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A BIOECONOMIC MODEL OF A RATIO-DEPENDENT PREDATOR-PREY SYSTEM AND OPTIMAL HARVESTING

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Abstract. This paper deals with the problem of a ratio-dependent prey-predator model with combined harvesting. The existence of steady states and their stability are studied using eigenvalue analysis. Boundedness of the exploited system is examined. We derive conditions for persistence and global stability of the system. The possibility of existence of bionomic equilibria has been considered. The problem of optimal harvest policy is then solved by using Pontryagin's maximal principle.

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1. Introduction

Harvesting has a strong impact on the dynamic evolution of a population subjected to it. First of all, depending on the nature of the applied harvesting strategy, the long run stationary density of the population may be significantly smaller than the long run stationary density of a population in the absence of harvesting [12]. Therefore, while a population can in the absence of harvesting be free of extinction risk, harvesting can lead to the incorporation of a positive extinction probability and therefore, to potential extinction in finite time. Secondly, if a population is subject to a positive extinction rate then harvesting can drive the population density to a dangerously low level at which extinction becomes sure no matter how the harvester affects the population afterwards. In order to stabilize and conserve fish population in marine ecosystems, various dynamic models for commercial fishing have been proposed and analyzed by taking into account the economic and ecological factors (See Clark [12, 13], Chaudhuri [10], Kar and Chaudhuri [17], Kar [18], Chaudhuri and Saha Ray [11], Pradhan and Chaudhuri [24], Samanta et al.[27]). In particular, an extensive

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study has been conducted by Clark [12, 13] to study the effect of harvesting and other parameters on fishery resources by using ecological and economic models. He also studied the optimal harvesting of two ecologically independent species whose dynamics is governed by the logistic law of growth. Mesterton-Gibbons [20] also examined an optimal policy for combined harvest of two ecologically independent species which grow logistically and are harvested at a rate proportional to both stock and effort. Pradhan and Chaudhuri [23] studied the dynamics of a single species fishery in which fish species follows the Gompertz law of growth. Bhattacharya and Begum [5] proposed three types of models of two species system:

(i) a logistic growth model of two ecologically independent species
(ii) a logistic growth model of two competing species, and
(iii) a Lotka-Volterra model of one prey and one predator.

In each case they obtained the feasible bionomic equilibrium points. Ragozin and Brown [25] studied an optimal policy of a prey-predator system in which the predator is selectively harvested and prey has no commercial value. Mesterton-Gibbons [21] investigated an optimal policy for maximizing the present value from the combined harvest of two ecologically independent species, which would coexist as predator and prey in the absence of harvesting. Recently, Zhang et al. [29] studied the optimal harvesting policy of a stage-structured prey- predator model and obtained necessary and sufficient condition for the permanence of two species and the extinction of one species or two species. Jerry and Raissi [16], Brauer and Soudack [7-9], Dai and Tang [14], Gamito [15], Kar [18] and some other authors also have discussed the fishery model with harvesting.

Generally speaking, a bionomic model consists of a biological (or biophisical) model that describes the behaviour of a living system and an economic model that relates the biological system to market prices and resource and institutional constarints. Bioeconomic models often contain a single equation to represent biological process. The logistic equation is perhaps the most commonly used function to capture the essential features of population dynamics in fishery and forestry models. However, there is an increasing trend towards using simulation models developed by biologists and agricultural scientists. These models often attempt to make the models behaviour approximate reality as closely as possible and their complexity may preclude their use directly as part of optimal control models.

The management of renewable resources has been based on the MSY (maximum sustainable yield). The MSY is a simple way to manage resources taking into consideration that over exploiting resources lead to a loss in productivity. Therefore, the aim is to determine how much we can harvest without altering dangerously the harvested population. The main problem of the MSY is economical irrelevance. It is so since it takes into consideration the benefits of resource exploitation, but completely disregard the cost operation of resource exploitation. For example, it ignores the fact that if a species is harvested such that its population decreases to a certain level, then the cost of harvesting can become time consuming. This might lead to a situation where the cost of harvesting is higher than the benefit. Confronted with the inadequacy of the MSY, people tried to replace it by the OSY, that is , the optimum sustainable yield, which is based on the standard cost benefit criterion used to maximize revenues. Actually renewable resources management is complicated and constructing accurate mathematical models about the effect of harvesting is even more complicated. This is so because to have a perfect model we need to consider its size, growth rate , carrying capacity, competitors combined with the cost of harvesting and the price obtained for the harvesting species. More informations can be found about these factors in Clark [12].

