is proved that for any value s between the maximal and minimal values, there exists an element in $\mathscr{M}(\{n_k\}_{k\geqslant1},\{c_k\}_{k\geqslant1})$ such that its Hausdorff dimension is equal to s. The same results hold for packing dimension. In the meantime, some other properties of homogeneous Moran sets are discussed.

Keywords: homogeneous Moran set, homogeneous Cantor set, partial homogeneous Cantor set, Hausdorff dimension, packing dimension.

1 Homogeneous Moran sets

Let $\{n_k\}_{k\geqslant1}$ be a sequence of positive integers and $\{c_k\}_{k\geqslant1}$ be a sequence of positive numbers satisfying $n_k\geqslant2$, $0<c_k<1$, $n_1c_1\leqslant\delta$ and $n_kc_k\leqslant1$ $(k\geqslant2)$, where δ is a positive number. For any $k\geqslant1$, let $D_k=\{(i_1,\cdots,i_k);1\leqslant i_j\leqslant n_j,1\leqslant j\leqslant k\}$, $D=\bigcup_{k\geqslant0}D_k$, where $D_0=\emptyset$. If $\sigma=(\sigma_1,\cdots,\sigma_k)\in D_k$, $\tau=(\tau_1,\cdots,\tau_m)\in D_m$, let $\sigma*\tau=(\sigma_1,\cdots,\sigma_k,\tau_1,\cdots,\tau_m)$.

Definition 1.1. Suppose that J is a closed interval of length δ. The collection of closed subintervals $\mathscr{F}=\{J_\sigma;\sigma\in D\}$ of J has homogeneous Moran structure, if it satisfies:

(i) $J_\emptyset=J$;

(ii) $\forall\, k\geqslant0,\sigma\in D_k, J_{\sigma*1},J_{\sigma*2},\cdots,J_{\sigma*n_{k+1}}$ are subintervals of J_σ, and $\mathring{J}_{\sigma*i}\cap\mathring{J}_{\sigma*j}=\emptyset(i\neq j)$, where $\mathring{A}$ denotes the interior of A;

(iii) for any $k\geqslant1$ and any $\sigma\in D_{k-1}, 1\leqslant j\leqslant n_k$, we have
$$c_k=\frac{|J_{\sigma*j}|}{|J_\sigma|},$$
where $|A|$ denotes the diameter of A.

Suppose that $\mathscr{F}$ is a collection of closed subintervals of J having homogeneous Moran structure. We call $E(\mathscr{F}):=\bigcap_{k\geqslant1}\bigcup_{\sigma\in D_k}J_\sigma$ a homogeneous Moran set determined by $\mathscr{F}$, and call $\mathscr{F}_k=\{J_\sigma;\sigma\in D_k\}$ the k-order fundamental intervals of $E(\mathscr{F})$. J is called the original interval of $E(\mathscr{F})$.

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By the definition above, we can see that for any fixed closed interval J, $\{n_k\}_{k\geq 1}$, $\{c_k\}_{k\geq 1}$, if the positions of k-order fundamental intervals are different, different homogeneous Moran sets are obtained. We use $\mathscr{M}(J, \{n_k\}, \{c_k\})$ to denote the collection of homogeneous Moran sets determined by J, $\{n_k\}_{k\geq 1}$, $\{c_k\}_{k\geq 1}$. We use $\mathscr{M}$ for the sake of convenience if it does not cause any confusion.

Homogeneous Moran sets are very important fractal sets. Some special cases have been studied by Moran[1]. Under the condition $\inf_k c_k > 0$, Hua[2] and Marion[3] studied the generalized self-similar set, a special case of homogeneous Moran sets, and obtained the Hausdorff dimension. In ref. [4], Hua and Li obtained the packing dimension. In ref. [5], Feng *et al*. considered a class of homogeneous Moran sets, and called them homogeneous Cantor set and obtained their Hausdorff dimensions. For all the cases considered above, the positions of k-order fundamental intervals have been determined by the positions of $(k-1)$-order fundamental intervals. None of the above has considered the case where the positions of fundamental intervals can vary. On the other hand, as we mentioned above, the construction of a homogeneous Moran set depends tightly on the relative position of fundamental intervals. Therefore, a natural question is whether the dimension varies if the relative position of fundamental intervals are different. If it varies, does it vary "continuously"; i.e. for any value s in the varying scope, does there exist an element of $\mathscr{M}$ such that its dimension is equal to s? In this paper, we will answer these questions. In the meantime, we will discuss some other dimensional properties of homogeneous Moran sets.

