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A Model for the Macroscopic Description and Continual Observations in Quantum Mechanics.

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Summary. — Starting from the idea of generalized observables, related to effect-valued measures, as introduced by Ludwig, some examples of continual observations in quantum mechanics are discussed. A functional probability distribution, on the set of the trajectories which are obtained as output of the continual observation, is constructed in the form of a Feynman integral. Interesting connections with the theory of dynamical semi-groups are pointed out. The examples refer to small systems, but they are interesting for the light they may shed on the problem of the connections between the quantum and the macroscopic levels of description for a large body; the idea of continuous trajectories indeed seems to be essential for the macroscopic level of description.

1. – Introduction.

In the ordinary formulation of quantum mechanics the « preparation » of a system is represented by a statistical operator $\hat{W}$ (which is said to specify the « state of the system ») and an observable quantity A by a self-adjoint operator $\hat{A}$, both acting in an appropriate Hilbert space $\mathfrak{H}$. At a given time t the expectation value of any function $\Phi(A)$ of the quantity A is given by

$$\langle\Phi(A)\rangle = \operatorname{Tr}\{\Phi(\hat{A})\,\hat{W}_{\rm S}(t)\} = \operatorname{Tr}\{\Phi(\hat{A}_{\rm H}(t))\,\hat{W}\}\,, \tag{1.1}$$

where

$$\begin{cases} \hat{W}_{\rm S}(t) = \exp\left[-\dfrac{i}{\hbar}\hat{H}t\right]\hat{W}\exp\left[\dfrac{i}{\hbar}\hat{H}t\right], \\[2mm] \hat{A}_{\rm H}(t) = \exp\left[\dfrac{i}{\hbar}\hat{H}t\right]\hat{A}\exp\left[-\dfrac{i}{\hbar}\hat{H}t\right], \end{cases} \tag{1.2}$$

the first equality referring to the Schrödinger picture, the second one to the Heisenberg picture.

According to the spectral theorem with the operator $\hat{A}$ a « projection-valued » measure

$$\hat{E}(T) = \int_T \mathrm{d}\hat{E}(a)\,, \qquad \int_{-\infty}^{+\infty} \mathrm{d}\hat{E}(a) = 1 \tag{1.3}$$

is associated such that

$$\Phi(\hat{A}) = \int_{-\infty}^{+\infty} \mathrm{d}\hat{E}(a)\,\Phi(a)\,. \tag{1.4}$$

Specializing eq. (1.1) to the characteristic function of an interval $I \equiv (a, a+\Delta a)$, we obtain

$$P(W|A \in I, t) = \mathrm{Tr}\,\{\hat{E}(I)\,\hat{W}_{\mathrm{S}}(t)\} = \mathrm{Tr}\,\{\hat{E}_{\mathrm{H}}(I, t)\,\hat{W}\} \tag{1.5}$$

for the probability that, observing A at the time t, we find a value in the interval I.

Equation (1.5) is completely equivalent to eq. (1.1), which in turn can be derived from (1.5) via (1.4).

On the other side, eq. (1.1), or (1.5) corresponds in some way to an over-idealized situation in which it is possible to discriminate sharply between values of A belonging to I and values not belonging to it (1).

A more general mathematical characterization of an observable has been given by LUDWIG in the context of an axiomatic formulation of quantum mechanics.

According to LUDWIG (2) an observable A is associated with an « effect-valued » measure

$$\hat{F}(T) = \int_T \mathrm{d}\hat{F}(a)\,, \qquad \int_{-\infty}^{+\infty} \mathrm{d}\hat{F}(a) = 1\,, \tag{1.6}$$

rather than with a « projection-valued » one, and one has

$$P(W|A \in I, t) = \mathrm{Tr}\,\{\hat{F}(I)\,\hat{W}_{\mathrm{S}}(t)\} = \mathrm{Tr}\,\{\hat{F}_{\mathrm{H}}(I, t)\,\hat{W}\}\,. \tag{1.7}$$

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(1) E. P. WIGNER: *Z. Phys.*, **133**, 101 (1952); H. ARAKI and M. N. YANASE: *Phys. Rev.*, **120**, 622 (1960); M. N. YANASE: *Phys. Rev.*, **123**, 666 (1961); M. N. YANASE: in *Proc. S.I.F.*, Course IL, edited by B. D'ESPAGNAT (New York, N. Y., 1971), p. 77; K.-E. HELLWIG: in *Proc. S.I.F.*, Course IL, edited by B. D'ESPAGNAT (New York, N. Y., 1971), p. 338.

(2) G. LUDWIG: *Commun. Math. Phys.*, **4**, 331 (1967); **9**, 1 (1968); *Lecture Notes in Physics*, Vol. **4** (Berlin, 1970). See also E. B. DAVIES and J. T. LEWIS: *Commun. Math. Phys.*, **17**, 239 (1970); A. S. HOLEVO: *Trans. Moscow Math. Soc.*, **26**, 133 (1972); *J. Multivariate Anal.*, **3**, 337 (1973).

We recall that, according to Ludwig's terminology, an effect is a positive self-adjoint operator smaller than unity:

$$\hat{F}^+ = \hat{F}\,, \qquad 0 < \hat{F} < 1\,. \tag{1.8}$$

Note that, whilst from idempotency it follows immediately that

$$\hat{E}(T_1)\,\hat{E}(T_2) = \hat{E}(T_1 \cap T_2) = \hat{E}(T_2)\,\hat{E}(T_1)\,, \tag{1.9}$$

we have, in general,

$$\hat{F}(T_1)\,\hat{F}(T_2) \neq \hat{F}(T_2)\,\hat{F}(T_1)\,. \tag{1.10}$$

Significant examples of such generalized observables have been given by LUDWIG and collaborators (3), HOLEVO (4) and DAVIES (5).

The main advantage of the new definition in comparison with the ordinary one stays in its greater flexibility. Since generalized observables can represent observations of limited accuracy and discrimination power, they are suitable for describing inaccurate simultaneous measurements of incompatible quantities. In particular, they seem to be very convenient for describing continual macroscopic observations and it is just this aspect that interests us here.

In fact, LUDWIG has also given an axiomatic formulation of macroscopic physics (6) which for our present purpose can be stated in the following simplified way:

1) the state of a macroscopic system is specified at any time t of a given interval (t_i, t_f) in terms of a set of variables $z = (z_1, z_2, \ldots)$ which belongs to a « macroscopic phase space » Z (which can be given the uniform space structure);

2) a definite choice of the physical conditions imposed on the system corresponds to a probability distribution on the space Y of the continuous functions $z(t)$ on the interval (t_i, t_f);

3) the macroscopic dynamics must be specified by a set of rules for the actual construction of the probability distribution mentioned above.

In this context, in order that the quantum and the macroscopic descriptions of a large body be consistent, it should be possible to define an effect-

(3) See ref. (2) and K. KRAUS: in *The Uncertainty Principle and Foundations of Quantum Mechanics*, edited by C. PRICE and S. S. CHISSIK (London, 1977), p. 293.
(4) A. S. HOLEVO: *Rep. Math. Phys.*, **13**, 379 (1978); **16**, 385 (1979).
(5) E. B. DAVIES: *Quantum Theory of Open Systems* (London, 1976).
(6) G. LUDWIG: in *Lecture Notes in Physics*, Vol. **29** (Berlin, 1973), p. 122; *Makroskopische Systeme und Quantenmechanik*, in *Notes in Math. Phys.* (Marburg, 1972).

valued measure $\hat{F}(M)$ on an appropriate σ-algebra of subsets of Y such that the quantity

$$P(W|[\boldsymbol{z}(t)] \in M) = \mathrm{Tr}\{\hat{F}(M)\hat{W}\} \tag{1.11}$$

could be interpreted as the probability that the function $\boldsymbol{z}(t)$ describing the actual macroscopic behaviour of the body belongs to the specific subset M of Y.

Once an equation of the type (1.11) is assumed, it is possible, among other things, to give a satisfactory solution of the problem of measurement in quantum mechanics in terms of an interaction (7) between microscopic and macroscopic objects and all paradoxical aspects of quantum mechanics disappear. However, no explicit construction of such a functional probability distribution or « effect-valued » measure has been given so far.

Although a realistic solution of this problem should be given in the framework of the second-quantization formalism, it seems worthwhile beginning to study models with finitely many degrees of freedom.

In this paper we shall consider two different models which arise from the following generalized observables:

I) a « coarse-grained » position for a one-dimensional particle defined by the trivial « effect-valued » measure

$$\hat{F}(T) = \sqrt{\frac{\alpha}{\pi}} \int_T \mathrm{d}x \exp[-\alpha(x-\hat{q})^2], \qquad \alpha > 0, \tag{1.12}$$

where $\hat{q}$ denotes the ordinary position operator for the system;

II) a generalized observable which in ordinary language corresponds to a simultaneous inaccurate measurement of position and momentum (again for a one-dimensional particle) and which is defined by (4)

$$\hat{F}(T) = \int_T \frac{\mathrm{d}x\,\mathrm{d}p}{2\pi\hbar} \exp\left[\frac{i}{\hbar}(p\hat{q} - x\hat{p})\right] \hat{\varrho} \exp\left[-\frac{i}{\hbar}(p\hat{q} - x\hat{p})\right], \tag{1.13}$$

where $\hat{\varrho}$ is a given positive, trace-one operator; for instance, it can be chosen as

$$\hat{\varrho} = C \exp[-\alpha(\hat{q}^2 + \lambda\hat{p}^2)], \qquad \alpha, \lambda > 0. \tag{1.14}$$

(7) G. LUDWIG: *Lecture Notes in Physics*, Vol. **29** (Berlin, 1973), p. 122; G. LUDWIG: in *Proc. S.I.F.*, Course IL, edited by B. D'ESPAGNAT (New York, N. Y. 1971), p. 122; L. LANZ: in *The Uncertainty Principle and Foundations of Quantum Mechanics*, edited by C. PRICE and S. S. CHISSIK (London, 1977), p. 87; for a review of different approaches on this problem see also G. M. PROSPERI: in *Lecture Notes in Physics*, Vol. **29** (Berlin, 1973), p. 163.

The fact that measure (1.13) satisfies the second of equations (1.6) is shown in appendix A.

For both models we shall consider repetitions of the measurement at subsequent regularly separated times between t_i and t_f and shall evaluate the probability of a certain sequence of results by an obvious generalization of the Wigner formula for repeated observations ([8]). Then we set

$$\alpha = \gamma\tau \tag{1.15}$$

(τ being the time interval between two subsequent measurements) and show that the limit for $\tau \to 0$ can be performed and a functional density of probability or density of effect for a continual observation can be defined.

For instance, for model I) we can write

$$p(W|[x(t)]) = \mathrm{Tr}\{\hat{f}[x(t)]\hat{W}\}, \qquad \int \mathrm{d}\mu_G[x(t)]\hat{f}[x(t)] = 1, \tag{1.16}$$

$$P(W|[x(t)] \in M) = \int_M \mathrm{d}\mu_G[x(t)]\, p(W|[x(t)]), \tag{1.17}$$

where $\int \mathrm{d}\mu_G[x(t)]$ denotes an appropriate functional integral and M a subset of Y.

Quite unexpectedly it turns out that a set M' of the type

$$M' = \{x(t)|a(t) < x(t) < b(t)\} \tag{1.18}$$

occurs with zero probability. On the contrary, a set

$$M'' = \left\{x(t)\left|\bar{a}_s < \frac{1}{\bar{t}_s - \bar{t}_{s-1}} \int_{\bar{t}_{s-1}}^{\bar{t}_s} \mathrm{d}t\, x(t) < \bar{b}_s\,,\; s = 1, 2, \ldots, \nu\right.\right\} \tag{1.19}$$

for given $\bar{t}_s$,

$$\bar{t}_0 = t_i < \bar{t}_1 < \ldots < \bar{t}_{\nu-1} < \bar{t}_\nu = t_f\,, \tag{1.20}$$

has a finite probability.

Actually, for what concerns the second statement, we prove that the ordinary density of probability

$$p(W|\bar{t}_0; \bar{x}_1, \bar{t}_1; \bar{x}_2, \bar{t}_2; \ldots; \bar{x}_\nu, \bar{t}_\nu) = \int \mathrm{d}\mu_G[x(t)] \cdot \prod_{s=1}^{\nu} \delta\left(\bar{x}_s - \frac{1}{\bar{t}_s - \bar{t}_{s-1}} \int_{\bar{t}_{s-1}}^{\bar{t}_s} \mathrm{d}t\, x(t)\right) p(W|[x(t)]) \tag{1.21}$$

exists and is finite.

([8]) E. P. Wigner: *Am. J. Phys.*, **31**, 6 (1963); R. M. F. Houtappel, H. van Dam and E. P. Wigner: *Rev. Mod. Phys.*, **37**, 595 (1965); E. P. Wigner: in *Proc. S.I.F.*, Course IL, edited by B. d'Espagnat (New York, N. Y., 1971), p. 1.

As we shall see, it is also possible to write eq. (1.21) in the form

$$p(W|\bar{t}_0;\bar{x}_1,\bar{t}_1;\bar{x}_2,\bar{t}_2;\ldots;\bar{x}_\nu,\bar{t}_\nu)=\mathrm{Tr}\{\mathscr{F}(\bar{x}_\nu,\bar{t}_\nu-\bar{t}_{\nu-1})\ldots\mathscr{F}(\bar{x}_1,\bar{t}_1-\bar{t}_0)\hat{W}_S(t_0)\}, \tag{1.22}$$

where $\mathscr{F}(x,t)$ is an operator acting in the space $\tau c(\mathfrak{H})$ of trace class operators on the space $\mathfrak{H}$. Then it is particularly significant to consider the expression

$$\mathscr{G}(t)=\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathscr{F}(x,t), \tag{1.23}$$

which provides the evolution of the statistical operator under continual observation when no notice is taken of the result. It is worthwhile mentioning that the operator $\mathscr{G}(t)$ satisfies the semi-group condition

$$\mathscr{G}(t_1)\,\mathscr{G}(t_2)=\mathscr{G}(t_1+t_2), \qquad t_1,t_2\geqslant 0, \tag{1.24}$$

and the differential equation

$$\frac{\mathrm{d}\mathscr{G}(t)}{\mathrm{d}t}=\mathscr{L}\mathscr{G}(t), \tag{1.25}$$

where $\mathscr{L}$ is again an operator in $\tau c(\mathfrak{H})$ defined by

$$\mathscr{L}\hat{W}=-\frac{i}{\hbar}[\hat{H},\hat{W}]-\frac{\gamma}{4}[\hat{q},[\hat{q},\hat{W}]]. \tag{1.26}$$

Equations (1.25) and (1.26) define typically a time evolution which satisfies the requirement of preserving the trace and the positivity of the operator $\hat{W}$ to which it is applied (9).

We note that the parameter α occuring in eqs. (1.12) and (1.15) specifies the degree of accuracy in the observation related to the considered « effect-valued » measure. For instance, in terms of the eigenstates of $\hat{q}$,

$$\hat{q}|q\rangle=q|q\rangle, \tag{1.27}$$

eq. (1.12) can be written as

$$\hat{F}(T)=\sqrt{\frac{\alpha}{\pi}}\int_T\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}q\;|q\rangle\exp[-\alpha(x-q)^2]\langle q|, \tag{1.28}$$

(9) V. GORINI, A. KOSSAKOWSKI and E. C. G. SUDARSHAN: *J. Math. Phys. (N. Y.)*, **17**, 821 (1976); G. LINDBLAD: *Commun. Math. Phys.*, **48**, 119 (1976).

which for $\alpha \to +\infty$ becomes

$$\hat{F}(T) = \int_T \mathrm{d}q |q\rangle\langle q| \tag{1.29}$$

and our « coarse-grained » position x becomes identical to the « microscopic » position q. On the contrary, as $\alpha \to 0$, the information implied in the statement $x \in T$ becomes progressively poorer for what concerns the microscopic position q.

The meaning of eq. (1.15) is that, in order to introduce consistently a continual observation of a quantity starting from a sequence of subsequent discrete observations, the accuracy of the single observation has to decrease as the number of the repeated observations increases. It is just this circumstance that enables us to circumvent the difficulties that have been found in the preceding attempts of introducing a continual observation for ordinary observables and, in particular, the curious paradox, that has been called Zeno paradox, according to which a system is frozen in a definite state by such an observation ([10]). The quantity γ simulates what should be, in a certain order of ideas, a fundamental constant occurring in the relationship between the macroscopic and the quantum level of description.

As a generalization of models I) and II), finally, we consider the case of a continual coarse-grained observation of a set of noncommuting quantities $A_1, A_2, \ldots, A_n$ for an arbitrary system with a finite number of degrees of freedom.

In conclusion, it should be mentioned that, in a different context, a theory of continual observations (tailor-made for photon counting experiments) has been developed by DAVIES ([5]) and DAVIES and SRINIVAS ([11]). Strictly connected with the concept of continual observation is also the treatment of unstable systems of FONDA, GHIRARDI and RIMINI ([12]).

The plane of the paper is the following one.

In sect. **2** we discuss the probability distribution for the outcome of repeated measurements of an ordinary or a generalized observable according to the usual rules on the reduction of the state in quantum mechanics and we derive the Wigner formula and its generalization.

In sect. **3** we perform the limit $\tau \to 0$ for model I) and evaluate $p(W|[x(t)])$ and $p(W|\bar{t}_0; \bar{x}_1, \bar{t}_1; \ldots; \bar{x}_\nu, \bar{t}_\nu)$ in terms of Feynman integrals.

In sect. **4** we study the operator $\mathscr{F}(x, t)$ and its Fourier transform $\tilde{\mathscr{F}}(k, t)$ for which a simple differential equation can be given, in some way similar to eq. (1.25); then we consider the evolution operator $\mathscr{G}(t) = \tilde{\mathscr{F}}(0, t)$ and establish eqs. (1.25), (1.26).

In sect. **5** we treat model II) along the same lines and in sect. **6** we give the generalization to the case of noncommuting quantities $A_1, A_2, \ldots, A_n$.