This paper is organized as follows. In the next section we study the existence and local stability of the equilibria and their dependence on the harvesting efforts. We have concentrated more on the interior equilibrium of the system as we are interested in the existence of the species. Next we derive the conditions for persistence and global stability of the system. Taking simple economic considerations into account, we discuss the possibilities of existence of a bionomic equilibrium when the system is exploited. The optimal policy of exploitation is derived by using Pontryagin's maximum principle. Some numerical results have been done. The problem ends with a brief description of the principal results obtained here.

2. The model

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance [4]. These problems may appear to be simple mathematically at first sight, they are, in fact, often very challenging and complicated.

A milestone progress in the study of predator-prey interactions was the discovery of the now well known " paradox of enrichment," which states that according to the Lotka-Volterra type predator-prey theory (with Michaelis-Menten-Holling type functional response) enriching a predator-prey system will cause an increase in the equilibrium density of the predator but not in that of the prey and will destabilize the community equilibrium. Another similar paradox is the so-called " biological control paradox," stating that according to the classic predator-prey theory, you can not have both a low and stable prey equilibrium density. Berryman [4] show that the ratio-dependent models are capable of producing richer and more reasonable or acceptable dynamics.Beretta and Kuang [2] studied the global qualitative analysis of a ratio dependent prey-predator system. Stability analysis of some delayed ratio dependent prey predator system were also studied by Beretta and Kuang [3] and Rao and Rao [26]. It is easy to see that the paradox of biological control is no longer valid for ratio dependent systems. It can also be easily shown that the ratio dependent type models do not produce the so-called paradox of enrichment. For this reason, we will focus our attention here on the ratio-dependent type predator-prey model, which takes the form of

$$\begin{aligned} \frac{dx}{dt} &= ax\left(1-\frac{x}{k}\right)-\frac{cxy}{my+x} \\ \frac{dy}{dt} &= y\left(-d+\frac{fx}{my+x}\right) \end{aligned} \tag{1}$$

We consider only biological meaningful initial condition $x(0) = x_0$, $y(0) = y_0$.

Here $x(t)$, $y(t)$ represent the population density of prey and predator respectively, at any time t and a, k, c, m, d, f are all positive constants. More specifically, $a > 0$ is the *intrinsic growth rate* and $k > 0$ is the *carrying capacity* of the prey. $f > 0$ is a *conversion factor* specifying the number of newly born predators for each captured prey. $d > 0$ is the death rate of the predator. The function $g(x) = a\left(1-\frac{x}{k}\right)$ is the *specific growth rate* of the prey in the absence of any predator. g(x) is the so-called *logistic growth function* [12], which is the simplest and perhaps the most useful growth function.

Assuming that both the prey and predator are subjected to a combined harvesting effort E, we may write

$$\begin{aligned} \frac{dx}{dt} &= ax\left(1-\frac{x}{k}\right)-\frac{cxy}{my+x}-q_1Ex \\ \frac{dy}{dt} &= y\left(-d+\frac{fx}{my+x}\right)-q_2Ey \end{aligned} \tag{2}$$

We consider the initial condition $x(0) = x_0, y(0) = y_0$. Here q_1 and q_2 are *catchability co-efficients* of the two species. The catch rate functions q_1Ex and q_2Ey are based on the **CPUE** (catch-per-unit-effort) hypothesis [12] .

3. The steady states

The possible equilibria of the dynamical system (2) are

$$P_0(0,0), P_1\left(k\left(1-\frac{q_1E}{a}\right),0\right), P_2(x*,y*)$$

where

$$\begin{aligned} x^* &= k\frac{\Big[c(d+q_2E)-f(c-m(a-q_1E)\Big]}{amf} \\ y^* &= \frac{x^*(f-d-q_2E)}{(d+q_2E)\,m}. \end{aligned}$$

The unique positive equilibrium $P_2(x^*,y^*)$ exist if and only if any one of the following two conditions is true :

(i) $(d+q_2E) < f < \dfrac{(d+q_2E)c}{(c-m(a-q_1E))}$, when $E > \dfrac{(ma-c)}{(mq1)}$

(ii) $f > (d+q_2E)$, when $E \leq \dfrac{(ma-c)}{(mq_1)}$.