For the definitions and properties of Hausdorff measure, Hausdorff dimension, packing measure, packing dimension and Bouligand dimension, please refer to reference [6].

2 Hausdorff dimensions of homogeneous Moran sets

In order to discuss the dimensions of sets in $\mathscr{M}(J, \{n_k\}, \{c_k\})$, we first consider the maximal value and minimal value of the dimension of elements in $\mathscr{M}(\{n_k\}_{k\geq 1}, \{c_k\}_{k\geq 1})$. By the definition of dimension, considering the economic covering, the set whose fundamental intervals have homogeneous gaps may get the maximal value, and the set of which the total gap of fundamental intervals is minimal may get the minimal value (of course, the two numbers may be equal). For this reason, we introduce two special homogeneous Moran sets $C := C(J, \{n_k\}, \{c_k\})$ and $C^* = C^*(J, \{n_k\}, \{c_k\})$, and call them homogeneous Cantor set, partial homogeneous Cantor set, respectively, with respect to J, $\{n_k\}_{k\geq 1}$, $\{c_k\}_{k\geq 1}$. The definitions are as follows.

Suppose $E \in \mathscr{M}(J, \{n_k\}, \{c_k\})$ and $I \in \mathscr{F}_k$, $k \geq 1$. Let $I_1, I_2, \cdots, I_{n_{k+1}}$ be the n_{k+1}-order fundamental intervals contained in I which are spaced from left to right.

(i) If the gaps between I_i and I_{i+1} $(1 \leq i < n_{k+1})$ are equal, and the left endpoint of I_1 is the same as the left endpoint of I, and the right endpoint of $I_{n_{k+1}}$ is the same as the right endpoint of I, then E is called homogeneous Cantor set.

(ii) If the left endpoint of I_1 is the same as the left endpoint of I, and the left endpoint of I_{i+1} is the same as the right endpoint of I_i, $1 \leq i \leq n_{k+1} - 1$ (i.e. the gap between two adjoining fundamental intervals is equal to zero), then E is called partial homogeneous Cantor set.

Suppose that $C^* := C^*(J, \{n_k\}, \{c_k\})$ is a partial homogeneous Cantor set. For any integer $l \geqslant 0$, let

$$u_l := \sum_{k=l+1}^{\infty} (n_k - 1) \prod_{i=1}^{k} c_i;$$

denote $J' = [0, u_0]$ and $d_k = \frac{u_k}{u_{k-1}}$, $k \geqslant 1$. By the definitions of homogeneous Cantor set and partial homogeneous Cantor set, we have

Lemma 2.1. *Following the above notation, we have*

$$C^*(J, \{n_k\}, \{c_k\}) = C(J', \{n_k\}, \{d_k\}),$$

i.e. the partial homogeneous Cantor set in $\mathcal{M}(J, \{n_k\}, \{c_k\})$ *is the homogeneous Cantor set in* $\mathcal{M}(J', \{n_k\}, \{c_k\})$.

Remark 2.1. Suppose $J = [0, 1]$, $n_k = 2$, $c_k = 1/3$, $k \geqslant 1$. In this case, the homogeneous Cantor set C is the classical ternary Cantor set, and the partial homogeneous Cantor set satisfies

$$C^*([0,1], \{2\}, \{1/3\}) = C([0, 1/2], \{2\}, \{1/3\}).$$

By an easy calculating, we can get

$$C([0,1], \{2\}, \{1/3\}) = 2C([0, 1/2], \{2\}, \{1/3\}).$$

By ref. [6], we have

$$\dim_H C([0,1], \{2\}, \{1/3\}) = \frac{\log 2}{\log 3}, \quad \mathcal{H}^s(C([0,1], \{2\}, \{1/3\})) = 1, \text{ where } s = \frac{\log 2}{\log 3};$$

therefore

$$\dim_H C^*([0,1], \{2\}, \{1/3\}) = \frac{\log 2}{\log 3}, \quad \mathcal{H}^s(C^*) = 2^{-s}\mathcal{H}^s(C([0,1], \{2\}, \{1/3\})) = 2^{-s},$$

where $\dim_H$ and $\mathcal{H}^s(E)$ denote the Hausdorff dimension of E and s-dimensional Hausdorff measure.