([10]) B. MISRA and E. C. G. SUDARSHAN: *J. Math. Phys.* (*N. Y.*), **18**, 756 (1977).
([11]) E. B. DAVIES and M. D. SRINIVAS: *Opt. Acta*, **28**, 981 (1981).
([12]) L. FONDA, G. C. GHIRARDI and A. RIMINI: *Rep. Prog. Phys.*, **41**, 587 (1978).

In sect. **7** expectation values and correlation functions are treated and a generalized Ehrenfest theorem is obtained.

In sect. **8** finally the explicit results are worked out for the free particle and the harmonic oscillator in the one-dimensional case.

2. – Repeated observations.

In the ordinary formulation of quantum mechanics the probability for observing a value in the interval $I_1 = (a_1, a_1 + \Delta a_1)$ for the quantity A at time t_1 is given by eq. (1.5), as we have recalled. Taking into account the idempotency of the projections ($\hat{E}^2 = \hat{E}$) and the cyclic property of the trace operation, we can write, *e.g.* in the Heisenberg picture,

$$P(W|I_1, t_1) = \mathrm{Tr}\,\{\hat{E}_{\mathrm{H}}(I_1, t_1)\,\hat{W}\hat{E}_{\mathrm{H}}(I_1, t_1)\}\,. \tag{2.1}$$

We may ask now what is the probability that starting from a given initial condition one jointly observes $A \in I_1$ at the time t_1 and $A \in I_2$ at the subsequent time t_2. The answer to this problem depends on the assumption that we make on the measuring apparatus. If we assume that the disturbance introduced by the apparatus at the time t_1 is the minimum one consistent with the discrimination between the two situations $A \in I_1$ and $A \notin I_1$, the usual reduction postulate can be applied. Then, if the result $A \in I_1$ at t_1 is found, the probability of observing $A \in I_2$ at t_2 is given by

$$P(W_1|I_2, t_2) = \mathrm{Tr}\,\{\hat{E}_{\mathrm{H}}(I_2, t_2)\,\hat{W}_1\,\hat{E}_{\mathrm{H}}(I_2, t_2)\}\,, \tag{2.2}$$

where

$$\hat{W}_1 = \frac{\hat{E}_{\mathrm{H}}(I_1, t_1)\,\hat{W}\hat{E}_{\mathrm{H}}(I_1, t_1)}{\mathrm{Tr}\,\{\hat{E}_{\mathrm{H}}(I_1, t_1)\,\hat{W}\hat{E}_{\mathrm{H}}(I_1, t_1)\}}\,, \tag{2.3}$$

and for the joint probability one has

$$\begin{aligned} P(W|I_1, t_1; I_2, t_2) &= P(W|I_1, t_1)\,P(W_1|I_2, t_2) = \\ &= \mathrm{Tr}\,\{\hat{E}_{\mathrm{H}}(I_2, t_2)\,\hat{E}_{\mathrm{H}}(I_1, t_1)\,\hat{W}\hat{E}_{\mathrm{H}}(I_1, t_1)\,\hat{E}_{\mathrm{H}}(I_2, t_2)\}\,. \end{aligned} \tag{2.4}$$

By repeated application of such an equation, we find the Wigner formula [8] for the case of many subsequent observations:

$$\begin{aligned} &P(W|I_1, t_1; I_2, t_2; \dots; I_N, t_N) = \\ &= \mathrm{Tr}\,\{\hat{E}_{\mathrm{H}}(I_N, t_N) \dots \hat{E}_{\mathrm{H}}(I_2, t_2)\,\hat{E}_{\mathrm{H}}(I_1, t_1)\,\hat{W}\hat{E}_{\mathrm{H}}(I_1, t_1)\,\hat{E}_{\mathrm{H}}(I_2, t_2) \dots \hat{E}_{\mathrm{H}}(I_N, t_N)\}\,. \end{aligned} \tag{2.5}$$

In a similar way for a generalized observable we can write eq. (1.7) in a form parallel to eq. (2.1):

$$P(W|I_1, t_1) = \mathrm{Tr}\,\{\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)\,\hat{W}\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)\}\,. \tag{2.6}$$

An obvious generalization of eq. (2.5) is then

$$(2.7)\qquad P(W|I_1, t_1; I_2, t_2; \ldots; I_N, t_N) = \\ = \mathrm{Tr}\{\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_N, t_N) \ldots \hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_2, t_2)\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)\hat{W}\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_2, t_2) \ldots \hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_N, t_N)\}\,.$$

Equation (2.7) corresponds to restating the reduction postulate for a generalized observable in the form

$$(2.8)\qquad \hat{W}_1 = \frac{\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)\hat{W}\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)}{\mathrm{Tr}\{\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)\hat{W}\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)\}}\,.$$

The meaning of such an assumption for the pure-state case is illustrated in appendix B.

One of the most appealing aspects of eqs. (2.5) and (2.7) is that they could be assumed as a fundamental postulate for repeated observations, independently of the way they have been derived, avoiding any explicit reference to a reduction postulate.

We note that both equations (2.5) and (2.7) can be written as

$$(2.9)\qquad P(W|I_1, t_1; \ldots; I_N, t_N) = \mathrm{Tr}\{\hat{F}(I_1, t_1; \ldots; I_N, t_N)\hat{W}\}\,,$$

where

$$(2.10)\qquad \hat{F}(I_1, t_1; \ldots; I_N, t_N) = \\ = \hat{E}_{\mathrm{H}}(I_1, t_1) \ldots \hat{E}_{\mathrm{H}}(I_{N-1}, t_{N-1})\hat{E}_{\mathrm{H}}(I_N, t_N)\hat{E}_{\mathrm{H}}(I_{N-1}, t_{N-1}) \ldots \hat{E}_{\mathrm{H}}(I_1, t_1)$$

for an ordinary observable and as

$$(2.11)\qquad \hat{F}(I_1, t_1; \ldots; I_N, t_N) = \\ = \hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1) \ldots \hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_{N-1}, t_{N-1})\hat{F}_{\mathrm{H}}(I_N, t_N)\hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_{N-1}, t_{N-1}) \ldots \hat{F}^{\frac{1}{2}}_{\mathrm{H}}(I_1, t_1)$$

for a generalized one.

In both cases $\hat{F}(I_1, t_1; \ldots; I_N, t_N)$ is an effect and eq. (2.9) is similar to eq. (1.7) under this respect, showing once more the much greater generality and concrete relevance of effects as compared with projections.

Such an effect, however, does not generate a measure on $\boldsymbol{R}^N$, in fact, if I_1' and I_1'' are two disjoint intervals, we have

$$(2.12)\qquad \hat{F}(I_1' \cup I_1'', t_1; I_2, t_2; \ldots; I_N, t_N) \neq \\ \neq \hat{F}(I_1', t_1; I_2, t_2; \ldots; I_N, t_N) + \hat{F}(I_1'', t_1; I_2, t_2; \ldots; I_N, t_N)$$

apart from trivial cases. Thus repeated observations cannot be considered as a generalized observable in general. Having in mind eq. (1.12) or (1.13), however, we are led to consider « effect-valued » measures which derive from densities of effects, *i.e.*

$$\mathrm{d}\hat{F}(a)=\hat{f}(a)\,\mathrm{d}\mu(a) \tag{2.13}$$

with

$$\int_{-\infty}^{+\infty}\mathrm{d}\mu(a)\hat{f}(a)=1\,, \tag{2.14}$$

where $\mathrm{d}\mu(a)$ is a numerical measure and $\hat{f}(a)$ is a function on the real axis (or, more generally, on a real space $\boldsymbol{R}^r$) with values in the set of the positive operators in $\mathfrak{H}$. In this case, replacing eq. (2.13) into (2.11), we obtain for infinitesimal intervals

$$\begin{aligned}\hat{F}(\mathrm{d}I_1,t_1;\ldots;\mathrm{d}I_N,t_N)&=\hat{f}(a_1,t_1;\ldots;a_N,t_N)\,\mathrm{d}\mu(a_1)\ldots\mathrm{d}\mu(a_N)=\\&=\hat{f}^{\frac{1}{2}}_{\mathrm{H}}(a_1,t_1)\ldots\hat{f}^{\frac{1}{2}}_{\mathrm{H}}(a_{N-1},t_{N-1})\hat{f}_{\mathrm{H}}(a_N,t_N)\hat{f}^{\frac{1}{2}}_{\mathrm{H}}(a_{N-1},t_{N-1})\ldots\hat{f}^{\frac{1}{2}}_{\mathrm{H}}(a_1,t_1)\,\mathrm{d}\mu(a_1)\ldots\mathrm{d}\mu(a_N)\end{aligned} \tag{2.15}$$

and from (2.9)

$$\begin{aligned}P(W|\mathrm{d}I_1,t_1;\ldots;\mathrm{d}I_N,t_N)&=p(W|a_1,t_1;\ldots;a_N,t_N)\,\mathrm{d}\mu(a_1)\ldots\mathrm{d}\mu(a_N)=\\&=\mathrm{Tr}\{\hat{f}(a_1,t_1;\ldots;a_N,t_N)\,\hat{W}\}\,\mathrm{d}\mu(a_1)\ldots\mathrm{d}\mu(a_N)\,.\end{aligned} \tag{2.16}$$

From eq. (2.14) it follows that

$$\begin{aligned}&\int_{-\infty}^{-\infty}\mathrm{d}\mu(a_N)\hat{f}(a_1,t_1;\ldots;a_{N-1},t_{N-1};a_N,t_N)=\\&=\hat{f}^{\frac{1}{2}}_{\mathrm{H}}(a_1,t_1)\ldots\hat{f}^{\frac{1}{2}}_{\mathrm{H}}(a_{N-1},t_{N-1})\hat{f}^{\frac{1}{2}}_{\mathrm{H}}(a_{N-1},t_{N-1})\ldots\hat{f}^{\frac{1}{2}}_{\mathrm{H}}(a_1,t_1)=\hat{f}(a_1,t_1;\ldots;a_{N-1},t_{N-1})\end{aligned} \tag{2.17}$$

and, by iteration,

$$\int\mathrm{d}\mu(a_1)\ldots\mathrm{d}\mu(a_N)\hat{f}(a_1,t_1;\ldots;a_N,t_N)=1\,, \tag{2.18}$$

$$\int\mathrm{d}\mu(a_1)\ldots\mathrm{d}\mu(a_N)\,p(W|a_1,t_1;\ldots;a_N,t_N)=1\,. \tag{2.19}$$

Thus we arrive at a normalized density of effects for repeated observations and at a density of probability which actually generate an effect-valued measure on $\boldsymbol{R}^N$:

$$\hat{F}'(I_1,t_1;\ldots;I_N,t_N)=\int_{I_1}\mathrm{d}\mu(a_1)\ldots\int_{I_N}\mathrm{d}\mu(a_N)\hat{f}(a_1,t_1;\ldots;a_N,t_N) \tag{2.20}$$

and a new probability distribution

$$\begin{aligned}P'(W|I_1,t_1;\ldots;I_N,t_N)&=\int_{I_1}\mathrm{d}\mu(a_1)\ldots\int_{I_N}\mathrm{d}\mu(a_N)\,p(W|a_1,t_1;\ldots;a_N,t_N)=\\&=\mathrm{Tr}\{\hat{F}'(I_1,t_1;\ldots;I_N,t_N)\,\hat{W}\}\end{aligned} \tag{2.21}$$

having all the required properties.

We note that the result we have obtained depends strictly on having considered generalized observables for the measurement at the single times $t_1, t_2, \ldots, t_N$. For an ordinary observable a significant limit exists for vanishing intervals $\mathrm{d}I_1, \ldots, \mathrm{d}I_N$ only in the case of a purely discrete spectrum. In the presence of a continuous spectrum, if one tries to set $\mathrm{d}\hat{E}(a) = \hat{e}(a)\,\mathrm{d}a$, eq. (1.9) implies

$$\hat{e}(a)\hat{e}(a') = \delta(a - a')\hat{e}(a)\,, \tag{2.22}$$

thus $\hat{e}(a)$ cannot be an ordinary operator-valued function, but must be an operator-valued distribution and an equation similar to (2.15) cannot hold (*).

From the physical point of view, the difference between the cases of an ordinary and a generalized observable can be traced back to the fact that, in the first case, as the amplitudes $\Delta a_1, \Delta a_2, \ldots, \Delta a_N$ of the intervals $I_1, I_2, \ldots, I_N$ vanish, the disturbance produced by every observation on the subsequent ones explodes; on the contrary, in the second case for an effect-valued measure, *e.g.* of the type defined by eq. (1.12), the disturbance remains limited due to the intrinsic inaccuracy introduced by the parameter α.

Note also that there is a physical difference between eqs. (2.20), (2.21) and eqs. (2.11), (2.9). In the case of eqs. (2.20), (2.21) the measurement apparatus is supposed to be as accurate as possible consistently with the form of $\hat{F}(T)$: large intervals $I_1, I_2, \ldots, I_N$ simply mean that we are asking for less information than that in principle available. Instead, in the case of eqs. (2.11), (2.9), the apparatus is assumed to be built in such a way that the minimal disturbance on the system is produced compatible with the required information: diminishing $\Delta a_1, \Delta a_2, \ldots, \Delta a_N$ amounts to modifying the apparatus and to increasing the disturbance.

Equations (2.13)-(2.21) will be at the basis of our subsequent development.

To close this section, we note that, by iterated application of eq. (2.17), it follows that

$$\int \mathrm{d}\mu(a_{p+1}) \ldots \mathrm{d}\mu(a_N)\, p(W|a_1, t_1; \ldots; a_p, t_p; a_{p+1}, t_{p+1}; \ldots; a_N, t_N) = \\ = p(W|a_1, t_1; \ldots; a_p, t_p)\,, \qquad p < N; \tag{2.23}$$

(*) Actually, in this case too it is possible to set

$$\hat{F}(\mathrm{d}I_1, t_1; \ldots; \mathrm{d}I_N, t_N) = \hat{E}_{\mathrm{H}}(\mathrm{d}I_1, t_1) \ldots \hat{E}_{\mathrm{H}}(\mathrm{d}I_{N-1}, t_{N-1}) \cdot \\ \cdot \hat{E}_{\mathrm{H}}(\mathrm{d}I_N, t_N)\, \hat{E}_{\mathrm{H}}(\mathrm{d}I_{N-1}, t_{N-1}) \ldots \hat{E}_{\mathrm{H}}(\mathrm{d}I_1, t_1)$$

and correspondingly to define an effect-valued measure $\hat{F}'(I_1, t_1; \ldots; I_N, t_N)$, but for this measure it is not possible to perform the limit $\tau \to 0$, $N \to \infty$, discussed in the next section.

instead, we have

$$\int \mathrm{d}\mu(a_1) \ldots \mathrm{d}\mu(a_p)\, p(W|a_1, t_1; \ldots; a_p, t_p; a_{p+1}, t_{p+1}; \ldots; a_N, t_N) \neq \neq p(W|a_{p+1}, t_{p+1}; \ldots; a_N, t_N)\,. \tag{2.24}$$

The difference between (2.23) and (2.24) expresses the irreversibility in the time evolution introduced by the repeated observations.

3. – Continual observation of the coarse-grained position.

We will now analyse the case of the « coarse-grained » position for a one-dimensional particle defined by eq. (1.12). Setting

$$\mathrm{d}\mu(x) = \sqrt{\frac{\alpha}{\pi}}\,\mathrm{d}x\,, \qquad \hat{f}(x) = \exp[-\alpha(\hat{q} - x)^2]\,, \tag{3.1}$$

we can write

$$\hat{f}(x_1, t_1; \ldots; x_N, t_N) = = \exp\left[-\frac{\alpha}{2}(\hat{q}_{\mathrm{H}}(t_1) - x_1)^2\right] \ldots \exp\left[-\frac{\alpha}{2}(\hat{q}_{\mathrm{H}}(t_{N-1}) - x_{N-1})^2\right] \cdot \exp[-\alpha(\hat{q}_{\mathrm{H}}(t_N) - x_N)^2] \exp\left[-\frac{\alpha}{2}(\hat{q}_{\mathrm{H}}(t_{N-1}) - x_{N-1})^2\right] \ldots \exp\left[-\frac{\alpha}{2}(\hat{q}_{\mathrm{H}}(t_1) - x_1)^2\right] \tag{3.2}$$

and

$$p(W|x_1, t_1; \ldots; x_N, t_N) = \mathrm{Tr}\{\hat{f}(x_1, t_1; \ldots; x_N, t_N)\hat{W}\} = = \mathrm{Tr}\left\{\exp\left[-\frac{\alpha}{2}(\hat{q}_{\mathrm{H}}(t_N) - x_N)^2\right] \ldots \exp\left[-\frac{\alpha}{2}(\hat{q}_{\mathrm{H}}(t_1) - x_1)^2\right]\hat{W} \cdot \exp\left[-\frac{\alpha}{2}(\hat{q}_{\mathrm{H}}(t_1) - x_1)^2\right] \ldots \exp\left[-\frac{\alpha}{2}(\hat{q}_N(t_N) - x_N)^2\right]\right\}. \tag{3.3}$$

Let us then consider a definite interval of time (t_i, t_f) and a continuous function $x(t)$ on it; then set in eqs. (3.2), (3.3)

$$t_s = t_i + s\tau\,, \qquad \tau = \frac{1}{N}(t_f - t_i)\,, \qquad x_s = x(t_s)\,. \tag{3.4}$$

If we further assume $\alpha = \gamma\tau$ (cf. eq. (1.15)) and perform the limit for $N \to \infty$, we obtain

$$\hat{f}[x(t)] = T^* \exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f}\mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t) - x(t))^2\right] \cdot T\exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f}\mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t) - x(t))^2\right] \tag{3.5}$$

and

$$(3.6)\qquad p(W|[x(t)]) = \mathrm{Tr}\{\hat{f}[x(t)]\hat{W}\} =$$

$$= \mathrm{Tr}\left\{T \exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f} \mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t) - x(t))^2\right]\hat{W} T^* \exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f} \mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t) - x(t))^2\right]\right\}.$$

Here T denotes the time-ordering prescription and T^* the prescription of reverse order, that is equivalent in this case to considering the adjoint operator of the expression with T.