We shall point out here that although (0, 0) is defined for system (2), it can not be linearised there. So, local stability of (0, 0) cannot be studied. Indeed, this singularity at the origin, while causes much difficulty in our analysis of the system, contributes significantly to the richness of dynamics of the model.

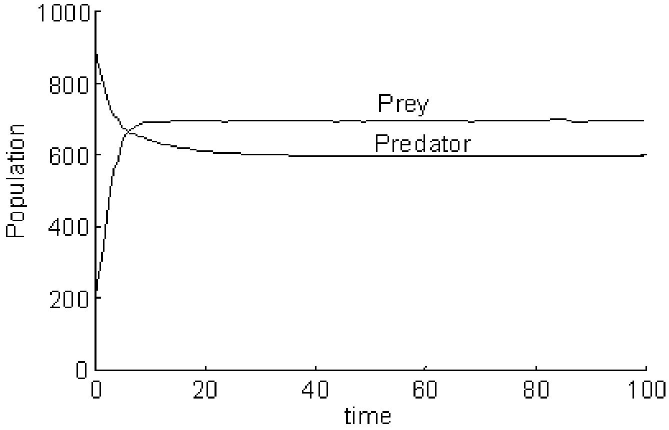

Figure 1. Both the prey and predator populations converge to their equilibrium values.

It is easy to check that, whenever the positive steady state P_2 exists, $P_1\Big(k\Big(1-\dfrac{q_1E}{a}\Big), 0\Big)$ is unstable. Now following Beretta and Kuang[2], after some straightforward computation, it is easy to show that P_2 is locally asymptotically stable if

$$-a+q_1E+\left(1-\frac{d+q_2E}{f}\right)\frac{\left[c+\dfrac{d(c-mf)}{f}\right]}{m} < 0$$

Example 1. Let $k = 1000, a = 1, c = 0.9, m = 6, d = 0.07, f = 0.8, q_1 = 0.09, q_2 = 0.03, E = 2$. For these values of the parameters it is found that

(i) (0, 0) is unstable
(ii) (1000, 0) is unstable but
(iii) the only interior steady state (698, 598) is stable.

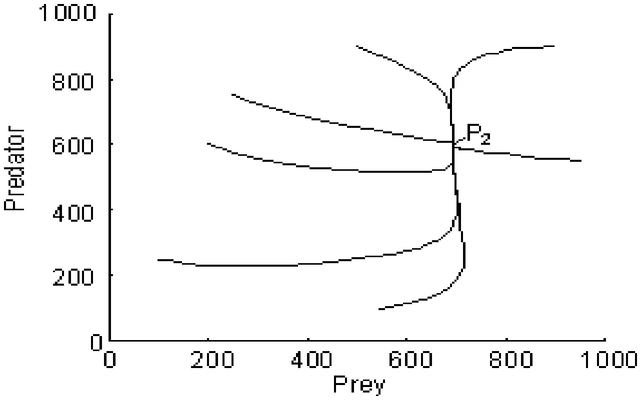

Figure 2. Phase plane trajectories with reference to different initial levels. The graph shows that the interior equilibrium point P_2 (698, 598) is globally asymptotically stable.

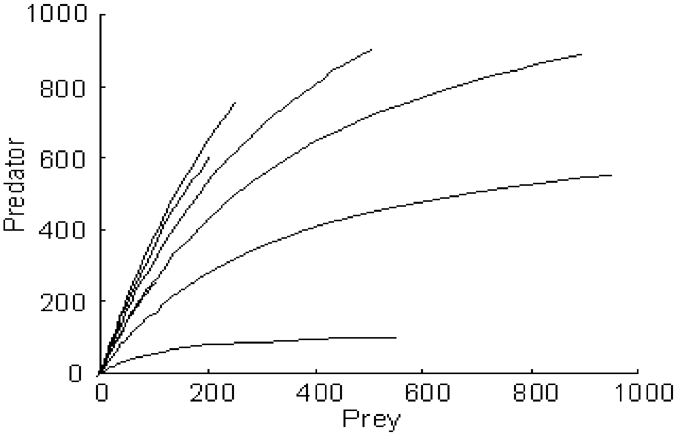

Figure 3. For E=15, both the populations tending to the origin, showing P_0(0, 0) is a global attractor.