Remark 2.1 shows that even if the sets are very regular, their measures (although the dimensions are equal) depend tightly on the relative positions of fundamental intervals.

For convenience, let $J = [0, 1]$ and

$$s_1 = \liminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}, \quad s_2 = \liminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}}.$$

Lemma 2.2. *Suppose* C, C^* *are homogeneous Cantor set, partial homogeneous Cantor set, respectively in* $\mathcal{M}(J, \{n_k\}, \{c_k\})$. *Then*

$$\dim_H C = s_1, \quad \dim_H C^* = s_2.$$

Proof. By Theorem 2 in ref. [5], we can get $\dim_H C = s_1$. By this result and Lemma 2.1, we have

$$\dim_H C^* = \liminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log d_1 d_2 \cdots d_k} = \liminf_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log\left(\sum_{j=k+1}^{\infty} (n_j - 1) \prod_{i=1}^{j} c_i\right)}, \tag{1}$$

where d_i is defined in Lemma 2.1.

Since $n_k \geqslant 2$, $n_k c_k \leqslant 1$ $(k \geqslant 1)$, we have $c_k \leqslant 1/2$ and

$$\frac{1}{2} n_{k+1} \prod_{i=1}^{k+1} c_i \leqslant (n_{k+1} - 1) \prod_{i=1}^{k+1} c_i \leqslant \sum_{j=k+1}^{\infty} (n_j - 1) \prod_{i=1}^{j} c_i \leqslant n_{k+1} \prod_{i=1}^{k+1} c_i .$$

By (1), we have

$$\dim_H C^* = \liminf_{k \to \infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}} = s_2 .$$

Theorem 2.1. *Suppose* $E \in \mathcal{M}(J, \{n_k\}, \{c_k\})$. *Then we have*

$$s_2 \leqslant \dim_H E \leqslant s_1 .$$

Therefore $\sup_{E \in \mathcal{M}} \dim_H E = \dim_H C = s_1$, $\inf_{E \in \mathcal{M}} \dim_H E = \dim_H C^* = s_2$.

Proof. For any $k \geqslant 1$, E can be covered by $n_1 n_2 \cdots n_k$ intervals of length $c_1 c_2 \cdots c_k$. Thus $\dim_H E \leqslant s_1$.

Now we estimate the lower bound of dimension of E. If $s_2 = 0$, we have nothing to prove.

Suppose $s_2 > 0$. For any $0 < s < s_2$, suppose that μ is the probability measure supported on E such that for any $A \in \mathcal{F}_k$, $\mu(A) = (n_1 n_2 \cdots n_k)^{-1}$. By the definition of s_2, there exists a $c > 0$ such that for any $k \geqslant 1$, we have

$$n_1 n_2 \cdots n_k (c_1 c_2 \cdots c_{k+1} n_{k+1})^s \geqslant c . \tag{2}$$

For any closed interval $U \subset [0, 1]$, $|U| \leqslant c_1$, there exists a positive integer k such that $c_1 c_2 \cdots c_{k+1} \leqslant |U| < c_1 c_2 \cdots c_k$. Therefore we have

(i) U intersects at most $\dfrac{3|U|}{c_1 c_2 \cdots c_{k+1}}$ $(k+1)$-order fundamental intervals;

(ii) U intersects at most 2 k-order fundamental intervals.

By (2) and inequality $\min(a, b) \leqslant a^{1-s} b^s (0 \leqslant s \leqslant 1)$, we have

$$\mu(U) \leqslant \min\left(\frac{2}{n_1 n_2 \cdots n_k}, \frac{3|U|}{c_1 c_2 \cdots c_{k+1} n_{k+1}} \times \frac{1}{n_1 n_2 \cdots n_{k+1}} \right)$$
$$\leqslant \frac{1}{n_1 n_2 \cdots n_k} \left(\frac{3|U|}{c_1 c_2 \cdots c_{k+1} n_{k+1}} \right)^s 2^{1-s} \leqslant \frac{1}{c} 3^s 2^{1-s} |U|^s \leqslant \frac{6}{c} |U|^s .$$

Thus $\dim_H E \geqslant s$. By the arbitrariness of s, we have $\dim_H E \geqslant s_2$.