Equations (3.5) and (3.6) define functional densities of effect and the probability corresponding to a *continual observation* in the considered time interval. Such densities refer to the functional integration which formally can be derived from the « measure »

$$(3.7)\qquad \mathrm{d}\mu_{\mathrm{G}}[x(t)] = \lim_{N\to\infty}\prod_{s=1}^{N}\mathrm{d}\mu(x_s) = \lim_{N\to\infty}\left(\frac{\gamma\tau}{\pi}\right)^{N/2}\prod_{s=1}^{N}\mathrm{d}x_s$$

and are normalized with respect to it. *E.g.* we have

$$(3.8)\qquad \int \mathrm{d}\mu_{\mathrm{G}}[x(t)]\,p(W|[x(t)]) = \lim_{N\to\infty}\int \mathrm{d}\mu(x_1)\ldots\int \mathrm{d}\mu(x_N)\,p(W|x_1,t_1;\ldots;x_N,t_N) = 1\,.$$

As we mentioned, the density of probability $p(W|[x(t)])$ is such that a subset M' of the type defined by eq. (1.18) of the functional space Y has zero probability:

$$(3.9)\qquad \int_{a(t)}^{b(t)} \mathrm{d}\mu_{\mathrm{G}}[x(t)]\,p(W|[x(t)]) = \lim_{N\to\infty}\int_{a(t_1)}^{b(t_1)} \mathrm{d}\mu(x_1)\ldots\int_{a(t_N)}^{b(t_N)} \mathrm{d}\mu(x_N)\,p(W|x_1,t_1;\ldots;x_N,t_N) = 0\,.$$

On the contrary, a subset M'' of the type of eq. (1.19) has a positive probability or, what is the same, the density of probability in eq. (1.21)

$$(3.10)\qquad p(W|\bar{t}_0;\bar{x}_1,\bar{t}_1;\bar{x}_2,\bar{t}_2;\ldots;\bar{x}_\nu,\bar{t}_\nu) =$$

$$= \int \mathrm{d}\mu_{\mathrm{G}}[x(t)]\prod_{l=1}^{\nu}\delta\left(\bar{x}_l - \frac{1}{\bar{t}_l - \bar{t}_{l-1}}\int_{\bar{t}_{l-1}}^{\bar{t}_l}\mathrm{d}t\,x(t)\right)p(W|[x(t)]) =$$

$$= \int \mathrm{d}\mu_{\mathrm{G}}[x(t)]\prod_{l=1}^{\nu}\delta\left(\bar{x}_l - \frac{1}{\bar{t}_l - \bar{t}_{l-1}}\int_{\bar{t}_{l-1}}^{\bar{t}_l}\mathrm{d}t\,x(t)\right)\cdot$$

$$\cdot\,\mathrm{Tr}\left\{T \exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f} \mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t) - x(t))^2\right]\hat{W} T^* \exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f} \mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t) - x(t))^2\right]\right\}$$

is well defined and nonzero.

In order to prove the above statement, it is convenient to use the Feynman integral formalism.

We have

$$(3.11)\qquad \langle q_f, t_f | T \exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f} dt\,(\hat{q}_H(t) - x(t))^2\right] | q_i, t_i\rangle =$$

$$= \int d\mu_F[q(t)] \exp\left[\int_{t_i}^{t_f} dt \left[\frac{i}{\hbar} L(q(t), \dot{q}(t)) - \frac{\gamma}{2}(q(t) - x(t))^2\right]\right],$$

where $L(q, \dot{q})$ is the «classical» Lagrangian of the particle,

$$(3.12)\qquad |q, t\rangle = \exp\left[\frac{i}{\hbar}\hat{H}(t)\right]|q\rangle$$

and, formally,

$$(3.13)\qquad d\mu_F[q(t)] = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\tau}\right)^{N/2} \prod_{s=1}^{N-1} dq(t_s)$$

with

$$(3.14)\qquad q(t_0) = q_i, \qquad q(t_N) = q_f.$$

Replacing eq. (3.11) in (3.6), we obtain

$$(3.15)\qquad p(W|[x(t)]) = \int dq_i\, dq'_i\, dq_f\, dq'_f\, \delta(q_f - q'_f)\langle q_i, t_i|\hat{W}|q'_i, t_i\rangle \cdot$$

$$\cdot \int d\mu_F[q(t)] \int d\mu_F^*[q'(t)] \exp\left[\int_{t_i}^{t_f} dt \left[\frac{i}{\hbar}(L(q,\dot{q}) - L(q',\dot{q}')) - \frac{\gamma}{2}((q-x)^2 + (q'-x)^2)\right]\right].$$

We can then perform explicitly the integration over the function $x(t)$ in eqs. (3.9) and (3.10).

We have

$$(3.16)\qquad \int_{a(t)}^{b(t)} d\mu_G[x(t)] \exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f} dt[(q-x)^2 + (q'-x)^2]\right] =$$

$$= \lim_{N\to\infty}\left(\frac{\gamma\tau}{\pi}\right)^{N/2} \int_{a(t_1)}^{b(t_1)} dx_1 \ldots \int_{a(t_N)}^{b(t_N)} dx_N \cdot$$

$$\cdot \exp\left[-\gamma\tau \sum_{s=1}^{N}\left[\left(x_s - \frac{q(t_s) + q'(t_s)}{2}\right)^2 + \frac{1}{4}(q(t_s) - q'(t_s))^2\right]\right] =$$

$$= \exp\left[-\frac{\gamma}{4}\int_{t_i}^{t_f} dt\,(q(t) - q'(t))^2\right] \cdot \lim_{N\to\infty}[\phi(\sqrt{\gamma\tau}B)]^N = 0,$$

$$\phi(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\xi}^{+\xi} d\eta \exp[-\eta^2] < 1 , \qquad B = \max \frac{b(t) - a(t)}{2} , \tag{3.17}$$

and eq. (3.9) follows.

In a similar way, taking advantage of the identity

$$\int dx_1 \dots dx_n \left(\frac{\alpha}{\pi}\right)^{n/2} \delta\left(\bar{x} - \frac{1}{n}\sum_{s=1}^{n} x_s\right) \exp\left[-\alpha \sum_{s=1}^{n} (x_s - \xi_s)^2\right] = \left(\frac{n\alpha}{\pi}\right)^{\frac{1}{2}} \exp\left[-n\alpha\left(\bar{x} - \frac{1}{n}\sum_{s=1}^{n}\xi_s\right)^2\right], \tag{3.18}$$

we obtain

$$\int d\mu_G[x(t)] \prod_{l=1}^{\nu} \delta\left(\bar{x}_l - \frac{1}{\Delta t_l}\int_{\bar{t}_{l-1}}^{\bar{t}_l} dt\, x(t)\right) \exp\left[-\frac{\gamma}{2}\int_{t_i}^{t_f} dt[(q-x)^2 + (q'-x)^2]\right] = \prod_{l=1}^{\nu}\left\{\left(\frac{\gamma \Delta t_l}{\pi}\right)^{\frac{1}{2}} \exp\left[-\gamma\, \Delta t_l\left(\bar{x}_l - \frac{1}{\Delta t_l}\int_{\bar{t}_{l-1}}^{\bar{t}_l} dt \frac{q+q'}{2}\right)^2\right]\right\} \exp\left[-\frac{\gamma}{4}\int_{t_i}^{t_f} dt\,(q-q')^2\right], \tag{3.19}$$

where we have set $\Delta t_l = \bar{t}_l - \bar{t}_{l-1}$. In conclusion, eq. (3.10) becomes

$$\begin{aligned} p(W|\bar{t}_0; \bar{x}_1, \bar{t}_1; \dots; \bar{x}_\nu, \bar{t}_\nu) = &\int dq_i\, dq_i'\, dq_f\, dq_f'\, \delta(q_f - q_f') \cdot \\ &\cdot \langle q_i, t_i|\hat{W}|q_i', t_i\rangle \int d\mu_F[q(t)] \int d\mu_F^*[q'(t)] \cdot \\ &\cdot \prod_{l=1}^{\nu}\left\{\left(\frac{\gamma\Delta t_l}{\pi}\right)^{\frac{1}{2}} \exp\left[-\gamma \Delta t_l\left(\bar{x}_l - \frac{1}{\Delta t_l}\int_{\bar{t}_{l-1}}^{\bar{t}_l} dt \frac{q+q'}{2}\right)^2\right]\right\} \cdot \\ &\cdot \exp\left[\int_{t_i}^{t_f} dt\left[\frac{i}{\hbar}(L(q,\dot{q}) - L(q',\dot{q}')) - \frac{\gamma}{4}(q-q')^2\right]\right], \end{aligned} \tag{3.20}$$

in which only ordinary Feynman integrals occur.

In the above manipulations for clarity we have formally exchanged the functional integrals in $d\mu_F[q(t)]$ and in $d\mu_G[x(t)]$. Actually one should proceed in the following way:

1) introduce a partition of the time interval (t_i, t_f) according to eq. (3.4),

2) replace the integral in $d\mu_G[x(t)]$ in eq. (3.10) by the corresponding discrete integrals,

3) write each of the two exponentials in eq. (3.10) as a product of the N factors corresponding to the partition of (t_i, t_f),

4) introduce between the factors completenesses of the form $\int dq_s |q_s, t_s\rangle \cdot$ $\cdot \langle q_s, t_s|$,

5) use the Feynman asymptotic formula for small intervals,

6) perform the limit $N \to \infty$.

Therefore, a single limit $N \to \infty$ has to be used for defining globally all the functional integrations; the exchange of the integrals is made before the limit and formula (3.18) is used (13).

Equation (3.20) is our basic result.

Note that a so significant result depends strictly on assumption (1.15). Had we taken α fixed rather than γ, we should have replaced γ by α/τ in eqs. (3.16) and (3.19) and we would have obtained vanishing results also for $p(W|\bar{t}_0; \bar{x}_1, \bar{t}_1; \dots; \bar{x}_\nu, \bar{t}_\nu)$.

Note also that eq. (3.20) in terms of a density of effect can be equivalently written as

$$(3.21) \quad \langle q'_i, t_i | \hat{f}(\bar{t}_0; \bar{x}_1, \bar{t}_1; \dots; \bar{x}_\nu, \bar{t}_\nu) | q_i, t_i \rangle = \\ = \int dq_f \, dq'_f \, \delta(q_f - q'_f) \int d\mu_F[q] \int d\mu_F^*[q'] \prod_{l=1}^{\nu} \left\{ \left(\frac{\gamma \Delta t_l}{\pi} \right)^{\frac{1}{2}} \exp \left[-\gamma \Delta t_l \cdot \right.\right. \\ \left.\left. \cdot \left(\bar{x}_l - \frac{1}{\Delta t_l} \int_{\bar{t}_{l-1}}^{\bar{t}_l} dt \, \frac{q + q'}{2} \right)^2 \right] \right\} \exp \left[\int_{t_i}^{t_f} dt \left[\frac{i}{\hbar} \left(L(q, \dot{q}) - L(q', \dot{q}') \right) - \frac{\gamma}{4} (q - q')^2 \right] \right].$$

Finally let us consider a finer subdivision of the macroscopic time intervals $(\bar{t}_{l-1}, \bar{t}_l)$ into smaller ones and set

$$(3.22) \qquad \bar{t}_{l0} = \bar{t}_{l-1} < \bar{t}_{l1} < \dots < \bar{t}_{l\mu_l} = \bar{t}_l, \qquad \Delta t_{lj} = \bar{t}_{lj} - \bar{t}_{lj-1}.$$

By using the obvious identity

$$(3.23) \quad \int d\bar{x}_{l1} \dots d\bar{x}_{l\mu_l} \, \delta \left(\bar{x}_l - \sum_{j=1}^{\mu_l} \frac{\Delta t_{lj}}{\Delta t_l} \bar{x}_{lj} \right) \cdot \\ \cdot \prod_{j=1}^{\mu_l} \delta \left(\bar{x}_{lj} - \frac{1}{\Delta t_{lj}} \int_{\bar{t}_{lj-1}}^{\bar{t}_{lj}} dt \, x(t) \right) = \delta \left(\bar{x}_l - \frac{1}{\Delta t_l} \int_{\bar{t}_{l-1}}^{\bar{t}_l} dt \, x(t) \right),$$

---

(13) A rigorous treatment of the whole matter can be given using the Albeverio Hoeg-Krohn definition of the Feynman integral (G. LUPIERI: *Generalized stochastic processes and continual measurements in quantum mechanics* (preprint, Milano)).

we obtain from eq. (3.10)

$$(3.24)\qquad \int \mathrm{d}\bar{x}_{11}\,\mathrm{d}\bar{x}_{12}\ldots\mathrm{d}\bar{x}_{\nu\mu_\nu}\,p(W|\bar{t}_0;\bar{x}_{11},\bar{t}_{11};\bar{x}_{12},\bar{t}_{12};\ldots;\bar{x}_{\nu\mu_\nu},\bar{t}_{\nu\mu_\nu})\cdot \cdot\prod_{l=1}^{\nu}\delta\left(\bar{x}_l-\sum_{j=1}^{\mu_l}\frac{\Delta t_{lj}}{\Delta t_l}\bar{x}_{lj}\right)=p(W|\bar{t}_0;\bar{x}_1,\bar{t}_1;\bar{x}_2,\bar{t}_2;\ldots;\bar{x}_\nu,\bar{t}_\nu)\,,$$

or a similar equation for the density of effect $\hat{f}(\bar{t}_0;\bar{x}_1,\bar{t}_1;\ldots;\bar{x}_\nu,\bar{t}_\nu)$.

Equation (3.24) is the analogous for the case of the continual observation of the additivity of the effect $\hat{F}'(I_1,t_1;\ldots;I_N,t_N)$, or the corresponding probability $P'(W|I_1,t_1;\ldots;I_N,t_N)$ as defined by eqs. (2.20), (2.21).

Equation (3.24) will be referred as the *consistency property*.

4. – The probability density and semi-group properties.

As stated in the introduction (eq. (1.22)), the probability density $p(W|\bar{t}_0;\bar{x}_1,\bar{t}_1;\bar{x}_2,\bar{t}_2;\ldots;\bar{x}_\nu,\bar{t}_\nu)$ can be built up by means of operators $\mathscr{F}(x,\tau)$ acting on the $\tau c(\mathfrak{H})$ space.

Actually, we define

$$(4.1)\qquad \mathscr{F}(x,\tau)\,\hat{W}=\int \mathrm{d}\mu_{\mathrm{G}}[x(t)]\,\delta\left(x-\frac{1}{\tau}\int_0^\tau \mathrm{d}t\;x(t)\right)\exp\left[-\frac{i}{\hbar}\hat{H}\tau\right]T\cdot \cdot\exp\left[-\frac{\gamma}{2}\int_0^\tau \mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t)-x(t))^2\right]\hat{W}T^*\exp\left[-\frac{\gamma}{2}\int_0^\tau \mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t)-x(t))^2\right]\exp\left[\frac{i}{\hbar}\hat{H}\tau\right],$$

where now the functional integral has to be understood as defined on the continuous functions in the interval $(0,\tau)$. From this definition, it is trivial to see that we have

$$(4.2)\qquad \mathscr{F}(\bar{x}_l;\bar{t}_l-\bar{t}_{l-1})\,\hat{W}=\int \mathrm{d}\mu_{\mathrm{G}}[x(t)]\,\delta\left(\bar{x}_l-\frac{1}{\Delta t_l}\int_{\bar{t}_{l-1}}^{\bar{t}_l}\mathrm{d}t\;x(t)\right)\exp\left[-\frac{i}{\hbar}\hat{H}\bar{t}_l\right]T\cdot \cdot\exp\left[-\frac{\gamma}{2}\int_{\bar{t}_{l-1}}^{\bar{t}_l}\mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t)-x(t))^2\right]\exp\left[\frac{i}{\hbar}\hat{H}\bar{t}_{l-1}\right]\hat{W}\exp\left[-\frac{i}{\hbar}\hat{H}\bar{t}_{l-1}\right]\cdot \cdot T^*\exp\left[-\frac{\gamma}{2}\int_{\bar{t}_{l-1}}^{\bar{t}_l}\mathrm{d}t\,(\hat{q}_{\mathrm{H}}(t)-x(t))^2\right]\exp\left[\frac{i}{\hbar}\hat{H}\bar{t}_l\right],$$

so that eq. (3.10) can be rewritten as

$$(4.3)\qquad p(W|\bar{t}_0;\bar{x}_1,\bar{t}_1;\ldots;\bar{x}_{\nu-1},\bar{t}_{\nu-1};\bar{x}_\nu,\bar{t}_\nu)= =\mathrm{Tr}\left\{\mathscr{F}(\bar{x}_\nu,\bar{t}_\nu-\bar{t}_{\nu-1})\,\mathscr{F}(\bar{x}_{\nu-1},\bar{t}_{\nu-1}-\bar{t}_{\nu-2})\ldots\mathscr{F}(\bar{x}_1,\bar{t}_1-\bar{t}_0)\,\hat{W}_{\mathrm{S}}(\bar{t}_0)\right\}.$$

By eq. (4.3) the study of the probability density can be reduced to the study of the operator $\mathscr{F}(x,\tau)$ (and of its Fourier transform $\tilde{\mathscr{F}}(k,\tau)$).