4. Boundness of the system

The positive quadrant R_2^+ is the domain of solutions of the system (2) and all the standard results on existence, uniqueness and continuous dependence on initial condition of solutions are evidently satisfied. Now, we shall show that the solutions of the system (2) are uniformly bounded.

Lemma 1. *All the solutions of (2) which initiate in R_2^+ are uniformly bounded.*

Proof. We define the function

$$W = x + \frac{c}{f}y \tag{3}$$

The time derivative along a solution of (2) is

$$\dot{W} = ax\left(1 - \frac{x}{k}\right) - q_1 Ex - \frac{cd}{f}y - \frac{c}{f}q_2 Ey$$

For each $\lambda > 0$, we have

$$\begin{aligned} \dot{W} + \lambda W &= ax\left(1 - \frac{x}{k}\right) - q_1 Ex - \frac{cd}{f}y - \frac{c}{f}q_2 Ey + \lambda x + \frac{\lambda c}{f}y \\ &\le \frac{k}{4a}\left(a - q_1 E + \lambda\right)^2 + \frac{c}{f}y\left(\lambda - d - q_2 E\right) \end{aligned} \tag{4}$$

Now, if we choose $\lambda < d + q_2 E$, then the right-hand side of (4) is bounded for all $(x, y) \in R_2^+$.

Thus we find a $\mu > 0$ with $\dot{W} + \lambda W < \mu$. Applying a theorem on differential inequality [6], we obtain

$$0 \le W(x, y) \le \frac{\mu}{\lambda} + \left\{W(x(0), y(0)) - \frac{k}{\lambda}\right\} e^{-\lambda t} \tag{5}$$

and for $t \to \infty$, we have $0 \le W \le \frac{\mu}{\lambda}$.

Therefore, we have

$$B = \left[(x, y) \in R_2^+ : W < \frac{\mu}{\lambda} + \in, \text{ for any } \in > 0\right],$$

where B is the region in which all the solutions of system Eq. (2) that start in R_2^+ are confined.

5. Permanence of the system

Before starting our theorem, we give some definitions:

Definition 1. System (2) is said to be *uniformly persistent* if there is an $\eta > 0$ (independent of initial data) such that every solution $(x(t), y(t))$ of system (2) with given initial condition satisfies

$$\lim_{t\to\infty} \inf x(t) \ge \eta, \quad \lim_{t\to\infty} \inf y(t) \ge \eta.$$

Definition 2. System (2) is said to be *permanent* if there exists a compact region $\Omega_0 \in \text{int } \Omega$ such that every solution of system (2) with given initial condition will eventually enter and remain in region Ω_0.

Clearly for a dissipative system uniform persistence is equivalent to permanence. Now following the Theorem 2.2 of Beretta and Kuang [3], we may state the following theorem.

Theorem. *If* $E < \min\left[\frac{1}{q_1}\left(a - \frac{c}{m}\right), \frac{1}{q_2}(f-d)\right]$*, then system of Eq. (2) is permanent.*

6. Global stability

In this section, we consider the global stability of the system (2) by constructing a suitable Lyapunov function. We define a Lyaponov function

$$V(x,y) = \left[(x-x^*) - x^* \log\left(\frac{x}{x^*}\right)\right] + \alpha\left[(y-y^*) - y^* \log\left(\frac{y}{y^*}\right)\right]$$

where α is a suitable constant to be determined in the subsequent steps. It can be easily verified that the function V is zero at the equilibrium (x^*, y^*) and is positive for all other positive values of x, y.
The time derivative of V along with the solutions of (2) is