Corollary 2.1. *All the sets in* $\mathcal{M}(J, \{n_k\}, \{c_k\})$ *have the same Hausdorff dimension if and only if* $s_1 = s_2$. *Especially, if* $\inf_k c_k > 0$ (*noticing that the homogeneous Moran sets in refs*. [2, 3] *satisfy this condition*), *then* $s_1 = s_2$.

Theorem 2.2. *Suppose* $s_2 < s_1$, $s_2 < s < s_1$. *Then there exists* $E \in \mathcal{M}(J, \{n_k\}, \{c_k\})$ *such that* $\dim_H E = s$.

Proof. Let $C = C(J, \{n_k\}, \{c_k\})$ be the homogeneous Cantor set, and let $\mathcal{F}_k$ $(k \geqslant 1)$ be the k-order fundamental intervals. By Theorem 2.1, we have $\dim_H C = s_1$. Since $s < s_1$, $\mathcal{H}^s(C) = \infty$. By Theorem 4.10 in ref. [6], there exists a compact subset F of C such that $\dim_H F = s$.

Let $\mathcal{T}_1 = \{A \in \mathcal{F}_1; A \cap F = \emptyset\}, \mathcal{T}_1^* = \{A \in \mathcal{F}_1; A \cap F \neq \emptyset\}$. For any $A \in \mathcal{T}_1$, let A^* be the partial homogeneous Cantor set $A^*(A, \{n_k\}_{k\geqslant 2}, \{c_k\}_{k\geqslant 2})$. Denote by $\hat{\mathcal{F}}_2$ the 2-order fundamental intervals generated by elements of $\mathcal{T}_1^*$, and let $\mathcal{T}_2 = \{A \in \hat{\mathcal{F}}_2; A \cap F = \emptyset\}, \mathcal{T}_2^* = \{A \in \hat{\mathcal{F}}_2; A \cap F \neq \emptyset\}$. For any $A \in \mathcal{T}_2$, let $A^*(A, \{n_k\}_{k\geqslant 3}, \{c_k\}_{k\geqslant 3})$ be the corresponding partial homogeneous Cantor set. In this way, for any l, we can define $\mathcal{T}_l, \mathcal{T}_l^*$ and partial homogeneous Cantor set $A^*(A, \{n_k\}_{k\geqslant l+1}, \{c_k\}_{k\geqslant l+1})$.

By the above construction, we have

(i) $F = \bigcap_{k\geqslant 1} \bigcup_{A \in \mathcal{T}_k^*} A$;

(ii) $\forall k \geqslant 1, A \in \mathcal{T}_k$; $\dim_H A^* = s_2$.

Let $F^* = F \cup (\bigcup_{k\geqslant 1} \bigcup_{A \in \mathcal{T}_k} A^*)$. Then $F^* \in \mathcal{M}(J, \{n_k\}, \{c_k\})$. By the σ-stability of Hausdorff dimension, we have $\dim_H F^* = s$.

Theorem 2.3. *Suppose* $0 \leqslant \alpha \leqslant \beta \leqslant 1$. *Then there exists* $\{n_k\}_{k\geqslant 1}$, $\{c_k\}_{k\geqslant 1}$ *such that* $\dim_H C^* = \alpha, \dim_H C = \beta$, *where* C *and* C^* *are the corresponding homogeneous Cantor set, and partial homogeneous Cantor set, respectively.*

Proof. We divide the proof into two cases: $\alpha \neq \beta$, $\alpha = \beta$.

Case 1. $\alpha \neq \beta$. In the following 4 cases, we construct $\{n_k\}_{k\geqslant 1}$ and $\{c_k\}_{k\geqslant 1}$. By Lemma 2.2 and Theorem 2.1, we can get the desired result easily.