By using the Feynman integral formalism, it is possible to explicitly perform the $\mathrm{d}\mu_G[x(t)]$ integration in eq. (4.1), as done in eqs. (3.18)-(3.20). We have

$$(4.4a)\qquad \langle q|(\mathscr{F}(x,\tau)\,\hat{W})|q'\rangle = \int_{-\infty}^{+\infty}\mathrm{d}q_0\,\mathrm{d}q_0'\,F(x,\tau|q,q';q_0,q_0')\,\langle q_0|\hat{W}|q_0'\rangle\,,$$

where

$$(4.4b)\qquad F(x,\tau|q,q';q_0,q_0') = \int\mathrm{d}\mu_F[q(t)]\int\mathrm{d}\mu_F^*[q'(t)]\,\sqrt{\frac{\gamma\tau}{\pi}}\exp\left[-\gamma\tau\left(x-\frac{1}{\tau}\int_0^\tau\mathrm{d}t\,\frac{q(t)+q'(t)}{2}\right)^2\right]\cdot$$
$$\cdot\exp\left[\int_0^\tau\mathrm{d}t\left\{\frac{i}{\hbar}\left[L(q(t),\dot{q}(t))-L(q'(t),\dot{q}'(t))\right]-\frac{\gamma}{4}\,(q(t)-q'(t))^2\right\}\right].$$

By using eqs. (4.4) and (4.3), we immediately reobtain eq. (3.20).

From definition (4.1) and eq. (3.23), we see that for the operators $\mathscr{F}(x,\tau)$ the « consistency property » takes the form

$$(4.5)\qquad \mathscr{F}(x,\tau_1+\tau_2) = \int_{-\infty}^{+\infty}\mathrm{d}x_1\,\mathrm{d}x_2\,\delta\left(x-\frac{\tau_1x_1+\tau_2x_2}{\tau_1+\tau_2}\right)\mathscr{F}(x_2,\tau_2)\,\mathscr{F}(x_1,\tau_1)\,.$$

Note that, if we discard all information referring to a time interval (t_{i-1},t_i), we must integrate the probability density with respect to the variable $\bar{x}_i$ and consider the expression

$$(4.6)\qquad \int_{-\infty}^{+\infty}\mathrm{d}\bar{x}_i\,p(W|\bar{t}_0;\bar{x}_1,\bar{t}_1;\dots;\bar{x}_i,\bar{t}_i;\dots;\bar{x}_\nu,\bar{t}_\nu) =$$
$$= \mathrm{Tr}\left\{\mathscr{F}(\bar{x}_\nu,\bar{t}_\nu-\bar{t}_{\nu-1})\dots\mathscr{F}(\bar{x}_{i+1},\bar{t}_{i+1}-\bar{t}_i)\left[\int_{-\infty}^{+\infty}\mathrm{d}\bar{x}_i\,\mathscr{F}(\bar{x}_i,\bar{t}_i-\bar{t}_{i-1})\right]\cdot\right.$$
$$\left.\cdot\,\mathscr{F}(\bar{x}_{i-1},\bar{t}_{i-1}-\bar{t}_{i-2})\dots\mathscr{F}(\bar{x}_1,\bar{t}_1-\bar{t}_0)\,\hat{W}_S(\bar{t}_0)\right\}.$$

We see that $\mathscr{G}(t-t')$, defined by

$$(4.7)\qquad \mathscr{G}(t-t') = \int_{-\infty}^{+\infty}\mathrm{d}x\,\mathscr{F}(x,t-t')\,,$$

acquires the meaning of « evolution operator » for the time interval (t', t). Moreover, from property (4.5) we have

$$\mathscr{G}(t_1 + t_2) = \mathscr{G}(t_2)\,\mathscr{G}(t_1)\,, \qquad t_1, t_2 \geqslant 0\,, \tag{4.8}$$

so that the set of operators $\{\mathscr{G}(t)\}_{t\geqslant 0}$ is a semi-group of completely positive, trace-preserving transformations on the $\tau c(\mathfrak{H})$ space. In the literature such a set of operators is called « quantum-dynamical semi-group » (see, for instance, ref. ([5,9,14])).

Obviously the Feynman integrals in eq. (4.4*b*) can be performed only in special cases (*e.g.* free particle and harmonic oscillator). However, it is possible to give a significant differential equation for the Fourier transform of $\mathscr{F}(x, \tau)$. Let us put

$$\tilde{\mathscr{F}}(k, \tau) = \int\limits_{-\infty}^{+\infty} \mathrm{d}x\, \mathscr{F}(x, \tau) \exp[-ikx]\,. \tag{4.9}$$

In terms of $\tilde{\mathscr{F}}(k, \tau)$ the consistency property (4.5) becomes

$$\tilde{\mathscr{F}}(k, \tau_1 + \tau_2) = \tilde{\mathscr{F}}\left(\frac{k\tau_2}{\tau_1 + \tau_2}, \tau_2\right) \tilde{\mathscr{F}}\left(\frac{k\tau_1}{\tau_1 + \tau_2}, \tau_1\right). \tag{4.10}$$

This equation takes a simpler form if we introduce a new operator $\mathscr{G}(\xi, \tau)$ as

$$\mathscr{G}(\xi, \tau) = \tilde{\mathscr{F}}(\xi\tau, \tau)\,. \tag{4.11}$$

In terms of this operator eq. (4.10) becomes

$$\mathscr{G}(\xi, \tau_1 + \tau_2) = \mathscr{G}(\xi, \tau_2)\,\mathscr{G}(\xi, \tau_1)\,. \tag{4.12}$$

Again $\mathscr{G}(\xi, \tau)$ defined a semi-group. If we denote by $\mathscr{K}(\xi)$ its generator, the $\mathscr{G}$-operators satisfy the differential equation

$$\frac{\partial}{\partial \tau}\,\mathscr{G}(\xi, \tau) = \mathscr{K}(\xi)\,\mathscr{G}(\xi, \tau)\,. \tag{4.13a}$$

Then, using the initial condition

$$\mathscr{G}(\xi, 0) = \tilde{\mathscr{F}}(0, 0) = \mathbf{1}\,, \tag{4.13b}$$

([14]) V. Gorini, A. Frigerio, M. Verri, A. Kossakowski and E. C. G. Sudarshan: *Rep. Math. Phys.*, **13**, 149 (1978).

we can write

$$\mathscr{G}(\xi,\tau)=\exp\left[\tau\mathscr{K}(\xi)\right], \tag{4.14a}$$

or

$$\tilde{\mathscr{F}}(k,\tau)=\exp\left[\tau\mathscr{K}(k/\tau)\right]. \tag{4.14b}$$

Note that, for the « evolution operator », we have

$$\mathscr{G}(t)=\mathscr{G}(0,t)=\tilde{\mathscr{F}}(0,t)=\exp\left[\mathscr{L}t\right] \tag{4.15a}$$

where

$$\mathscr{L}=\mathscr{K}(0). \tag{4.15b}$$

In order to obtain an explicit expression for the generators $\mathscr{K}(\xi)$ and $\mathscr{L}$, it is convenient to use the Feynman integral formalism. From eqs. (4.4), (4.9) and (4.11) we have

$$\langle q|\left(\mathscr{G}(\xi,\tau)\hat{W}\right)|q'\rangle=\int_{-\infty}^{+\infty}\mathrm{d}q_0\mathrm{d}q_0'\langle q_0|\hat{W}|q_0'\rangle\, G(\xi,\tau|q,q';q_0,q_0') \tag{4.16a}$$

with

$$G(\xi,\tau|q,q';q_0,q_0')=\exp\left[-\frac{\tau}{4\gamma}\xi^2\right]\cdot\int\mathrm{d}\mu_{\mathrm{F}}[q(t)]\int\mathrm{d}\mu_{\mathrm{F}}^*[q'(t)]\cdot$$
$$\cdot\exp\left[\int_0^\tau\mathrm{d}t\left\{\frac{i}{\hbar}\Big(L(q(t),\dot{q}(t))-L(q'(t),\dot{q}'(t))\Big)-\frac{\gamma}{4}\left(q(t)-q'(t)\right)^2-\frac{i}{2}\xi\left(q(t)+q'(t)\right)\right\}\right], \tag{4.16b}$$

where the Feynman integrals must be performed with the boundary conditions

$$q(\tau)=q\,,\qquad q'(\tau)=q'\,,\qquad q(0)=q_0\,,\qquad q'(0)=q_0'\,. \tag{4.16c}$$

From eq. (4.16*b*), in the limit $\tau\to 0$, using the Feynman asymptotic formula, we have

$$G(\xi,\tau|q,q';q_0,q_0')\overset{\tau\to 0}{\simeq}\delta(q-q_0)\,\delta(q'-q_0')\cdot$$
$$\cdot\left[1-\frac{\tau}{4\gamma}\xi^2-\frac{\gamma\tau}{4}(q-q')^2-\frac{i}{2}\tau\xi(q+q')\right]-$$
$$-\frac{i}{\hbar}\tau\langle q|\hat{H}|q_0\rangle\,\delta(q'-q_0')+\frac{i}{\hbar}\tau\langle q_0'|\hat{H}|q'\rangle\,\delta(q-q_0)\,. \tag{4.17}$$

Introducing this expression into eq. (4.16a), we have finally

$$\mathscr{L}\hat{W} = -\frac{i}{\hbar}[\hat{H}, \hat{W}] - \frac{\gamma}{4}[\hat{q}, [\hat{q}, \hat{W}]], \tag{4.18a}$$

$$\mathscr{K}(\xi)\hat{W} = \mathscr{L}\hat{W} - \frac{i}{2}\xi\{\hat{q}, \hat{W}\} - \frac{1}{4\gamma}\xi^2\hat{W}, \tag{4.18b}$$

where $\{\hat{A}, \hat{B}\}$ denotes the anticommutator between $\hat{A}$ and $\hat{B}$. Note that $\mathscr{L}$ has the typical form of the generators of completely positive and trace-preserving semi-groups (9). This is apparently not true for $\mathscr{K}(\xi)$. Indeed, $\mathscr{G}(\xi, \tau)$ transforms trace class operators into trace class ones, but not positive operators into positive ones.

Finally, we observe that the density of the effect-valued measure introduced into eq. (3.21) can be written as

$$\hat{f}(t_0; x_1, t_1; \dots; x_\nu, t_\nu) = \mathscr{F}'(x_1, t_1 - t_0)\, \mathscr{F}'(x_2, t_2 - t_1) \dots \mathscr{F}'(x_\nu, t_\nu - t_{\nu-1})\hat{1}, \tag{4.19}$$

where $\hat{1}$ is the identity operator on the space $\mathfrak{H}$ and $\mathscr{F}'(x, \tau)$ is the adjoint operator of $\mathscr{F}(x, \tau)$, which acts on the space of the bounded operators on $\mathfrak{H}$ (the dual space of $\tau c(\mathfrak{H})$).

Note that the trace-preserving property of $\mathscr{G}(t)$ is equivalent to the following equation for its adjoint $\mathscr{G}'(t)$:

$$\mathscr{G}'(t)\hat{1} = \hat{1}. \tag{4.20}$$

5. – Continual observation of coarse-grained position and momentum.

Let us consider now model II) of the introduction defined by eq. (1.13) and (1.14). In this case, we have an effect-valued density for the simultaneous measurement of a coarse-grained position and momentum, which can be written as

$$\hat{f}(x, p) = C \exp[-\alpha[(\hat{q} - x)^2 + \lambda(\hat{p} - p)^2]]. \tag{5.1}$$

We can repeat the whole construction done for model I) starting from the new effect-valued density and build up a theory for the continual simultaneous observation of position and momentum. In particular, we can construct a probability density $p(W|\bar{t}_0; \bar{x}_1, \bar{p}_1, \bar{t}_1; \dots; \bar{x}_\nu, \bar{p}_\nu, \bar{t}_\nu)$, where the variables $\bar{x}_i$ and $\bar{p}_i$ have the meaning of position and momentum time averages in the interval $(\bar{t}_{i-1}, \bar{t}_i)$.

Using the formalism of sect. 4 we obtain

$$\begin{aligned} p(W|\bar{t}_0; \bar{x}_1, \bar{p}_1, \bar{t}_1; \dots; \bar{x}_\nu, \bar{p}_\nu, \bar{t}_\nu) &= \\ = \operatorname{Tr} \mathscr{F}(\bar{x}_\nu, \bar{p}_\nu; \bar{t}_\nu - \bar{t}_{\nu-1})\, \mathscr{F}(\bar{x}_{\nu-1}, \bar{p}_{\nu-1}; \bar{t}_{\nu-1} - \bar{t}_{\nu-2}) \dots \mathscr{F}(\bar{x}_1, \bar{p}_1; \bar{t}_1 - \bar{t}_0)\hat{W}_S(\bar{t}_0)&, \end{aligned} \tag{5.2}$$

where

$$(5.3)\qquad \mathscr{F}(x, p; \tau)\,\hat{W} = \int \mathrm{d}\mu_{\mathrm{G}}[x(t), p(t)]\cdot$$
$$\cdot\,\delta\left(x - \frac{1}{\tau}\int_0^\tau \mathrm{d}t\; x(t)\right)\delta\left(p - \frac{1}{\tau}\int_0^\tau \mathrm{d}t\; p(t)\right)\exp\left[-\frac{i}{\hbar}\hat{H}\tau\right]\cdot$$
$$\cdot\, T\exp\left[-\frac{\gamma}{2}\int_0^\tau \mathrm{d}t\,[(\hat{q}_{\mathrm{H}}(t) - x(t))^2 + \lambda(\hat{p}_{\mathrm{H}}(t) - p(t))^2]\right]\hat{W}\cdot$$
$$\cdot\, T^*\exp\left[-\frac{\gamma}{2}\int_0^\tau \mathrm{d}t\,[(\hat{q}_{\mathrm{H}}(t) - x(t))^2 + \lambda(\hat{p}_{\mathrm{H}}(t) - p(t))^2]\right]\exp\left[\frac{i}{\hbar}\hat{H}\tau\right]$$

and

$$(5.4)\qquad \mathrm{d}\mu_{\mathrm{G}}[x(t), p(t)] = \lim_{N\to\infty}\left(\frac{\gamma\tau\sqrt{\lambda}}{\pi N}\right)^N \prod_{s=1}^N \mathrm{d}x(t_s)\,\mathrm{d}p(t_s)\,.$$

The consistency property now is

$$(5.5)\qquad \mathscr{F}(x, p; \tau_1 + \tau_2) = \int_{-\infty}^{+\infty}\mathrm{d}x_1\,\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}p_1\,\mathrm{d}p_2\,\delta\left(x - \frac{\tau_1 x_1 + \tau_2 x_2}{\tau_1 + \tau_2}\right)\cdot$$
$$\cdot\,\delta\left(p - \frac{\tau_1 p_1 + \tau_2 p_2}{\tau_1 + \tau_2}\right)\mathscr{F}(x_2, p_2; \tau_2)\,\mathscr{F}(x_1, p_1; \tau_1)$$

and the evolution semi-group is given by

$$(5.6)\qquad \mathscr{G}(t) = \int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}p\,\mathscr{F}(x, p; t)\,.$$

If we set

$$(5.7)\qquad \mathscr{G}(\xi, \eta; \tau) = \int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}p\,\exp[-i\xi\tau x - i\eta\tau p]\,\mathscr{F}(x, p; \tau)\,,$$

we have again $\mathscr{G}(t) = \mathscr{G}(0, 0; t)$ and

$$(5.8)\qquad \mathscr{G}(\xi, \eta; \tau_1 + \tau_2) = \mathscr{G}(\xi, \eta; \tau_2)\,\mathscr{G}(\xi, \eta; \tau_1)\,.$$

As before we can obtain an expression for $\mathscr{G}(\xi, \eta; \tau)$ in the Feynman integral formalism. We have

$$(5.9a)\qquad \langle q|(\mathscr{G}(\xi,\eta;\tau)\,\hat{W})|q'\rangle = \int_{-\infty}^{+\infty} \mathrm{d}q_0\,\mathrm{d}q_0'\,\langle q_0|\hat{W}|q_0'\rangle\, G(\xi,\eta;\tau|q,q';q_0,q_0')\,,$$

$$(5.9b)\qquad G(\xi,\eta;\tau|q,q';q_0,q_0') =$$

$$= \int \mathrm{d}\mu_F[q(t),p(t)] \int \mathrm{d}\mu_F[q'(t),p'(t)] \exp\left[-\frac{\tau\xi^2}{4\gamma} - \frac{\tau\eta^2}{4\gamma\lambda}\right]\cdot$$

$$\cdot \exp\int_0^\tau \mathrm{d}t \left\{-\frac{i}{\hbar}\left[H(q(t),p(t)) - H(q'(t),p'(t))\right] + \frac{i}{\hbar}\left[p(t)\dot{q}(t) - p'(t)\dot{q}(t)\right] - \right.$$

$$\left. -\frac{\gamma}{4}(q(t)-q'(t))^2 - \frac{\gamma\lambda}{4}(p(t)-p'(t))^2 - \frac{i}{2}\xi(q(t)+q'(t)) - \frac{i}{2}\eta(p(t)+p'(t))\right\},$$

where the following conditions on $q(t)$ and $q'(t)$ have to be understood:

$$(5.10a)\qquad q(\tau) = q\,,\quad q'(\tau) = q'\,,\quad q(0) = q_0\,,\quad q'(0) = q_0'$$

and the Feynman «measure» in phase space

$$(5.10b)\qquad \mathrm{d}\mu_F[q(t),p(t)] = \lim_{N\to\infty} \prod_{s=1}^{N-1} \mathrm{d}q(t_s) \prod_{s=1}^{N} \frac{\mathrm{d}p(t_s)}{2\pi\hbar}$$

has to be used.

Then, proceeding as in the previous section, we can write

$$(5.11)\qquad \mathscr{G}(\xi,\eta;\tau) = \exp\left[\tau\mathscr{K}(\xi,\eta)\right],$$

where

$$(5.12a)\qquad \mathscr{K}(\xi,\eta)\,\hat{W} = \mathscr{L}\hat{W} - \frac{i}{2}\xi\{\hat{q},\hat{W}\} - \frac{i}{2}\eta\{\hat{p},\hat{W}\} - \frac{1}{4\gamma}\left(\xi^2 + \frac{1}{\lambda}\eta^2\right)\hat{W}$$

and

$$(5.12b)\qquad \mathscr{L}\hat{W} = \mathscr{K}(0,0)\,\hat{W} = -\frac{i}{\hbar}[\hat{H},\hat{W}] - \frac{\gamma}{4}\left([\hat{q},[\hat{q},\hat{W}]] + \lambda[\hat{p},[\hat{p},\hat{W}]]\right).$$

Note that all the results of the previous section can be obtained from the present ones; for example,

$$(5.13)\qquad \mathscr{G}(\xi,\tau) = \lim_{\lambda\to 0} \mathscr{G}(\xi,0;\tau)\,.$$

6. – Continual observation of a finite number of noncommuting quantities.