$$\begin{aligned}
\frac{dV}{dt} &= (x-x^*)\left[-\frac{a}{k}(x-x^*) - \frac{c(yx^*-xy^*)}{(my+x)(my^*+x^*)}\right] \\
&\quad +\alpha(y-y^*)\left[-mf\frac{(yx^*-xy^*)}{(my+x)(my^*+x^*)}\right] \\
&= -\left[\frac{a}{k} - \frac{cy^*}{(my+x)(my^*+x^*)}\right](x-x^*)^2 \\
&\quad -\left[\frac{cx^*}{(my+x)(my^*+x^*)}\right](y-y^*)^2 \\
&\quad \left[\text{taking } \alpha = \frac{c}{mf}\right] \\
\Rightarrow &\quad \frac{dV}{dt} < 0 \text{ if } my + x > \frac{cy^*k}{a(my^*+x^*)}
\end{aligned}$$

Thus we find that the feasible region in which the interior equilibrium point (x^*, y^*) is globally asymptotically stable, is above the line $my + x = M$, of the first quadrant in the xy -phase plane, where $M = \frac{cy^*k}{a(my^*+x^*)}$.

7. Bionomic equilibrium

The term *bionomic equilibrium* is an amalgamation of the concepts of *biological equilibrium* and economic equilibrium. As we already saw, a biological equilibrium is given by $\dot{x} = 0 = \dot{y}$. The *economic equilibrium* is said to be achieved when TR (the total revenue obtained by selling the harvested biomass) equals TC (the total cost for the effort devoted to harvesting).

Let
c' = constant harvesting cost per unit effort,
p_1 = constant price per unit biomass of the prey species and
p_2 = constant price per unit biomass of the predator species.
The economic rent is given by

$$\pi\,(x,y,E) = p_1q_1xE + p_2q_2yE - c'E. \tag{6}$$

Now,

$$\dot{x} = 0 \Rightarrow x = 0 \;\text{ or, }\; E = \frac{a}{q_1} - \frac{a}{kq_1}x - \frac{c}{q_1}\frac{y}{my+x}$$

and

$$\dot{y} = 0 \Rightarrow y = 0 \;\text{ or, }\; E = -\frac{d}{q_2} + \frac{f}{q_2}\frac{x}{my+x}.$$

Hence the nontrivial equilibrium solution ($\dot{x} = \dot{y} = 0$) occurs at a point on the curve

$$\frac{a}{kq_1}x^2 + \frac{am}{kq_1}xy - \left(\frac{a}{q_1} + \frac{d}{q_2} - \frac{f}{q_2}\right)x - \left(\frac{am}{q_1} - \frac{c}{q_1} + \frac{dm}{q_2}\right)y = 0. \tag{7}$$

The bionomic equilibrium (x_∞, y_∞) is determined by (7) together with the condition

$$\pi = TR - TC = (p_1q_1x + p_2q_2y - c')\,E = 0. \tag{8}$$

Eliminating y from (7) and (8), we get

$$\left(\frac{a}{kq_1} - \frac{amp_1}{kp_2q_2}\right)x^2 + \left[\frac{acm}{kq_1p_2q_2} - \left(\frac{a}{q_1} + \frac{d}{q_2} - \frac{f}{q_2}\right) - \frac{p_1q_1}{p_2q_2}\Big(\frac{am}{q_1} - \frac{c}{q_1} + \frac{dm}{q_2}\Big)\right]x - \frac{c'}{p_2q_2}\left(\frac{am}{q_1} - \frac{c}{q_1} + \frac{dm}{q_2}\right) = 0$$

which is of the form

$$A_1x^2 + B_1x + C_1 = 0 \tag{9}$$

where

$$\begin{aligned} A_1 &= \frac{a}{kq_1} - \frac{amp_1}{kp_2q_2} \\ B_1 &= \frac{acm}{kq_1p_2q_2} - \left(\frac{a}{q_1} + \frac{d}{q_2} - \frac{f}{q_2}\right) + \frac{p_1q_1}{p_2q_2}\left(\frac{am}{q_1} - \frac{c}{q_1} + \frac{dm}{q_2}\right) \\ C_1 &= -\frac{c'}{p_2q_2}\left(\frac{am}{q_1} - \frac{c}{q_1} + \frac{dm}{q_2}\right). \end{aligned}$$