(i) $0 < \alpha < \beta < 1$, let $n_1 = 2$, $n_{k+1} = [((n_1 \cdots n_k)^{\frac{1}{\alpha} - \frac{1}{\beta}})^{\frac{\beta}{1-\beta}}]$, $k \geqslant 1$; $c_k = n_k^{-\frac{1}{\beta}}$;

(ii) $0 = \alpha < \beta < 1$, let $n_1 = 2$, $n_{k+1} = (n_1 \cdots n_k)^{n_1 \cdots n_k}$, $k \geqslant 1$; $c_k = n_k^{-\frac{1}{\beta}}$;

(iii) $\alpha = 0$, $\beta = 1$, let $n_1 = 2$, $n_{k+1} = (n_1 \cdots n_k)^{n_1 \cdots n_k}$, $k \geqslant 1$; $c_k = n_k^{-\frac{1}{\beta_k}}$, where $\beta_{k+1} = \frac{n_k}{n_k + 1}$, $k \geqslant 1$, $\beta_1 = \frac{n_1}{n_1 + 1}$;

(iv) $0 < \alpha < \beta = 1$, let $n_1 \geqslant 2$ such that $\frac{n_1}{n_1 + 1} > \alpha$, $\beta_1 = \frac{n_1}{n_1 + 1}$ $n_2 = [(n_1^{\frac{1}{\alpha}})^{\frac{\beta_1}{1-\beta_1}}]$, $\beta_2 = \frac{n_1}{1 + n_1}$; for any $k \geqslant 2$, suppose that n_k and β_k have been defined; let $\beta_{k+1} = \frac{n_k}{n_k + 1}$, $n_{k+1} = [(n_1^{\frac{1}{\alpha} - \frac{1}{\beta_1}} n_2^{\frac{1}{\alpha} - \frac{1}{\beta_2}} \cdots n_k^{\frac{1}{\alpha} - \frac{1}{\beta_k}})^{\beta_k (1-\beta_k)^{-1}}]$, $c_k = n_k^{-\frac{1}{\beta_k}}$.

Case 2. $\alpha = \beta$.

(i) $\alpha = \beta > 0$, let $n_k = 2$, $c_k = 2^{-\frac{1}{\beta}}$, $k \geqslant 1$;

(ii) $\alpha = \beta = 0$, let $n_k = 2$, $c_k = 2^{-k}$, $k \geqslant 1$.

3 Packing dimensions of homogeneous Moran sets

In this section, we use the same notations as in section 2.

Theorem 3.1. *Let* C, C^* *be the homogeneous Cantor set in* $\mathscr{M}(J, \{n_k\}, \{c_k\})$, *and the partial homogeneous Cantor set, respectively. Then*

$$\dim_P C = \limsup_{k\to\infty} \frac{\log n_1 n_2 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}}, \tag{3}$$

$$\dim_P C^* = \limsup_{k\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}, \tag{4}$$

where $\dim_P$ *denotes packing dimension.*

Proof. By Theorem 1 in ref. [7], we have

$$\overline{\dim}_B C = \inf\left\{ s > 0; \sum_{k\geq 1} n_1 \cdots n_k (n_{k+1} - 1)\left(\frac{(1 - n_{k+1}c_{k+1})\prod_{i=1}^{k} c_i}{n_{k+1} - 1}\right)^s < \infty \right\}, \tag{5}$$

where $\overline{\dim}_B$ denotes upper Bouligand dimension.

For any $s > \limsup_{k\to\infty} \frac{\log n_1 n_2 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}}$, there exists $\varepsilon > 0$, $k_0 > 0$, such that for any $k \geq k_0$, we have

$$a_k := s - \frac{\log n_1 n_2 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}} > \varepsilon.$$

By $n_{k+1}c_{k+1} \leq 1$ and $n_{k+1} - 1 \geq \frac{n_{k+1}}{2}$, we have

$$\sum_{k\geq k_0} n_1 \cdots n_k (n_{k+1} - 1)\left((1 - n_{k+1}c_{k+1})\prod_{i=1}^{k} c_i (n_{k+1} - 1)^{-1}\right)^s$$

$$\leq 2^s \sum_{k\geq k_0} n_1 \cdots n_{k+1}\left(\left(\prod_{i=1}^{k} c_i\right) n_{k+1}^{-1}\right)^s = 2^s \sum_{k\geq k_0}\left(\left(\prod_{i=1}^{k} c_i\right) n_{k+1}^{-1}\right)^{a_k} \leq 2^s \sum_{k\geq k_0} 2^{-(k+1)\varepsilon} < \infty.$$