Let us consider any system with a finite number of degrees of freedom and a set of ordinary observables A_j, $j = 1, 2, \dots, n$, associated with the non-commuting operators $\hat{A}_j$.

Equations (5.2), (5.7), (5.11) and (5.12) suggest a possible generalization of the results of the previous sections to the case of the A_j's.

Let us introduce

$$\mathscr{G}(\boldsymbol{\xi}; \tau) = \exp[\tau \mathscr{K}(\boldsymbol{\xi})], \tag{6.1}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_n)$ and

$$\mathscr{K}(\boldsymbol{\xi})\, \hat{W} = \mathscr{L}\hat{W} - \frac{i}{2} \sum_{j=1}^{n} \xi_j \{\hat{A}_j, \hat{W}\} - \sum_{j=1}^{n} \frac{\xi_j^2}{4\gamma_j} \hat{W}, \tag{6.2}$$

$$\mathscr{L}\hat{W} = -\frac{i}{\hbar}[\hat{H}, \hat{W}] - \frac{1}{4} \sum_{j=1}^{n} \gamma_j [\hat{A}_j, [\hat{A}_j, \hat{W}]]. \tag{6.3}$$

Here $\hat{H}$ is the Hamiltonian of the system and $\gamma_j > 0$.

Note that $\mathscr{G}(\boldsymbol{\xi}; \tau)$, defined by eqs. (6.1)-(6.3), satisfies the equation

$$\mathscr{G}(\boldsymbol{\xi}; \tau_1 + \tau_2) = \mathscr{G}(\boldsymbol{\xi}; \tau_2)\, \mathscr{G}(\boldsymbol{\xi}; \tau_1). \tag{6.4}$$

Then the operator

$$\mathscr{F}(\boldsymbol{x}; \tau) = \left(\frac{\tau}{2\pi}\right)^n \int \mathrm{d}^n \boldsymbol{\xi} \exp\left[i\tau \sum_{j=1}^{n} \xi_j x_j\right] \mathscr{G}(\boldsymbol{\xi}; \tau) \tag{6.5}$$

is the analog of the operators defined in eqs. (4.1) or (5.3) and the quantity

$$\begin{aligned} p(W|t_0; \boldsymbol{x}_1, t_1; \boldsymbol{x}_2, t_2; \dots; \boldsymbol{x}_\nu, t_\nu) &= \\ &= \operatorname{Tr} \mathscr{F}(\boldsymbol{x}_\nu; t_\nu - t_{\nu-1})\, \mathscr{F}(\boldsymbol{x}_{\nu-1}; t_{\nu-1} - t_{\nu-2}) \dots \mathscr{F}(\boldsymbol{x}_1; t_1 - t_0)\, \hat{W}_S(t_0) \end{aligned} \tag{6.6}$$

can be interpreted as the probability density for the time averages of the values of the coarse-grained A_j's.

Naturally, in this case, we have to prove that this probability density is well defined. It must be, obviously, positive and normalized; moreover, in order that the partition of the time interval can be arbitrary, the consistency property (3.24) must hold.

The three mentioned properties are consequences of the analogous ones for the operator $\mathscr{F}(\boldsymbol{x}; \tau)$:

i) $\mathscr{F}(\boldsymbol{x}; \tau)$ is a positive map on $\tau c(\mathfrak{H})$, *i.e.*

$$\hat{W} \geqslant 0 \Rightarrow \mathscr{F}(\boldsymbol{x}; \tau)\, \hat{W} \geqslant 0; \tag{6.7}$$

ii) $\mathscr{F}(\boldsymbol{x};\tau)$ is normalized:

$$\mathrm{Tr}\int \mathrm{d}^n\boldsymbol{x}\,\mathscr{F}(\boldsymbol{x};\tau)\,\hat{W} = \mathrm{Tr}\,\hat{W}\,, \qquad \forall \hat{W}\in \tau c(\mathfrak{H})\,; \tag{6.8}$$

iii) the following consistency property holds:

$$\mathscr{F}(\boldsymbol{x};\tau_1+\tau_2) = \int \mathrm{d}^n\boldsymbol{x}_1 \int \mathrm{d}^n\boldsymbol{x}_2 \prod_{j=1}^n \delta\left(x_j - \frac{\tau_1 x_{j1} + \tau_2 x_{j2}}{\tau_1+\tau_2}\right) \mathscr{F}(\boldsymbol{x}_2;\tau_2)\,\mathscr{F}(\boldsymbol{x}_1;\tau_1)\,. \tag{6.9}$$

The proof of properties ii) and iii) is trivial. Equation (6.8) follows from the fact that the evolution operator

$$\mathscr{G}(t) = \int \mathrm{d}^n\boldsymbol{x}\,\mathscr{F}(\boldsymbol{x};t) = \exp[t\mathscr{L}] \tag{6.10}$$

is a trace-preserving map, as is clear from the structure of $\mathscr{L}$. Equation (6.9) is an immediate consequence of eq. (6.4).

By iteration of eq. (6.9) it is then obvious that it is sufficient to prove the positivity property i) for an infinitesimal time. We insert eqs. (6.1)-(6.3) into (6.5) and expand the exponential in the following way:

$$\begin{aligned}\mathscr{F}(\boldsymbol{x};\varepsilon)\,\hat{W} = {} & \left(\frac{\varepsilon}{2\pi}\right)^n \int \mathrm{d}^n\boldsymbol{\xi}\,\exp\left[-\varepsilon\sum_{j=1}^n \frac{\xi_j^2}{4\gamma_j}\right]\cdot \\ & \cdot\left(\hat{W} - \frac{i\varepsilon}{\hbar}[\hat{H},\hat{W}] - \frac{\varepsilon}{4}\sum_{j=1}^n \gamma_j[\hat{A}_j,[\hat{A}_j,\hat{W}]] + \frac{i\varepsilon}{2}\sum_{j=1}^n \xi_j\{x_j-\hat{A}_j,\hat{W}\} - \right. \\ & \left. -\frac{\varepsilon^2}{8}\sum_{j=1}^n \xi_j^2\{x_j-\hat{A}_j,\{x_j-\hat{A}_j,\hat{W}\}\} + \dots\right).\end{aligned} \tag{6.11}$$

Note that, after integration, the terms with $\varepsilon^2\xi_j^2$ turn out to be of the same order as the terms with ε; this is not the case for the other ε^2-terms. Performing the integrals, we obtain

$$\begin{aligned}\mathscr{F}(\boldsymbol{x};\varepsilon)\,\hat{W} = {} & \left(\frac{\varepsilon}{\pi}\right)^{n/2}\Big(\prod_{j=1}^n \gamma_j\Big)^{\frac12}\left(\hat{W} - \frac{i\varepsilon}{\hbar}[\hat{H},\hat{W}] - \frac{\varepsilon}{4}\sum_{j=1}^n \gamma_j[\hat{A}_j,[\hat{A}_j,\hat{W}]] - \right. \\ & \left. -\frac{\varepsilon}{4}\sum_{j=1}^n \gamma_j\{x_j-\hat{A}_j,\{x_j-\hat{A}_j,\hat{W}\}\} + O(\varepsilon^2)\right) = \\ = {} & \left(\frac{\varepsilon}{\pi}\right)^{n/2}\Big(\prod_{j=1}^n \gamma_j\Big)^{\frac12}\left(\hat{W} - \frac{i\varepsilon}{\hbar}[\hat{H},\hat{W}] - \frac{\varepsilon}{2}\sum_{j=1}^n \gamma_j\{(x_j-\hat{A}_j)^2,\hat{W}\} + O(\varepsilon^2)\right).\end{aligned} \tag{6.12}$$

Now, one could prove directly that $\mathscr{F}(\boldsymbol{x};\varepsilon)$ is positive apart from terms of relative order ε^2. Equivalently, note that eq. (6.12) can be written as

$$(6.13)\qquad \mathscr{F}(\boldsymbol{x};\varepsilon)\,\hat{W}\overset{\varepsilon\to 0}{\simeq}\left(\frac{\varepsilon}{\pi}\right)^{n/2}\left(\prod_{j=1}^{n}\gamma_j\right)^{\frac{1}{2}}\exp\left[-\frac{\varepsilon}{2}\gamma_n(x_n-\hat{A}_n)^2\right]\cdot\ldots$$

$$\ldots\cdot\exp\left[-\frac{\varepsilon}{2}\gamma_1(x_1-\hat{A}_1)^2\right]\exp\left[-\frac{i}{\hbar}\varepsilon\hat{H}\right]\hat{W}\exp\left[\frac{i}{\hbar}\varepsilon\hat{H}\right]\cdot$$

$$\cdot\exp\left[-\frac{\varepsilon}{2}\gamma_1(x_1-\hat{A}_1)^2\right]\ldots\exp\left[-\frac{\varepsilon}{2}\gamma_n(x_n-\hat{A}_n)^2\right].$$

The order chosen for the exponentials on the r.h.s. of eq. (6.13) is unimportant: a change of order changes this expression only for ε^2-terms. As written in eq. (6.13), $\mathscr{F}(\boldsymbol{x};\varepsilon)$ is apparently positive; this closes the proof of the positivity of $\mathscr{F}(\boldsymbol{x};\tau)$.

Note that now we have no explicit expression for the effect related to the simultaneous measurement at a single time of our n observables as in eqs. (1.12)-(1.14) for models I) and II). This effect is implicitly given by

$$(6.14a)\qquad \hat{F}(T)=\int_T \mathrm{d}^n\boldsymbol{x}\,\frac{1}{(2\pi)^n}\int\mathrm{d}^n\boldsymbol{k}\,\hat{g}(\boldsymbol{k})\exp\left[i\sum_{j=1}^{n}k_jx_j\right],$$

where

$$(6.14b)\qquad \hat{g}(\boldsymbol{k})=\exp\left[-\frac{1}{4}\sum_{j=1}^{n}\frac{k_j^2}{\alpha_j}-\frac{i}{2}\sum_{j=1}^{n}k_j\{\hat{A}_j,\ldots\}-\frac{1}{4}\sum_{j=1}^{n}\alpha_j[\hat{A}_j,[\hat{A}_j,\ldots]]\right]\hat{1}\,.$$

Here the normalization $(\hat{F}(\mathbf{R}^n)=\hat{1})$ is trivial; the proof of the positivity is similar to that the positivity of $\mathscr{F}(\boldsymbol{x};\tau)$.

7. – Mean values and correlation functions.

Mean values and correlation functions can be introduced as usual in probabilistic theories and obtained from derivatives of the Fourier transform of the probability densities.

In our case, from

$$(7.1)\qquad \mathscr{G}(\boldsymbol{\xi};\Delta t)=\int\mathrm{d}^n\boldsymbol{x}\,\mathscr{F}(\boldsymbol{x};\Delta t)\exp\left[-i\Delta t\sum_{j=1}^{n}\xi_jx_j\right],$$

we have

$$(7.2)\qquad \int\mathrm{d}^n\boldsymbol{x}\,x_1^{p_1}x_2^{p_2}\ldots x_n^{p_n}\,\mathscr{F}(\boldsymbol{x};\Delta t)=\left(\frac{i}{\Delta t}\right)^{p_1+p_2+\ldots+p_n}\frac{\partial^{p_1+p_2+\ldots+p_n}\mathscr{G}(\boldsymbol{\xi};\Delta t)}{\partial\xi_1^{p_1}\partial\xi_2^{p_2}\ldots\partial\xi_n^{p_n}}\bigg|_{\boldsymbol{\xi}=\boldsymbol{0}}.$$

We recall that derivatives must be calculated from the formula ([15])

$$(7.3) \qquad \frac{\partial}{\partial \xi_j} \exp[\tau \mathscr{K}(\boldsymbol{\xi})] = \int_0^\tau \mathrm{d}\tau' \exp[(\tau - \tau')\mathscr{K}(\boldsymbol{\xi})] \frac{\partial \mathscr{K}(\boldsymbol{\xi})}{\partial \xi_j} \exp[\tau' \mathscr{K}(\boldsymbol{\xi})] .$$

First, let us consider mean values referring to the time interval $(t, t + \Delta t)$

$$(7.4) \qquad \langle x_{j,\Delta t}\rangle_t = \int \mathrm{d}^n \boldsymbol{x}\, x_j \operatorname{Tr}\left(\mathscr{F}(\boldsymbol{x}; \Delta t)\, \mathscr{G}(t)\, \hat{W}\right),$$

where

$$(7.5) \qquad \mathscr{G}(t) = \mathscr{G}(\mathbf{0}; t) = \exp[t\mathscr{L}]$$

is the evolution operator (here and in the following $t_i = t_0 = 0$). Using eqs. (7.2), (6.1) and (6.2), we obtain

$$(7.6) \qquad \langle x_{j,\Delta t}\rangle_t = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathrm{d}t' \operatorname{Tr}\left(\hat{A}_j\, \mathscr{G}(t')\, \hat{W}\right).$$

Note that for mean values the limit $\Delta t \to 0$ exists and gives the usual quantum formula

$$(7.7) \qquad \langle x_{j,\Delta t = 0}\rangle_t = \operatorname{Tr}\left(\hat{A}_j\, \mathscr{G}(t)\, \hat{W}\right).$$

For the time derivative of mean values we have

$$(7.8) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle x_{j,\Delta t}\rangle_t = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathrm{d}t' \operatorname{Tr}\left(\hat{A}_j \frac{\mathrm{d}\mathscr{G}(t')}{\mathrm{d}t'} \hat{W}\right) = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathrm{d}t' \operatorname{Tr}\left(\hat{A}_j\, \mathscr{L}\mathscr{G}(t')\, \hat{W}\right),$$

where $\mathscr{L}$ is given by eq. (6.3). In the case of model II) (sect. **5**) for $\hat{H} = (1/2m)\hat{p}^2 + \hat{V}(q)$, the action of the adjoint of $\mathscr{L}$ on $\hat{q}$ and $\hat{p}$ can be calculated and a generalized Ehrenfest theorem is obtained:

$$(7.9a) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle x_{\Delta t}\rangle_t = \frac{1}{m}\langle p_{\Delta t}\rangle_t ,$$

$$(7.9b) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle p_{\Delta t}\rangle_t = \frac{1}{\Delta t}\int_t^{t+\Delta t} \mathrm{d}t' \operatorname{Tr}\left[\left(-\frac{\mathrm{d}\hat{V}(q)}{\mathrm{d}q}\right) \mathscr{G}(t')\, \hat{W}\right].$$

For the free particle and the harmonic oscillator it turns out that $\langle x_{\Delta t}\rangle_t$ and $\langle p_{\Delta t}\rangle_t$ satisfy exactly the classical equations of motion.

([15]) See, for instance, R. M. WILCOX: *J. Math. Phys.* (*N. Y.*), **8**, 962 (1967).

In a similar way, for the variances we have

$$(7.10)\qquad (\Delta x_{j,\Delta t})_t^2 \equiv \int \mathrm{d}^n \boldsymbol{x}\, (x_j - \langle x_{j,\Delta t}\rangle_t)^2 \mathrm{Tr}\left(\mathscr{F}(\boldsymbol{x};\,\Delta t)\,\mathscr{G}(t)\,\hat{W}\right) = \frac{1}{2\gamma_j \Delta t} +$$

$$+ \frac{1}{(\Delta t)^2}\int_t^{t+\Delta t}\mathrm{d}t_1\int_t^{t_1}\mathrm{d}t_2\,\mathrm{Tr}\left[(\hat{A}_j - \langle x_{j,\Delta t}\rangle_t)\,\mathscr{G}(t_1 - t_2)\{(\hat{A}_j - \langle x_{j,\Delta t}\rangle_t),\,(\mathscr{G}(t_2)\,\hat{W})\}\right].$$

For small Δt, we can write

$$(7.11)\qquad (\Delta x_{j,\Delta t})_t^2 \simeq \frac{1}{2\gamma_j \Delta t} + \mathrm{Tr}\left[(\hat{A}_j - \langle x_{j,\Delta t=0}\rangle_t)^2\,\mathscr{G}(t)\,\hat{W}\right],$$

showing that the actual limit $\Delta t \to 0$ cannot be performed. The second term on the r.h.s. of eq. (7.11) is the usual quantum expression for the squared variance (when the time evolution is given by $\mathscr{G}(t)$). Note that eq. (7.11) can also be written as a kind of uncertainty relation between time resolution and variances:

$$(7.12)\qquad \Delta t\,(\Delta x_{j,\Delta t})_t^2 \geqslant 1/2\gamma_j\,.$$

Finally, for the one-time correlation functions, we have $(j \neq j')$

$$(7.13)\qquad \langle \Delta x_{j,\Delta t},\, \Delta x_{j',\Delta t}\rangle_t \equiv \int \mathrm{d}^n \boldsymbol{x}\,(x_j - \langle x_{j,\Delta t}\rangle_t)(x_{j'} - \langle x_{j',\Delta t}\rangle_t)\cdot$$

$$\cdot\mathrm{Tr}\left(\mathscr{F}(\boldsymbol{x};\Delta t)\,\mathscr{G}(t)\,\hat{W}\right) = \frac{1}{2(\Delta t)^2}\int_t^{t+\Delta t}\mathrm{d}t_1\int_t^{t_1}\mathrm{d}t_2\,\mathrm{Tr}\Big((\hat{A}_j - \langle x_{j,\Delta t}\rangle_t)\,\mathscr{G}(t_1 - t_2)\cdot$$

$$\cdot\{(\hat{A}_{j'} - (\langle x_{j',\Delta t}\rangle_t),\,(\mathscr{G}(t_2)\,\hat{W})\} + (\hat{A}_{j'} - \langle x_{j',\Delta t}\rangle_t)\,\mathscr{G}(t_1 - t_2)\cdot$$

$$\cdot\{(\hat{A}_j - \langle x_{j,\Delta t}\rangle_t),\,(\mathscr{G}(t_2)\,\hat{W})\}\Big)\,.$$

Note that

$$(7.14)\qquad \lim_{\Delta t\to 0}\langle \Delta x_{j,\Delta t},\, \Delta x_{j',\Delta t}\rangle_t = \tfrac{1}{2}\,\mathrm{Tr}\left(\{(\hat{A}_j - \langle x_{j,\Delta t=0}\rangle_t),\,(\hat{A}_{j'} - \langle x_{j',\Delta t=0}\rangle_t)\}\,\mathscr{G}(t)\,\hat{W}\right).$$

In the literature this expression is sometimes called « quantum (symmetrized) correlation function » (see, for instance, ref. [16]).