In equation (9), we have
sum of the roots = - B_1/A_1 and product of the roots = C_1/A_1 .
Now, the following cases may arise :

Case I. Let $p_2q_2 > mp_1q_1$. In this case, $A_1 > 0$. We have one positive root when $C_1 < 0$. Then we must have

$$\frac{c}{q_1} < m\left(\frac{a}{q_1} + \frac{d}{q_2}\right). \tag{10}$$

Case II. Let $p_2q_2 < mp_1q_1$. In this case, $A_1 < 0$. We have one positive root when $C_1 > 0$. Then we must have

$$\frac{c}{q_1} > m\left(\frac{a}{q_1} + \frac{d}{q_2}\right). \tag{11}$$

In both these cases,

$$p_2q_2y_\infty = c' - p_1q_1x_\infty \Rightarrow y_\infty = \frac{c' - p_1q_1x_\infty}{p_2q_2} > 0 \;\text{ provided }\; x_\infty < \frac{c'}{p_1q_1}.$$

Case III. $p_2q_2 = mp_1q_1$. Then $A_1 = 0$, and $x = -C_1/B_1$. We have the following two subcases:

(i) For $\frac{c}{q_1} < m\left(\frac{a}{q_1} + \frac{d}{q_2}\right)$, we have $C_1 < 0$. Now, $B_1 > 0$, provided $c > kp_1q_1$ and $f > d$. In this case, (7) and (8) intersect at a unique point (x_∞, y_∞) in the first quadrant provided

$$x_\infty = -\frac{C_1}{B_1} < \frac{c}{p_1q_1}.$$

(ii) For $\frac{c}{q_1} > m\left(\frac{a}{q_1} + \frac{d}{q_2}\right)$, we have $C_1 > 0$. Now, $B_1 < 0$, provided $c < kp_1q_1$ and $f < d$. Then (7) and (8) intersect at a unique point (x_∞, y_∞) provided

$$x_\infty = -C_1/B_1 < c/p_1q_1.$$

8. Optimal harvesting policy

Wilen [28] pointed out that, from the point of view of humans, the ultimate users of natural resources, "population of natural organisms are not conveniently viewed as stocks of capital or assets which provide potential flows of services. Determining how to maximize benefits from these resources thus becomes a problem of capital theory deciding mainly how to use this portfolio of stocks over time". Hence optimal control theory provides the correct approach to natural resource management.

The present value J of a continuous time-stream of revenues is given by

$$J = \int_0^\infty \pi(x, y, E, t)e^{-\delta t}dt \tag{12}$$

where $\pi(x, y, E, t) = (p_1q_1x + p_2q_2y - c')E$ and δ denotes the instanteneous annual rate of discount. Our problem is to maximise J subject to the state equation (2) by invoking Pontryagin's Maximal principle [22]. The control variable E(t) is subjected to the constraints $0 \leq E(t) \leq E_{max}$, so that $V_t = [0, E_{max}]$ is the control set. E_{max}, stands for a feasible upper limit of the harvesting effort.
Let us now construct the Hamiltonian

$$\begin{aligned} H \;=\; & e^{-\delta t}\left[p_1q_1x + p_2q_2y - c'\right]E + \lambda_1\left[ax\left(1 - \frac{x}{k}\right) - \frac{cxy}{my + x} - q_1Ex\right] \\ & + \lambda_2\left[y\left(-d + \frac{fx}{my + x}\right) - q_2Ey\right] \end{aligned} \tag{13}$$

where $\lambda_1(t)$ and $\lambda_2(t)$ are the adjoint variables. By the maximal principle, there exists adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ for all $t \geq 0$ such that

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\frac{\partial H}{\partial x} \\
&= -\left\{[e^{-\delta t}p_1q_1 - \lambda_1 q_1]E + \lambda_1\left[a - \frac{2ax}{k} - \frac{mcy^2}{(my+x)^2}\right]\right\} \\
&\quad -\lambda_2\frac{famy^2}{(my+x)^2} \qquad (14) \\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial y} = -\left\{[e^{-\delta t}p_2q_2 - \lambda_2 q_2]E + \lambda_1\frac{cx^2}{(my+x)^2}\right\} \\
&\quad -\lambda_2\left[-d + \frac{fx^2}{(my+x)^2}\right] \qquad (15)
\end{aligned}$$