By (5), we have $\overline{\dim}_B C \leq s$. By the arbitrariness of s, we have

$$\overline{\dim}_B C \leq \limsup_{k\to\infty} \frac{\log n_1 n_2 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}}. \tag{6}$$

For any $\delta > 0$, let $M(\delta, C)$ be the minimal numbers of closed intervals of length δ needed to cover C. For any $A \in \mathscr{F}_k$, we divide it equally into n_{k+1} closed intervals. By the construction of homogeneous Cantor set, each closed interval intersects C. Therefore $M\left(\frac{c_1 \cdots c_k}{n_{k+1}}, C\right) \geq \frac{1}{2} n_1 \cdots n_{k+1}$. Thus

$$\overline{\dim}_B C \geq \limsup_{k\to\infty} \frac{\log n_1 n_2 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}}. \tag{7}$$

On the other hand, by Corollary 3.9 in ref. [6], we have $\overline{\dim}_B C = \dim_P C$. By (6) and (7), we have (3).

By (3), Lemma 2.1 and the method used for proving Theorem 2.1, we can get (4) easily.

Lemma 3.1. *For any* $E \in \mathcal{M}(J, \{n_k\}, \{c_k\})$, *we have*

$$\overline{\dim}_B E \leqslant \limsup_{k \to \infty} \frac{\log n_1 n_2 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}}.$$

Proof. Suppose that $\sigma \in D_k$ and J_σ is a k-order fundamental interval. Let $J_\sigma(1), J_\sigma(2), \cdots, J_\sigma(n_{k+1}-1)$ be $n_{k+1}-1$ gaps of the adjoining $(k+1)$-order fundamental intervals. By the concavity of $x^s (0 \leqslant s \leqslant 1)$, we have

$$\sum_{i=1}^{n_{k+1}-1} |J_\sigma(i)|^s \leqslant (n_{k+1}-1)\left(\frac{1}{n_{k+1}-1}\sum_{i=1}^{n_{k+1}-1}|J_\sigma(i)|\right)^s = (n_{k+1}-1)\left(\frac{(1-n_{k+1}c_{k+1})\prod_{i=1}^{k} c_i}{n_{k+1}-1}\right)^s$$

By Theorem 1 in ref. [7] and the proof of (6) in Theorem 3.1, we can get the desired result.

Lemma 3.2. *For any* $E \in \mathcal{M}(J, \{n_k\}, \{c_k\})$, *we have*

$$\dim_P E \geqslant \limsup_{k \to \infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_k}.$$

Proof. Let μ be a probability measure supported on E, such that for any k-order fundamental interval I, $\mu(I) = (n_1 \cdots n_k)^{-1}$. Notice that for any $x \in E$, the ball $B(x, c_1 \cdots c_k)$ at least contains one k-order fundamental interval and intersects at most 3 k-order fundamental intervals. Therefore

$$(n_1 \cdots n_k)^{-1} \leqslant \mu(B(x, c_1 \cdots c_k)) \leqslant 3(n_1 \cdots n_k)^{-1}.$$

Thus

$$\limsup_{k \to \infty} \sup_{x \in E} \frac{\log \mu(B(x, c_1 \cdots c_k))}{\log c_1 c_2 \cdots c_k} = \limsup_{k \to \infty} \inf_{x \in E} \frac{\log \mu(B(x, c_1 \cdots c_k))}{\log c_1 c_2 \cdots c_k}$$
$$= \limsup_{k \to \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 c_2 \cdots c_k}. \tag{8}$$

Now for any $0 \leqslant \alpha < \limsup_{k \to \infty} \frac{\log n_1 \cdots n_k}{-\log c_1 c_2 \cdots c_k}$, by (8), there exists $l_k \uparrow \infty$ such that when k is large enough, we have

$$\inf_{x \in E} \frac{\log \mu(B(x, \delta_{l_k}))}{-\log \delta_{l_k}} > \alpha,$$