8. – Continual position measurement on a free particle.

Our models I) and II) are completely solvable when the system is a free particle or a harmonic oscillator. In these cases everything can be explicitly calculated in several ways:

([16]) R. KUBO: *Statistical mechanics of equilibrium and nonequilibrium*, in *Proceedings of the I.U.P.A.P. Symposium, Aachen, 1964*, edited by J. MEIXNER (Amsterdam, 1965), p. 81.

a) the construction given in sect. **3** and **4** can be followed, starting from a discrete set of repeated coarse-grained measurements, by assuming condition (1.15) and taking the limit $\tau \to 0$;

b) path integrations in eqs. (4.4*b*) or (5.9*b*) can be performed directly;

c) the semi-group structure of $\mathscr{G}(\xi;t)$ or $\mathscr{G}(\xi,\eta;t)$ can be exploited.

As an example we work out the case of continual position measurement on a free particle by using the last procedure.

We shall start by solving eq. (4.13*a*) when $\mathscr{G}(\xi;t)$ is applied to a suitable statistical operator $\hat{W}_{\boldsymbol{\lambda}}$. Let us choose this statistical operator to be given in the position representation by

$$\langle q|\hat{W}_{\boldsymbol{\lambda}}|q'\rangle = \psi_{\boldsymbol{\lambda}}(q)\,\psi^*_{\boldsymbol{\lambda}}(q')\exp\left[-\frac{\varkappa_1}{\sigma}(q-q')^2\right], \tag{8.1a}$$

where

$$\psi_{\boldsymbol{\lambda}}(q) = \frac{1}{\sqrt[4]{\pi\sigma}}\exp\left[-\frac{1-i\varkappa_2}{2\sigma}(q-a)^2+\frac{i}{\hbar}pq\right] \tag{8.1b}$$

and

$$0<\sigma<\infty, \quad 0\leqslant\varkappa_1<\infty, \quad a,\ p,\ \varkappa_2 \text{ real}.$$

This operator, for $\varkappa_1 = 0$, becomes a pure state; in this case the meaning of the other parameters is apparent from the relations

$$\left\{\begin{array}{l} \langle\hat{q}\rangle \equiv \mathrm{Tr}(\hat{q}\hat{W}) = a\,, \quad \langle(\hat{q}-a)^2\rangle = \dfrac{\sigma}{2}\,, \\ \langle\hat{p}\rangle = p\,, \quad \langle(\hat{p}-p)^2\rangle = \dfrac{\hbar^2}{2\sigma}(1+\varkappa_2^2)\,; \end{array}\right. \tag{8.2}$$

in particular, $\varkappa_2 = 0$ corresponds to minimal uncertainties.

We will prove that $\mathscr{G}(t)$ transforms the statistical operator (8.1) into an operator of the same kind and, moreover, that $\mathscr{G}(\xi;t)$ transforms it into a trace class operator $\hat{X}_{\boldsymbol{\lambda}}$ having the following slightly more general structure:

$$\langle q|\hat{X}_{\boldsymbol{\lambda}}|q'\rangle = \psi_{\boldsymbol{\lambda}}(q)\,\psi^*_{\boldsymbol{\lambda}^*}(q')\exp\left[-\frac{\varkappa_1}{\sigma}(q-q')^2+\chi\right], \tag{8.3}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^*$ are shorthand notations for $(\sigma, \varkappa_1, \varkappa_2, a, p, \chi)$ and for $(\sigma, \varkappa_1, \varkappa_2, a^*, p^*, \chi^*)$ respectively, a and p are now complex numbers, χ is a new complex parameter, due to the fact that $\mathscr{G}(\xi;t)$ does not conserve the trace.

Precisely, for the operator $\hat{X}_{\boldsymbol{\lambda}}$ one has

$$\mathscr{G}(\xi;t)\hat{X}_{\boldsymbol{\lambda}} = \hat{X}_{\boldsymbol{\lambda}(\xi;t)}, \tag{8.4}$$

where

$$(8.5a)\quad \begin{cases} \sigma(t) = \dfrac{\hbar^2\gamma}{3m^2}t^3 + \dfrac{\hbar^2}{m^2}\dfrac{1+4\varkappa_1}{\sigma}t^2 + \dfrac{1}{\sigma}\left(\dfrac{\hbar}{m}\varkappa_2 t + \sigma\right)^2, \\ \varkappa_1(t) = \varkappa_1 + \dfrac{1}{4}\gamma\displaystyle\int_0^t \mathrm{d}t'\,\sigma(t'), \\ \varkappa_2(t) = \dfrac{m}{2\hbar}\dfrac{\mathrm{d}\sigma(t)}{\mathrm{d}t}, \end{cases}$$

$$(8.5b)\quad \begin{cases} p(\xi,t) = p - \dfrac{i}{4}m\xi\big(\sigma(t)-\sigma\big), \\ a(\xi,t) = a + \dfrac{p}{m}t + \dfrac{i}{4}\xi\left(\sigma t - 3\displaystyle\int_0^t \mathrm{d}t_1\,\sigma(t_1)\right), \\ \chi(\xi,t) = \chi - i\xi\left(at + \dfrac{p}{2m}t^2\right) - \xi^2\left(\dfrac{t}{4\gamma} - \dfrac{\sigma}{8}t^2 + \dfrac{3}{4}\displaystyle\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,\sigma(t_2)\right). \end{cases}$$

The proof of eqs. (8.4), (8.5) is given by the following points.

i) By setting $\hat{X}_\lambda(\xi,t) \equiv \mathscr{G}(\xi,t)\hat{X}_\lambda$, the differential equation (cf. eqs. (4.13a) and (4.18))

$$(8.6)\quad \frac{\mathrm{d}}{\mathrm{d}t}\hat{X}_\lambda(\xi,t) = -\frac{i}{\hbar}\left[\frac{\hat{p}^2}{2m}, \hat{X}_\lambda(\xi,t)\right] - \frac{\gamma}{4}[\hat{q},[\hat{q},\hat{X}_\lambda(\xi,t)]] - \\ -\frac{i}{2}\xi\{\hat{q},\hat{X}_\lambda(\xi,t)\} - \frac{\xi^2}{4\gamma}\hat{X}_\lambda(\xi,t)$$

can be written in terms of matrix elements as

$$(8.7a)\quad \left(\frac{\partial}{\partial t} - \frac{i\hbar}{m}\frac{\partial^2}{\partial Q\,\partial r} + \frac{1}{4}\gamma r^2 + i\xi Q - \frac{\xi^2}{4\gamma}\right)\left\langle Q + \frac{r}{2}\right|\hat{X}_\lambda(\xi,t)\left|Q - \frac{r}{2}\right\rangle = 0,$$

where we have set

$$(8.7b)\quad Q = \tfrac{1}{2}(q+q'), \qquad r = q - q'.$$

ii) $\langle Q + r/2|\hat{X}_\lambda(\xi,t)|Q - r/2\rangle$ satisfies the initial condition

$$(8.8)\quad \left\langle Q + \frac{r}{2}\right|\hat{X}_\lambda(\xi,0)\left|Q - \frac{r}{2}\right\rangle = \\ = \frac{1}{\sqrt{\pi\sigma}}\exp\left[-\frac{1}{\sigma}(Q-a)^2 - \frac{1}{\sigma}\left(\frac{1}{4}+\varkappa_1\right)r^2 + i\frac{\varkappa_2}{\sigma}(Q-a)r + \frac{i}{\hbar}pr + \chi\right].$$

iii) An expression of the type $\langle Q + r/2|\hat{X}_{\lambda(\xi,t)}|Q - r/2\rangle$ is a solution of eq. (8.7*a*) if $\boldsymbol{\lambda}(\xi, t)$ is given by eqs. (8.5). In fact, inserting $\langle Q + r/2|\hat{X}_{\lambda(\xi,t)}|Q - r/2\rangle$ into eq. (8.7*a*), one has

$$(8.9)\quad \left(\frac{\partial}{\partial t} - \frac{i\hbar}{m}\frac{\partial^2}{\partial Q\,\partial r} + \frac{1}{4}\gamma r^2 + i\xi Q - \frac{1}{4\gamma}\xi^2\right)\frac{1}{\sqrt{\pi\sigma(t)}}\cdot$$

$$\cdot\exp\left[-\frac{1}{\sigma(t)}(Q - a(\xi,t))^2 - \frac{1}{\sigma(t)}\left(\frac{1}{4} + \varkappa_1(t)\right)r^2 + i\,\frac{\varkappa_2(t)}{\sigma(t)}(Q - a(\xi,t))r + \right.$$

$$\left. + \frac{i}{\hbar}p(\xi,t)r + \chi(\xi,t)\right] \equiv P(Q,r)\frac{1}{\sqrt{\pi\sigma(t)}}\exp\left[-\frac{1}{\sigma(t)}(Q - a(\xi,t))^2 - \dots\right] = 0\,,$$

where $P(Q, r)$ is a polynomial of 2nd degree in Q and r. Equating to zero the coefficients of such a polynomial, with some manipulations one obtains the following system of differential equations:

$$(8.10)\quad \begin{cases} \dfrac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \dfrac{2\hbar}{m}\varkappa_2(t)\,, \quad \dfrac{\mathrm{d}\varkappa_1(t)}{\mathrm{d}t} = \dfrac{1}{4}\gamma\sigma(t)\,, \quad \sigma(t)\dfrac{\mathrm{d}\varkappa_2(t)}{\mathrm{d}t} = \dfrac{\hbar}{m}\left(1 + 4\varkappa_1(t) + \varkappa_2^2(t)\right), \\[2mm] \dfrac{\partial a(\xi,t)}{\partial t} = \dfrac{1}{m}p(\xi,t) - \dfrac{i}{2}\xi\sigma(t)\,, \quad \dfrac{\partial p(\xi,t)}{\partial t} = -\dfrac{i}{4}m\xi\dfrac{\mathrm{d}\sigma(t)}{\mathrm{d}t}\,, \\[2mm] \dfrac{\partial\chi(\xi,t)}{\partial t} = -i\xi a(\xi,t) - \dfrac{\xi^2}{4\gamma}\,; \end{cases}$$

the three first equations are linked to the 2nd-order terms in $P(Q, r)$, they are independent of ξ and imply for $\sigma(t)$ the equation

$$(8.11)\quad \frac{\mathrm{d}^3\sigma(t)}{\mathrm{d}t^3} = \frac{2\hbar^2}{m}\gamma\,.$$

Starting from this equation and taking into account the initial conditions

$$(8.12)\quad \boldsymbol{\lambda}(\xi, 0) = \boldsymbol{\lambda}\,,$$

one obtains eqs. (8.5*a*) and (8.5*b*).

vi) $\langle Q + r/2|\hat{X}_{\lambda(\xi,t)}|Q - r/2\rangle$ with $\boldsymbol{\lambda}(\xi, t)$ given by eqs. (8.5*a*) and (8.5*b*) satisfies initial condition (8.8).

In conclusion, since eq. (8.6) is of the first order in time, the identity of $\mathscr{G}(\xi, t)\hat{X}_{\lambda}$ and $\hat{X}_{\lambda(\xi,t)}$ is proved.

We note that, in the case $\xi = 0$, eqs. (8.5*b*) become much simpler and, for $\chi = 0$ and a, p real, $\mathscr{G}(t)\hat{W}_{\lambda}$ is a statistical operator of the form $\hat{W}_{\lambda(t)}$ with $\sigma(t)$, $\varkappa_1(t)$, $\varkappa_2(t)$ given by eqs. (8.5*a*) and with

$$(8.13)\quad p(t) = p\,, \quad a(t) = a + \frac{p}{m}t\,, \quad \chi(t) = 0\,.$$

Finally, in order to show that $\hat{X}_{\boldsymbol{\lambda}(\xi,t)}$ is a trace class operator for $t\geqslant 0$, we note that $\sigma(t)>0$ and $\varkappa_1(t)\geqslant 0$; then, taking into account the Fourier representation

$$(8.14)\qquad \exp\left[-\frac{\varkappa_1(t)}{\sigma(t)}(q-q')^2\right]=\frac{1}{2}\sqrt{\frac{\sigma(t)}{\pi\varkappa_1(t)}}\int_{-\infty}^{+\infty}\mathrm{d}k\,\exp\left[-ik(q-q')-\frac{\sigma(t)}{4\varkappa_1(t)}k^2\right],$$

we can write

$$(8.15a)\quad \hat{X}_{\boldsymbol{\lambda}(\xi,t)}=\frac{1}{2}\sqrt{\frac{\sigma(t)}{\pi\varkappa_1(t)}}\int_{-\infty}^{+\infty}\mathrm{d}k\,|\boldsymbol{\lambda}(\xi,t),k\rangle\exp\left[-\frac{\sigma(t)}{4\varkappa_1(t)}k^2+\chi(\xi,t)\right]\langle\boldsymbol{\lambda}^*(\xi,t),k|,$$

where

$$(8.15b)\qquad \langle q|\boldsymbol{\lambda}(\xi,t),k\rangle=\psi_{\boldsymbol{\lambda}(\xi,t)}(q)\exp[-ikq].$$

Then, for the trace class norm $\|\hat{X}_{\boldsymbol{\lambda}(\xi,t)}\|_1=\mathrm{Tr}\,(\hat{X}_{\boldsymbol{\lambda}(\xi,t)}\hat{X}^\dagger_{\boldsymbol{\lambda}(\xi,t)})^{\frac{1}{2}}$, we obtain the inequality

$$(8.16)\quad \|\hat{X}_{\boldsymbol{\lambda}(\xi,t)}\|_1\leqslant|\exp[\chi(\xi,t)]|\,\frac{1}{2}\sqrt{\frac{\sigma(t)}{\pi\varkappa_1(t)}}\int_{-\infty}^{+\infty}\mathrm{d}k\,\exp\left[-\frac{\sigma(t)}{4\varkappa_1(t)}k^2\right]=|\exp[\chi(\xi,t)]|,$$

where the trivial inequality $\||\varphi\rangle\langle\psi|\|_1\leqslant\||\varphi\rangle\|\cdot\||\psi\rangle\|$ has been used.

Now we are in a position to explicitly calculate the «matrix elements» $G(\xi,t|q,q';q_0,q_0')$ of the operator $\mathscr{G}(\xi,t)$ (cf. eq. (4.16a)) or, more conveniently, changing the variables,

$$(8.17)\qquad \bar{G}(\xi,t|Q,r;Q_0,r_0)=G(\xi,t|Q+r/2,Q-r/2;Q_0+r_0/2,Q_0-r_0/2).$$

In fact, let us write eq. (8.4) for $\varkappa_1=\varkappa_2=\chi=0$, $\sigma=\sigma_0$,

$$(8.18)\qquad \int\mathrm{d}Q_0\,\mathrm{d}r_0\,\bar{G}(\xi,t|Q,r;Q_0,r_0)\frac{1}{\sqrt{\pi\sigma_0}}\cdot$$
$$\cdot\exp\left[-\frac{1}{\sigma_0}(Q_0-a)^2-\frac{1}{4\sigma_0}r_0^2+\frac{i}{\hbar}pr_0\right]=\left\langle Q+\frac{r}{2}\,|\hat{X}_{\boldsymbol{\lambda}(\xi,t)}|\,Q-\frac{r}{2}\right\rangle,$$

or, equivalently,

$$(8.19)\qquad \int\mathrm{d}Q_0\,\bar{G}(\xi,t|Q,r;Q_0,r_0)\frac{1}{\sqrt{\pi\sigma_0}}\exp\left[-\frac{1}{\sigma_0}(Q_0-a)^2\right]=$$
$$=\frac{1}{\sqrt{2\pi\hbar}}\exp\left[\frac{1}{4\sigma_0}r_0^2\right]\int_{-\infty}^{+\infty}\mathrm{d}p\,\left\langle Q+\frac{r}{2}\,|\hat{X}_{\boldsymbol{\lambda}(\xi,t)}|\,Q-\frac{r}{2}\right\rangle\exp\left[-\frac{i}{\hbar}pr_0\right].$$

Taking in eq. (8.19) the limit $\sigma_0 \to 0$, one obtains

$$\bar{G}(\xi, t|Q, r; a, r_0) = \lim_{\sigma_0 \to 0} \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{1}{4\sigma_0} r_0^2\right] \cdot \tag{8.20}$$

$$\cdot \int_{-\infty}^{+\infty} dp \left\langle Q + \frac{r}{2} |\hat{X}_{\lambda(\xi,t)}| Q - \frac{r}{2} \right\rangle \exp\left[-\frac{i}{\hbar} p r_0\right] =$$

$$= \lim_{\sigma_0 \to 0} \sqrt{\frac{1}{2\pi^2 \hbar \sigma(t)}} \int_{-\infty}^{+\infty} dp \exp\left[\frac{1}{4\sigma_0} r_0^2 - \frac{i}{\hbar} p r_0 - \frac{1}{\sigma(t)} (Q - a(\xi, t))^2 - \right.$$

$$\left. - \frac{1}{\sigma(t)} \left(\frac{1}{4} + \varkappa_1(t)\right) r^2 + i \frac{\varkappa_2(t)}{\sigma(t)} (Q - a(\xi, t)) r + \frac{i}{\hbar} p(\xi, t) r + \chi(\xi, t)\right].$$

Since by eqs. (8.5*b*) the functions $a(\xi, t)$, $p(\xi, t)$ and $\chi(\xi, t)$ depend linearly on p, the integral over p can be performed easily; the subsequent limit $\sigma_0 \to 0$ requires some calculations. The final result is

$$\bar{G}(\xi, t|Q, r; Q_0, r_0) = \frac{m}{2\pi\hbar t} \exp\left[-\frac{1}{12} \gamma t (r^2 + r r_0 + r_0^2) + \right. \tag{8.21}$$

$$\left. + \frac{im}{\hbar t} (Q - Q_0)(r - r_0) - \frac{i}{2} \xi t \left[Q + Q_0 - \frac{i\gamma\hbar^2}{24m} t^2 (r + r_0)\right] - \frac{1}{4} \xi^2 t^2 \left[\frac{1}{\gamma t} + \frac{\hbar^2 \gamma}{120 m^2} t^3\right]\right].$$

The probability densities (4.3) can be obtained by applying to $\hat{W}$ products of the Fourier transform with respect to ξ of the operator defined by (8.21). The calculation becomes rather simple if one chooses a statistical operator of the type (8.1). Let us give some results for this case. We set for simplicity $t_0 = 0$ and choose $\hat{W} = \hat{W}_{\lambda_0}$.