We now consider an *optimal equilibrium* solution of the above problem so that we may take

$$\begin{aligned}
E &= \frac{a}{q_1} - \frac{a}{kq_1}x - \frac{c}{q_1}\frac{y}{my+x} \\
&= -\frac{d}{q_2} + \frac{f}{q_2}\frac{x}{my+x} \qquad (16)
\end{aligned}$$

Using (16), (14) and (15) become

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -p_1q_1e^{-\delta t}E + \lambda_1\left(\frac{ax}{k} - \frac{cxy}{(my+x)^2}\right) - \lambda_2\frac{fmy^2}{(my+x)^2}, \qquad (17) \\
\frac{d\lambda_2}{dt} &= -p_2q_2e^{-\delta t}E - \lambda_1\frac{cx^2}{(my+x)^2} + \lambda_2\frac{fmxy}{(my+x)^2}. \qquad (18)
\end{aligned}$$

Now eliminating λ_2 from (17) and (18), we have

$$\begin{aligned}
&\frac{d^2\lambda_1}{dt^2} - \left[\frac{ax}{k} - \frac{cxy}{(my+x)^2} - \frac{fmxy}{(my+x)^2}\right]\frac{d\lambda_1}{dt} \\
&\qquad - \left[\frac{cmfx^2y^2}{(my+x)^4} + \frac{mfxy}{(my+x)^2}\left(\frac{ax}{k} - \frac{cxy}{(my+x)^2}\right)\right]\lambda_1 \\
&= M_1e^{-\delta t} \qquad (19)
\end{aligned}$$

where

$$M_1 = \delta p_1q_1E + \frac{p_2q_2Efmy^2}{(my+x)^2} - \frac{fmxy}{(my+x)^2}p_1q_1E$$

The auxiliary equation for (19) is

$$\begin{aligned}
&\mu^2 - \left[\frac{ax}{k} - \frac{cxy}{(my+x)^2} - \frac{fmxy}{(my+x)^2}\right]\mu \\
&\qquad - \left[\frac{cmfx^2y^2}{(my+x)^4} + \frac{mfxy}{(my+x)^2}\left(\frac{ax}{k} - \frac{cxy}{(my+x)^2}\right)\right] = 0 \qquad (20)
\end{aligned}$$

The complete solution of (19) is of the form

$$\lambda_1(t) = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t} + \frac{M_1}{N} e^{-\delta t} \tag{21}$$

where the A_i's (i = 1, 2) are arbitrary constants and μ_i's (i = 1, 2) are the roots of the equation (21) and

$$\begin{aligned} N \,=\, & \delta^2 + \left[\frac{ax}{k} - \frac{cxy}{(my+x)^2} - \frac{fmxy}{(my+x)^2}\right]\delta \\ & - \left[\frac{cmfx^2y^2}{(my+x)^4} + \frac{mfxy}{(my+x)^2}\left(\frac{ax}{k} - \frac{cxy}{(my+x)^2}\right)\right] \neq 0 \end{aligned}$$

It is clear from (21) that λ_1 is bounded iff $\mu_i < 0$ $(i = 1, 2)$ or the A_i $(i = 1, 2)$ are identically zero. For robust calculations we ignore the cases where $\mu_i < 0$ $(i = 1, 2)$ and take $A_i \equiv 0$ $(i = 1, 2)$.
Then we have

$$\lambda_1(t) = \frac{M_1}{N} e^{-\delta t} \tag{22}$$

Proceeding in a similar way, we have

$$\lambda_2(t) = \frac{M_2}{N} e^{-\delta t} \tag{23}$$

where

$$M_2 = \left[\delta p_2 q_2 E + \frac{cx^2}{(my+x)^2} p_1 q_1 E + p_2 q_2 E\left(\frac{ax}{k} - \frac{cxy}{(my+x)^2}\right)\right]$$