where $\delta_{l_k} = c_1 c_2 \cdots c_{l_k}$. Thus for any $x \in E$,

$$\mu(B(x, \delta_{l_k})) < \delta_{l_k}^{\alpha}. \tag{9}$$

For any subset $F \subset E$, denote by $N(\delta_{l_k}, F)$ the maximal numbers of disjoint balls with center in F and radius δ_{l_k}. Let $B_i(x_i, \delta_{l_k}), x_i \in F, i = 1, \cdots, N(\delta_{l_k}, F)$ be $N(\delta_{l_k}, F)$ disjoint ball. Then $\{B_i(x_i, 2\delta_{l_k})\}$ is a cover of F. Noticing that $\mu(B(x, \delta_{l_k})) \geqslant \frac{1}{5}\mu(B(x, 2\delta_{l_k}))$, by (9), we have

$$\mathscr{P}^{\alpha}_{\delta_{l_k}}(F) \geqslant N(\delta_{l_k}, F) \times 2^{\alpha} \times \delta_{l_k}^{\alpha} \geqslant N(\delta_{l_k}, F) \times \delta_{l_k}^{\alpha}$$

$$\geqslant \sum_{i=1}^{N(\delta_{l_k}, F)} \mu(B_i(x_i, \delta_{l_k})) \geqslant \frac{1}{5} \sum_{i=1}^{N(\delta_{l_k}, F)} \mu(B_i(x_i, 2\delta_{l_k})) \geqslant \frac{1}{5}\mu(F).$$

Let $k \to \infty$. We have $\mathscr{P}^{\alpha}_{0}(F) \geqslant \frac{1}{5}\mu(F)$. Thus

$$\mathscr{P}^{\alpha}(E) = \inf\left\{\sum_i \mathscr{P}^{\alpha}_0(E_i);\ \bigcup_i E_i \supset E, E_i \subset E\right\}$$

$$\geqslant \inf\left\{\sum_i \frac{1}{5}\mu(E_i);\ \bigcup_i E_i \supset E, E_i \subset E\right\} \geqslant \frac{1}{5}.$$

Therefore $\dim_{\mathrm{P}} E \geqslant \alpha$. By the arbitrariness of α, we get the desired result.

By Lemmas 3.1 and 3.2, we have

Theorem 3.2. *For any* $E \in \mathscr{M}(J, \{n_k\}, \{c_k\})$, *we have*

$$\limsup_{k\to\infty} \frac{\log n_1 \cdots n_k}{-\log c_1 c_2 \cdots c_k} \leqslant \dim_{\mathrm{P}} E \leqslant \overline{\dim}_{\mathrm{B}} E \leqslant \limsup_{k\to\infty} \frac{\log n_1 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}}.$$

By Theorems 2.1 and 3.2, we have

Corollary 3.1. (i) *For any* $E, F \in \mathscr{M}(J, \{n_k\}, \{c_k\})$, *we have* $\dim_{\mathrm{H}} E \leqslant \dim_{\mathrm{P}} F$.

(ii) *If* $\liminf\limits_{k\to\infty} \dfrac{\log n_1 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}} = \limsup\limits_{k\to\infty} \dfrac{\log n_1 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}}$, *then*

$$\dim_{\mathrm{H}} E = \dim_{\mathrm{P}} E = \dim_{\mathrm{B}} E.$$

By ref. [8], we know that if $\mathscr{P}^s(E) = \infty$, then there exists a compact subset F of E such that $\dim_{\mathrm{P}} F = s$. By the same methods used in Theorems 2.2 and 2.3, we have

Theorem 3.3. *Suppose s satisfies*

$$\limsup_{k\to\infty} \frac{\log n_1 \cdots n_k}{-\log c_1 c_2 \cdots c_k} \leqslant s \leqslant \limsup_{k\to\infty} \frac{\log n_1 \cdots n_{k+1}}{-\log c_1 c_2 \cdots c_k + \log n_{k+1}}.$$

Then there exists $E \in \mathscr{M}(J, \{n_k\}, \{c_k\})$ *such that* $\dim_{\mathrm{P}} E = s$.

Theorem 3.4. *Suppose* $0 \leqslant \alpha \leqslant \beta \leqslant 1$. *There exist* $\{n_k\}_{k\geqslant 1}$ *and* $\{c_k\}_{k\geqslant 1}$ *such that* $\dim_{\mathrm{P}} C^* = \alpha$, $\dim_{\mathrm{P}} C = \beta$, *where C and* C^* *are the corresponding homogeneous Cantor set, and the partial homogeneous Cantor set, respectively.*

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