First we consider the probability density reduced to only one of the time intervals into which $(t_0 = 0, t_\nu)$ is partitioned, *i.e.* one integrates $p(W|t_0; x_1, t_1; x_2, t_2; \dots; x_\nu, t_\nu)$ over all variables $x_1, x_2, \dots, x_\nu$, except the variable x_r referring to the selected interval (t_{r-1}, t_r). Taking into account the consistency property, one has

$$\int dx_1 \dots dx_{r-1} dx_{r+1} \dots dx_\nu \, p(W|t_0; x_1, t_1; x_2, t_2; \dots; x_\nu, t_\nu) = \tag{8.22}$$

$$= \mathrm{Tr}\, \mathscr{F}(x_r; t_r - r_{r-1}) \mathscr{G}(t_{r-1}) \hat{W}_{\lambda_0}.$$

Setting $t_{r-1} = t$, $t_r - t_{r-1} = \Delta t$ and $x_l = x$, one has

$$\mathrm{Tr}\, \mathscr{F}(x; \Delta t) \mathscr{G}(t) \hat{W}_{\lambda_0} = \frac{\Delta t}{2\pi} \int_{-\infty}^{+\infty} d\xi \exp[i \Delta t \xi x] \,\mathrm{Tr}\, \mathscr{G}(\xi; \Delta t) \mathscr{G}(0; t) \hat{W}_{\lambda_0}. \tag{8.23}$$

The parameters $\boldsymbol{\lambda}(t)$ of the statistical operator $\hat{W}_{\boldsymbol{\lambda}(t)} = \mathscr{G}(0; t)\,\hat{W}_{\boldsymbol{\lambda}_0}$ are given by eqs. (8.5*a*) and (8.13):

$$\left\{\begin{aligned} &\sigma(t) = \frac{\hbar^2\gamma}{3m^2}\,t^3 + \frac{\hbar^2}{m^2}\,\frac{1+4\varkappa_1^{(0)}}{\sigma_0}\,t^2 + \frac{1}{\sigma_0}\left(\frac{\hbar}{m}\,\varkappa_2^{(0)}\,t + \sigma_0\right)^2,\\ &\varkappa_1(t) = \varkappa_1^{(0)} + \frac{1}{4}\,\gamma\int_0^t \mathrm{d}t'\,\sigma(t'), \qquad \varkappa_2(t) = \frac{m}{2\hbar}\,\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t},\\ &p(t) = p_0, \qquad a(t) = a_0 + \frac{p_0}{m}\,t, \qquad \chi(t) = 0. \end{aligned}\right. \tag{8.24}$$

The parameters of $\mathscr{G}(\xi, \Delta t)\,\mathscr{G}(0, t)\,\hat{W}_{\boldsymbol{\lambda}_0}$ are given again by eqs. (8.5), if we set $t = \Delta t$ and take as initial values $\boldsymbol{\lambda}$ the $\boldsymbol{\lambda}(t)$ just calculated; the only parameter we need is

$$\chi(\xi, \Delta t) = -\,i\xi\left[\left(a_0 + \frac{p_0}{m}\,t\right)\Delta t + \frac{p_0}{2m}\,\Delta t^2\right] - \frac{\xi^2}{2}\,\tilde{\chi}_t(\Delta t), \tag{8.25}$$

where

$$\tilde{\chi}_t(\Delta t) = \frac{\Delta t}{2\gamma} + \frac{3}{2}\int_0^{\Delta t}\mathrm{d}t'\int_0^{t'}\mathrm{d}t''\,\sigma(t+t'') - \frac{1}{4}\,\sigma(t)\,\Delta t^2 \tag{8.26}$$

with $\sigma(t)$ given by eq. (8.24).

Then (8.23) becomes

$$\begin{aligned}\mathrm{Tr}\,\mathscr{F}(x; \Delta t)\,\mathscr{G}(t)\,\hat{W}_{\boldsymbol{\lambda}_0} &= \frac{\Delta t}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\exp[i\,\Delta t\,\xi x + \chi(\xi, \Delta t)] = \\ &= \sqrt{\frac{\Delta t^2}{2\pi\tilde{\chi}_t(\Delta t)}}\,\exp\left[-\frac{\Delta t^2}{2\tilde{\chi}_t(\Delta t)}\left[x - a_0 - \frac{p_0}{m}\left(t + \frac{1}{2}\,\Delta t\right)\right]^2\right];\end{aligned} \tag{8.27}$$

this is a Gaussian probability distribution with maximum at the classical time average

$$a_0 + \frac{p_0}{m}\left(t + \frac{1}{2}\,\Delta t\right) = \frac{1}{\Delta t}\int_t^{t+\Delta t}\mathrm{d}t'\,x_{\mathrm{cl}}(t'), \qquad x_{\mathrm{cl}}(t) = a_0 + \frac{p_0}{m}\,t, \tag{8.28}$$

and with squared variance given by $\tilde{\chi}_t(\Delta t)/\Delta t^2$.

If $\Delta t \ll t$, one can replace in eq. (8.26) $\sigma(t + t'')$ by $\sigma(t)$; then one has

$$(\Delta x)^2 \simeq \frac{1}{2\gamma\,\Delta t} + \frac{1}{2}\,\sigma(t). \tag{8.29}$$

In (8.29) the peculiar term $1/2\gamma\Delta t$ occurs that has already been discussed in sect. **7**.

Now let us consider the probability density reduced to two time intervals $(t_1, t_1+\Delta t_1)$ and $(t_2, t_2+\Delta t_2)$ with $t_2 \geqslant t_1+\Delta t_1$. It is given by

$$\text{(8.30)}\qquad \operatorname{Tr}\mathscr{F}(x_2;\Delta t_2)\,\mathscr{G}(t_2-t_1-\Delta t_1)\,\mathscr{F}(x_1;\Delta t_1)\,\mathscr{G}(t_1)\,\hat{W}_{\lambda_0} = \frac{\Delta t_1\Delta t_2}{(2\pi)^2}\int_{-\infty}^{+\infty}\!\mathrm{d}\xi_1\,\mathrm{d}\xi_2\exp[i(\Delta t_1\xi_1x_1+\Delta t_2\xi_2x_2)+\chi(\xi_1,\xi_2;\Delta t_1,\Delta t_2)]\,,$$

x_1, x_2 have the meaning of time averages, respectively, over $(t_1, t_1+\Delta t_1)$ and $(t_2, t_2+\Delta t_2)$,

$$\text{(8.31)}\qquad \chi(\xi_1,\xi_2;\Delta t_1,\Delta t_2) = -i\xi_1\Delta t_1\left[a_0+\frac{p_0}{m}\left(t_1+\frac{1}{2}\Delta t_1\right)\right]- -i\xi_2\Delta t_2\left[a_0+\frac{p_0}{m}\left(t_2+\frac{1}{2}\Delta t_2\right)\right]-\xi_1\xi_2\tilde{\chi}_{t_1,t_2}(\Delta t_1,\Delta t_2)-\frac{\xi_1^2}{2}\tilde{\chi}_{t_1}(\Delta t_1)-\frac{\xi_2^2}{2}\tilde{\chi}_{t_2}(\Delta t_2)\,,$$

where

$$\text{(8.32)}\qquad \tilde{\chi}_{t_1,t_2}(\Delta t_1,\Delta t_2) = \frac{\Delta t_1\Delta t_2}{4}\left[\frac{3}{\Delta t_1}\int_0^{\Delta t_1}\!\mathrm{d}t'\,\sigma(t_1+t')-\sigma(t_1)+\frac{\sigma(t_1+\Delta t_1)-\sigma(t_1)}{\Delta t_1}\left(t_2+\frac{1}{2}\Delta t_2-t_1-\Delta t_1\right)\right]$$

and $\tilde{\chi}_{t_1}(\Delta t_1)$, $\tilde{\chi}_{t_2}(\Delta t_2)$ are given by eq. (8.26).

The probability distribution (8.30) is a Gaussian with maximum at

$$\text{(8.33)}\qquad x_1 = a_0+\frac{p_0}{m}\left(t_1+\frac{1}{2}\Delta t_1\right),\qquad x_2 = a_0+\frac{p_0}{m}\left(t_2+\frac{1}{2}\Delta t_2\right)$$

and with correlation functions

$$\text{(8.34)}\qquad \langle x_1,x_1\rangle = \frac{\tilde{\chi}_{t_1}(\Delta t_1)}{\Delta t_1^2}\,,\quad \langle x_2,x_2\rangle = \frac{\tilde{\chi}_{t_2}(\Delta t_2)}{\Delta t_2^2}\,,\quad \langle x_1,x_2\rangle = \frac{\tilde{\chi}_{t_1,t_2}(\Delta t_1,\Delta t_2)}{\Delta t_1\Delta t_2}\,.$$

If $\Delta t_1 \ll t_1$, $\Delta t_2 \ll t_2$, the correlation functions become

$$\text{(8.35)}\qquad \begin{cases} \langle x_1,x_1\rangle \simeq \dfrac{1}{2\gamma\Delta t_1}+\dfrac{1}{2}\sigma(t_1)\,, & \langle x_2,x_2\rangle \simeq \dfrac{1}{2\gamma\Delta t_2}+\dfrac{1}{2}\sigma(t_2)\,,\\[2ex] \langle x_1,x_2\rangle \simeq \dfrac{1}{2}\sigma(t_1)+\dfrac{\hbar}{2m}(t_2-t_1)\,\varkappa_2(t_1)\,. & \end{cases}$$

Finally we quote some results in the case of model II) for the harmonic oscillator; obviously, the corresponding results for the free particle can be obtained by setting the frequency $\omega/2\pi$ equal to zero. Even in this case an equation similar to eq. (8.4) can be established; the parameters $\boldsymbol{\lambda}(\xi, \eta; t)$, corresponding to those of eqs. (8.5), are given by

$$\sigma(t) = \frac{\gamma\hbar^2}{4m^2\omega^3}(2\omega t - \sin 2\omega t) + \frac{\gamma\lambda\hbar^2}{4\omega}(2\omega t + \sin 2\omega t) + \frac{\hbar^2}{2m^2\omega^2}\frac{1+4\varkappa_1}{\sigma}(1-\cos 2\omega t) + \frac{1}{\sigma}\left(\frac{\hbar}{m\omega}\varkappa_2 \sin\omega t + \sigma\cos\omega t\right)^2, \tag{8.36a}$$

$$\varkappa_2(t) = \frac{m}{2\hbar}\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} - \frac{1}{2}\gamma\lambda\hbar m, \tag{8.36b}$$

$$\varkappa_1(t) = \varkappa_1 + \frac{\hbar^2\gamma^2}{16m^2\omega^4}[1 + m^2\omega^2(1+m^2\omega^2\lambda)](\omega^2t^2 - \sin^2\omega t) + \frac{\hbar^2\gamma^2\lambda}{8\omega^2}\sin^2\omega t + \frac{\hbar^2\gamma\lambda}{16\omega}\frac{1+4\varkappa_1}{\sigma}(2\omega t + \sin 2\omega t) + \frac{\hbar^2\gamma}{16m^2\omega^3}\frac{1+4\varkappa_1}{\sigma}(2\omega t - \sin 2\omega t) + \frac{\gamma\lambda m^2\omega^2}{4\sigma}\int_0^t \mathrm{d}t'\left(\frac{\hbar}{m\omega}\varkappa_2\cos\omega t' - \sigma\sin\omega t'\right)^2 + \frac{\gamma}{4\sigma}\int_0^t \mathrm{d}t'\left(\frac{\hbar}{m\omega}\varkappa_2\sin\omega t' + \sigma\cos\omega t'^2\right), \tag{8.36c}$$

$$a(\xi,\eta;t) = a\cos\omega t + \frac{p}{m\omega}\sin\omega t + \frac{i}{2\omega^2}\xi\left\{\frac{1}{2}\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} - \frac{\hbar^2\gamma}{m^2\omega^2}\left(1+\frac{1}{2}m^2\omega^2\lambda\right) + \frac{\hbar}{m}\left(\frac{\hbar\gamma}{m\omega^2} - \varkappa_2\right)\cos\omega t - \frac{\hbar^2}{m^2\omega\sigma}(1+4\varkappa_1+\varkappa_2^2)\sin\omega t\right\} + \frac{i}{2}\eta\left\{m\sigma\cos\omega t - m\sigma(t) + \frac{\hbar}{\omega}(\hbar m\lambda\gamma + \varkappa_2)\sin\omega t\right\}, \tag{8.37a}$$

$$p(\xi,\eta;t) = p\cos\omega t - am\omega\sin\omega t + \frac{i}{2}\xi\cdot\left\{\frac{m}{2\omega^2}\frac{\mathrm{d}^2\sigma(t)}{\mathrm{d}t^2} - \frac{\hbar}{\omega}\left(\frac{\hbar\gamma}{m\omega^2} - \varkappa_2\right)\sin\omega t - \frac{\hbar^2}{m\omega^2\sigma}(1+4\varkappa_1+\varkappa_2^2)\cos\omega t + m\sigma(t)\right\} + i\eta\frac{m}{2}\left\{-m\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} - m\omega\sigma\sin\omega t + \hbar(\hbar m\gamma\lambda + \varkappa_2)\cos\omega t + \hbar\varkappa_2(t)\right\}, \tag{8.37b}$$

$$\chi(\xi,\eta;t) = \chi - i\xi\left(a\frac{\sin\omega t}{\omega} + \frac{p}{m}\frac{1-\cos\omega t}{\omega^2}\right) - i\eta\left[p\frac{\sin\omega t}{\omega} - ma(1-\cos\omega t)\right] - \frac{\xi^2}{2}\left[\frac{t}{2\gamma} - \frac{\sigma(t)-\sigma}{2\omega^2} + \frac{\hbar^2}{m^2\omega^4}\gamma\left(1+\frac{1}{2}m^2\omega^2\lambda\right)t - \frac{\hbar}{m\omega^3}\left(\frac{\hbar\gamma\lambda}{m\omega^2} - \varkappa_2\right)\sin\omega t + \frac{\hbar^2}{m^2\omega^4\sigma}(1+4\varkappa_1+\varkappa_2^2)(1-\cos\omega t)\right] - \tag{8.37c}$$

$$-\frac{\eta^2}{2}\left[\frac{t}{2\gamma\lambda}+\frac{1}{2}\hbar^2 m^2\gamma\lambda t+\frac{1}{2}m^2(\sigma(t)+\sigma)-m^2\sigma\cos\omega t-\frac{\hbar m}{\omega}(\hbar m\lambda\gamma+\varkappa_2)\sin\omega t\right]-$$

$$-\xi\eta\left[-m\sigma\frac{\sin\omega t}{2\omega}-\frac{m}{4\omega^2}\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t}+\frac{\hbar^2\gamma}{2m\omega^4}(1-\cos\omega t)+\frac{\hbar}{\omega^2}\left(\frac{1}{2}\hbar m\lambda\gamma+\varkappa_2\right)\cdot\right.$$

$$\left.\cdot\left(\cos\omega t-\frac{1}{2}\right)+\frac{\hbar^2}{2m\omega^3\sigma}(1+4\varkappa_1+\varkappa_2^2)\sin\omega t\right].$$

Now we choose as before $\hat{W}=\hat{W}_{\lambda_0}$. The probability density, referring to the time interval $(t, t+\Delta t)$, can be read off from $\chi(\xi,\eta;t)$. We again obtain a Gaussian with the maximum at the classical time averages

$$(8.38a)\qquad x=a(t)\frac{\sin\omega\Delta t}{\omega\Delta t}+\frac{1}{m}p(t)\frac{1-\cos\omega\Delta t}{\Delta t\,\omega^2}\equiv\frac{1}{\Delta t}\int_t^{t+\Delta t}\mathrm{d}t'\,a(t')\,,$$

$$(8.38b)\qquad p=p(t)\frac{\sin\omega t}{\omega\Delta t}-ma(t)\frac{1-\cos\omega\Delta t}{\Delta t}\equiv\frac{1}{\Delta t}\int_t^{t+\Delta t}\mathrm{d}t'\,p(t')\,,$$

where

$$a(t)=a_0\cos\omega t+\frac{p_0}{m\omega}\sin\omega t\,,\qquad p(t)=p_0\cos\omega t-m\omega a_0\sin\omega t\,.$$