Thus $e^{\delta t}\lambda_i(t), (i = 1, 2)$, remains constant over time in optimal equilibrium when they strictly satisfy the transversality condition at ∞[1], i.e., they remain bounded as $t \to \infty$. The Hamiltonian (13) must be maximized for $E \in [0, E_{max}]$. Assuming that the control constraints are not binding (that is optimal solution does not occur at E=0 or $E = E_{max}$) we have singular control [12] given by

$$\frac{\partial H}{\partial E} = e^{-\delta t}\left(p_1 q_1 x + p_2 q_2 y - c'\right) - \lambda_1 q_1 x - \lambda_2 q_2 y = 0 \tag{24}$$

or,

$$\lambda_1 q_1 x + \lambda_2 q_2 y = e^{-\delta t} \frac{\partial \pi}{\partial E} \tag{25}$$

Therefore, we may conclude that the total user cost of harvest per unit effort must be equal to the discounted value of the future price at the steady state effort level. Substituting the values of λ_1 and λ_2 from (22) and (23) into (24), we get

$$q_1 x\left(p_1 - \frac{M_1}{N}\right) + q_2 x\left(p_2 - \frac{M_2}{N}\right) = c' \tag{26}$$

Equation (26), together with (16), determine the optimal equilibrium populations $x = x_\delta$, $y = y_\delta$ when $\delta \to +\infty$, it can be easily seen that $(\frac{M_1}{N})$, $(\frac{M_2}{N}) \to 0$ which imply that $p_1 q_1 x_\infty + p_2 q_2 y_\infty = c'$.

So, $\pi(x_\infty, y_\infty, E) = 0$ which shows that the economic rent is completely dissipated. This conclusion was also drawn by Clark [12] in the combined harvesting of two ecologically independent populations.

Using (26), we have

$$\pi = (p_1 q_1 x + p_2 q_2 y - c')\, E = \left(\frac{M_1 q_1 x + M_2 q_2 y}{N} \right) E.$$

Here, we note that each of M_1 and M_2 is $0(\delta)$ where N is $0(\delta^2)$ so that π is $0(\delta^{-1})$. Thus π is a decreasing function of $\delta(\geq 0)$. We therefore, conclude that $\delta = 0$ leads to maximization of π.

We have established here the existence of an optimal equilibrium solution that satisfies the necessary conditions of the maximum principle. As pointed out by Clark [12], it is extremely difficult to find an optimal approach path consisting of a combination of bang bang control and non-equilibrium singular controls. This difficulty was faced by Clark [12] even in the study of a simple model of two ecologically independent fish species populations. The present model is much more complicated than the said model of Clark. Due to these difficulties, we have considered an optimal equilibrium solution only.

Example 2. Let k = 1000, a = 1, c = 0.9, m = 6, d = 0.07, f = 0.8, $q_1 = 0.09$, $q_2 = 0.03$, $p_1 = 1$, $p_2 = 2$, $c' = 45$, $\delta = 1.5$. For these values of parameters, we find that the bionomic equilibrium (415, 128) and the optimal equilibrium (747, 141) exist.

Also we find that the harvesting effort E, which corresponds to the bionomic equilibrium (415, 128) is 7.02 units and the optimal harvesting effort, E(t), which leads the system to the optimal equilibrium (747, 141) is 10.17 units.

9. Concluding remarks

Though the problem related to harvesting is an important part of population dynamics, but this sort of important phenomenon has not yet received much attention to researchers. In this paper we have considered and analyzed a ratio-dependent predator-prey model with Michaelis-Menten type functional response, where both the species are subject to combined harvesting effort.

It has been observed that the persistence of the prey-predator species depend on the values of the fishing effort. We have proved the global behavior of the system constructing a suitable Lyapunov function. It has been found that the system is globally asymptotically stable in a certain region. We then examine the various possibilities of existence of bionomic equilibria of the exploited system. The problem of optimal harvesting policy has been solved by using Pontryagin's Maximal principle theory. It has been proved that the optimal equilibrium populations x_δ and y_δ lead to a situation where the total user's cost of harvest per unit effort equals the discounted value of the future profit. It is proved that zero discounting leads to maximization of economic revenue and that an infinite discount rate leads to complete dissipation of economic rent.

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