Similarly the one-time correlation functions $\langle x,x\rangle$, $\langle p,p\rangle$, $\langle x,p\rangle$ are obtained from the coefficients of the terms $-\xi^2/2$, $-\eta^2/2$, $-\xi\eta$ in expression (8.37c) of $\chi(\xi,\eta;t)$, through the following steps: i) t is replaced by Δt; ii) σ, $\varkappa_1$, $\varkappa_2$, are put equal to $\sigma(t)$, $\varkappa_1(t)$, $\varkappa_2(t)$, where these functions are given again by eqs. (8.36), in which now σ, $\varkappa_1$, $\varkappa_2$ are replaced by σ_0, $\varkappa_1^{(0)}$, $\varkappa_2^{(0)}$; iii) one must multiply by $1/\Delta t^2$. The result is

$$(8.39a)\qquad \langle x,x\rangle=\frac{\gamma\hbar^2}{m^2\omega^5\Delta t^2}\left(\frac{3}{4}\omega\Delta t-\sin\omega\Delta t+\frac{1}{8}\sin 2\omega\Delta t\right)+$$

$$+\frac{\gamma\lambda\hbar^2}{4\omega^3\Delta t^2}\left(\omega\Delta t-\frac{1}{2}\sin 2\omega\Delta t\right)+\frac{\hbar^2}{2m^2\omega^4}(1-\cos\omega\Delta t)^2\frac{1+4\varkappa_1(t)}{\sigma(t)}+$$

$$+\frac{1}{2\sigma(t)\Delta t^2}\left[\frac{\sigma(t)}{\omega}\sin\omega\Delta t-\frac{\hbar}{m\omega^2}(1-\cos\omega\Delta t)\varkappa_2(t)\right]^2+\frac{1}{2\gamma\Delta t}\overset{\Delta t\to 0}{\simeq}\frac{1}{2}\sigma(t)+\frac{1}{2\gamma\Delta t},$$

$$(8.39b)\qquad \langle p,p\rangle=m^2\omega^2\left[\frac{\gamma\hbar^2}{4m^2\omega^5\Delta t^2}\left(\omega\Delta t-\frac{1}{2}\sin 2\omega\Delta t\right)+\right.$$

$$+\frac{\gamma\lambda\hbar^2}{\omega^3\Delta t^2}\left(\frac{3}{4}\omega\Delta t-\sin\omega\Delta t+\frac{1}{8}\sin 2\omega\Delta t\right)+\frac{\hbar^2\sin^2\omega\Delta t}{2m^2\omega^4\Delta t^2}\frac{1+4\varkappa_1(t)}{\sigma(t)}+$$

$$\left.+\frac{1}{2\sigma(t)\Delta t^2}\left(\frac{1-\cos\omega\Delta t}{\omega\Delta t}\sigma(t)-\frac{\hbar}{m\omega^2}\varkappa_2(t)\sin\omega\Delta t\right)^2\right]+$$

$$+\frac{1}{2\gamma\lambda\Delta t}\overset{\Delta t\to 0}{\simeq}\frac{\hbar^2}{2\sigma(t)}(1+4\varkappa_1(t)+\varkappa_2^2(t))+\frac{1}{2\gamma\lambda\Delta t},$$

$$\langle x, p\rangle = \left[\frac{\gamma\hbar^2}{4m\omega^2}(1-\lambda m^2\omega^2)(1-\cos\omega\Delta t) + \varkappa_2(t)\hbar\cos\omega\Delta t - \right. \tag{8.39c}$$

$$\left. -\frac{1}{2}m\omega\sigma(t)\sin\omega\Delta t + \frac{\hbar^2}{2m\omega\sigma(t)}\left(1+4\varkappa_1(t)+\varkappa_2^2(t)\right)\sin\omega\Delta t\right]\frac{1-\cos\omega\Delta t}{\omega^2\Delta t^2} \overset{\Delta t\to 0}{\simeq} \frac{\hbar}{2}\varkappa_2(t).$$

The uncertainty principle takes the form

$$\sqrt{\langle x, x\rangle\langle p, p\rangle} \overset{\Delta t\to 0}{\simeq} \frac{\hbar}{2}\left(1+\frac{1}{\gamma\sigma(t)\Delta t}\right)^{\frac{1}{2}}\left(1+4\varkappa_1(t)+\varkappa_2^2(t)+\frac{\sigma(t)}{\gamma\lambda\Delta t}\right)^{\frac{1}{2}}. \tag{8.40}$$

Note also the very simple results that are obtained if one chooses $\Delta t = 2\pi/\omega$:

$$\begin{cases} \langle x, x\rangle^{\frac{1}{2}} = \hbar\left(\dfrac{3\gamma}{8\pi m^2\omega^3}+\dfrac{\gamma\lambda}{8\pi\omega}+\dfrac{\omega}{4\pi\gamma\hbar^2}\right)^{\frac{1}{2}}, \\ \langle x, p\rangle = 0\,, \\ \langle p, p\rangle^{\frac{1}{2}} = \hbar m\omega\left(\dfrac{\gamma}{8\pi m^2\omega^3}+\dfrac{3\gamma\lambda}{8\pi\omega}+\dfrac{1}{4\gamma\lambda\pi\hbar^2 m^2\omega}\right)^{\frac{1}{2}}. \end{cases} \tag{8.41}$$

9. – Concluding remarks.

In conclusion, we want to stress once more some few points.

i) In the probability densities for the time averages described by eqs. (1.21), (3.10), (5.2), (6.6) the time intervals Δt can be taken as small as one likes, reflecting the fact that they derive from functional probability densities. As is apparent from eq. (7.12), however, the dispersion of the time averages diverges as $\Delta t \to 0$ with a pattern typical of the so-called generalized stochastic processes.

ii) There exists a certain connection between our joint probability distribution for noncommuting quantities (eqs. (6.6), (6.5), (6.1)-(6.3)) and the (generalized) Wigner quasi-probability. In fact, our probability density for time-averaged observables in the interval $(t, t+\Delta t)$, which is given (when $t_0 = 0$) by

$$p(W(t)|\boldsymbol{x};\Delta t) = \mathrm{Tr}\{\mathscr{F}(\boldsymbol{x};\Delta t)\hat{W}(t)\}, \tag{9.1a}$$

where

$$\hat{W}(t) = \mathscr{G}(t)\hat{W}, \tag{9.1b}$$

can be written as

$$p(W(t)|\boldsymbol{x};\Delta t) = \left(\frac{\Delta t}{\pi}\right)^{n/2}\sqrt{\prod_{j=1}^{n}\gamma_i}\int \mathrm{d}^n\boldsymbol{y}\,\exp\left[-\Delta t\sum_{j=1}^{n}\gamma_j y_j^2\right]\tilde{p}(W(t)|\boldsymbol{x}-\boldsymbol{y};\Delta t), \tag{9.2}$$

where

$$(9.3)\qquad \tilde{p}(W(t)|\boldsymbol{x};\Delta t)=\left(\frac{1}{2\pi}\right)^n\int \mathrm{d}^n\boldsymbol{k}\exp\left[i\sum_{j=1}^n k_j x_j\right]\cdot \\ \cdot\operatorname{Tr}\left\{\exp\left[\Delta t\mathscr{L}-\frac{i}{2}\sum_{j=1}^n k_j\{\hat{A}_j,\cdot\}\right]\hat{W}(t)\right\}.$$

The function $\tilde{p}(W(t)|\boldsymbol{x};\Delta t)$ is something like a time-smeared Wigner quasi-probability density; more precisely, we have

$$(9.4)\qquad \tilde{p}(W(t)|\boldsymbol{x};0)=\tilde{p}_{\mathrm{Wig}}(W(t)|\boldsymbol{x})\equiv \\ \equiv\left(\frac{1}{2\pi}\right)^n\int \mathrm{d}^n\boldsymbol{k}\operatorname{Tr}\left\{\exp\left[-i\sum_{j=1}^n k_j(\hat{A}_j-x_j)\right]\hat{W}(t)\right\}.$$

$\tilde{p}_{\mathrm{Wig}}(W(t)|\boldsymbol{x})$ is the density of the Wigner quasi-probability when the statistical operator is $\hat{W}(t)$.

Note that $\tilde{p}(W|\boldsymbol{x};\Delta t)$ and $\tilde{p}_{\mathrm{Wig}}(W|\boldsymbol{x})$ are not positive functions of $\boldsymbol{x}$ for every statistical operator $\hat{W}$, in contrast with our probability density (9.1*a*). Since the convolution with the Gaussian in eq. (9.2) comes from the ξ_j^2-terms in eq. (6.2), the essential role of these terms is apparent. For related results, see ref. ([5]), subsect. **5**.5, and references quoted therein.

iii) The discussion of sect. **6** shows that the method we have employed for the construction of our probability distribution for trajectories is fairly general and significant extensions can be foreseen and are actually in progress.

iv) If our discussion is intended to provide a model for a fundamental theory of the connection between the macroscopic and the quantum level of description of the physical world, some form of « energy conservation » would be desirable. However, as is apparent from the expression of $\mathscr{L}$ as given by eqs. (1.26), (5.12*b*) or (6.3), the « microscopic » energy $\hat{H}$ is not conserved under the time evolution described by the operator $\mathscr{G}(t)$. Nevertheless, as discussed elsewhere by two of us ([17]), at least in certain cases, a conserved modified energy can be defined as a generalized observable.

More coherently, a « coarse-grained » energy should be introduced in the theory as a linear combination of the quantities A_j (see sect. **6**), but, in general, the probability distribution for this « coarse-grained » energy too is not constant in time. A detailed discussion about this point seems to be out of place here; our feeling is that the problem of energy conservation could be dealt with and possibly resolved only inside more realistic models, based on field theory.

([17]) A. BARCHIELLI and L. LANZ: *Nuovo Cimento B*, **44**, 241 (1978).

* * *

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APPENDIX A

Let us prove that the effect-valued measure defined by (1.13) (model II)) satisfies the equation

$$\hat{F}(\mathbf{R}^2) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}p \exp\left[\frac{i}{\hbar}(p\hat{q} - x\hat{p})\right]\hat{\varrho}\exp\left[-\frac{i}{\hbar}(p\hat{q} - x\hat{p})\right] = 1\,. \tag{A.1}$$

From the well-known identity

$$\exp[\hat{A} + \hat{B}] = \exp[\hat{A}]\cdot\exp[\hat{B}]\cdot\exp[-\tfrac{1}{2}[\hat{A},\hat{B}]]\,, \tag{A.2}$$

which holds true if $[\hat{A},\hat{B}]$ is a c-number ([15]), we can write

$$\exp\left[\frac{i}{\hbar}(p\hat{q} - x\hat{p})\right] = \exp\left[\frac{i}{\hbar}p\hat{q}\right]\exp\left[-\frac{i}{\hbar}x\hat{p}\right]\exp\left[-\frac{i}{2\hbar}px\right]. \tag{A.3}$$

Denoting by $|q\rangle$ the eigenstate of $\hat{q}$ and by $|k\rangle$ the eigenstates of $\hat{p}$, we have then

$$\langle q|\hat{F}(\mathbf{R}^2)|q'\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}p\,\langle q|\exp\left[\frac{i}{\hbar}p\hat{q}\right]\cdot \tag{A.4}$$

$$\cdot\exp\left[-\frac{i}{\hbar}x\hat{p}\right]\hat{\varrho}\exp\left[\frac{i}{\hbar}x\hat{p}\right]\exp\left[-\frac{i}{\hbar}p\hat{q}\right]|q'\rangle =$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}p\exp\left[\frac{i}{\hbar}p(q-q')\right]\langle q|\exp\left[-\frac{i}{\hbar}x\hat{p}\right]\hat{\varrho}\exp\left[\frac{i}{\hbar}x\hat{p}\right]|q'\rangle =$$

$$= \delta(q-q')\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}k\int_{-\infty}^{+\infty}\mathrm{d}k'\langle q|\exp\left[-\frac{i}{\hbar}x\hat{p}\right]|k\rangle\langle k|\hat{\varrho}|k'\rangle\langle k'|\exp\left[\frac{i}{\hbar}x\hat{p}\right]|q'\rangle =$$

$$= \delta(q-q')\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}k\int_{-\infty}^{+\infty}\mathrm{d}k'\frac{1}{\sqrt{2\pi\hbar}}\exp\left[\frac{i}{\hbar}(q-x)k\right]\cdot$$

$$\cdot\langle k|\hat{\varrho}|k'\rangle\frac{1}{\sqrt{2\pi\hbar}}\exp\left[\frac{i}{\hbar}(x-q')k'\right] = \delta(q-q')\int_{-\infty}^{+\infty}\mathrm{d}k\,\langle k|\hat{\varrho}|k\rangle = \delta(q-q')\,.$$

APPENDIX B

For a pure state $\hat{W} = |\psi\rangle\langle\psi|$ and in the Schrödinger picture eq. (2.8) takes the form

$$(B.1)\qquad |\psi_1\rangle = \frac{\hat{F}^{\frac{1}{2}}|\psi_S(t_1)\rangle}{\|\hat{F}^{\frac{1}{2}}|\psi_S(t_1)\rangle\|}, \qquad |\psi_s(t)\rangle = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]|\psi\rangle .$$

Equation (B.1) is a generalization of the usual postulate (cf. eq. (2.3)) on the collapse of the wave function as a consequence of a positive answer to a yes-no experiment related to an effect rather than to a projection.

In order to better understand the assumptions implied in eq. (B.1), let us consider an ordinary observable A associated with the self-adjoint operator

$$(B.2)\qquad \hat{A} = \sum_r |\alpha_r\rangle \alpha_r \langle\alpha_r|$$

and a related effect of the form

$$(B.3)\qquad \hat{F} = \sum_r |\alpha_r\rangle f_r \langle\alpha_r| , \qquad 0 \leqslant f_r \leqslant 1 .$$

With obvious notations we have

$$(B.4)\qquad P(\psi|A = \alpha_r, t_1) = |\langle\alpha_r|\psi_S(t_1)\rangle|^2$$

and

$$(B.5)\qquad P(\psi|F = \text{yes},\ t_1) = \langle\psi_S(t_1)|\hat{F}\psi_S(t_1)\rangle = \|\hat{F}^{\frac{1}{2}}|\psi_S(t_1)\rangle\|^2 = \sum_r f_r|\langle\alpha_r|\psi_S(t_1)\rangle|^2$$

and, in particular,

$$(B.6)\qquad P(|\alpha_r\rangle|F = \text{yes},\ 0) = f_r .$$

According to eq. (B.6) the quantity f_r can be interpreted as the efficiency of the device associated with the effect F in detecting the system if it is prepared in the state $|\alpha_r\rangle$. Correspondingly, if, at the time t_1, before we perform a *first-kind measurement* of the quantity A and *immediately later* we allow the device associated with F to act, the probability of obtaining an outcome $A = \alpha_r$ and $F = \text{yes}$ is given by

$$(B.7)\qquad P(\psi|A = \alpha_r, t_1)\cdot P(|\alpha_r\rangle|F = \text{yes},\ 0) = |\langle\alpha_r|\psi_S(t_1)\rangle|^2 f_r .$$

Equation (B.1) states that the same result is obtained if the two devices corresponding to A and F act in the reversed order

$$(B.8)\qquad P(\psi|F = \text{yes},\ t_1)\, P(\psi_1|A = \alpha_r, 0) = |\langle\alpha_r|\psi_S(t_1)\rangle|^2 f_r$$

and that the modification on the state vector of the system is the minimum one consistent with such properties (the phases of the eigenstates $|\alpha_r\rangle$ in the expansion of $|\psi_S(t_1)\rangle$ and $|\psi_1\rangle \propto \hat{F}^{\frac{1}{2}}|\psi_S(t_1)\rangle$ are the same).

A more general prescription which could be a substitute for eq. (B.1) is

$$\hat{W}_1 = \frac{\sum_r \hat{\Gamma}_r \hat{W}_S(t_1)\,\hat{\Gamma}_r^\dagger}{\operatorname{Tr}\left(\sum_r \hat{\Gamma}_r \hat{W}_S(t_1)\,\hat{\Gamma}_r^\dagger\right)}, \tag{B.9}$$

where $\hat{\Gamma}_1, \hat{\Gamma}_2, \ldots$ is a sequence of bounded operators which decompose $\hat{F}$:

$$\hat{F} = \sum_r \hat{\Gamma}_r^\dagger \hat{\Gamma}_r\,. \tag{B.10}$$

Obviously, in general, the mapping defined by (B.9), which is called an *operation*, changes pure states into mixtures.

An equation of the form (B.9) is obtained if the effect $\hat{F}$ is defined as the result of the interaction of the system with an apparatus and the observation of an effect $\hat{F}'$ on the apparatus [2,5,18]. The specific sequence $\hat{\Gamma}_1, \hat{\Gamma}_2, \ldots$ is then a consequence of the nature of the apparatus, its interaction with the system and the choice of the effect $\hat{F}'$.

Obviously for a given $\hat{F}$ a decomposition of the form (B.10) is not unique corresponding to the fact that many different devices can be used to detect the *same* effect F.

Equation (B.1) is a particular case of eq. (B.9) and corresponds to a highly idealized « device » which induces the minimum possible disturbance on the system. In a certain sense we can say that eq. (B.1) or (2.8) refers to a *first-kind* observation of an effect.

([18]) K. KRAUS: in *Lecture Notes in Physics*, Vol. **29** (Berlin, 1973), p. 206.

● RIASSUNTO

Partendo dal concetto di osservabile generalizzata, associata a una misura a valori di effetto, come formulato da Ludwig, si discutono alcuni esempi di osservazione continuata in meccanica quantistica. Usando il formalismo dell'integrale di Feynman, si costruisce una distribuzione di probabilità funzionale sull'insieme delle traiettorie che rappresentano i possibili risultati dell'osservazione continuata. Si mettono inoltre in evidenza interessanti connessioni con la teoria dei semigruppi dinamici. Gli esempi si riferiscono a sistemi con pochi gradi di libertà; il loro interesse è tuttavia sopratutto in ordine alla luce che possono gettare sul problema del rapporto tra il livello di descrizione quanto-meccanica e di descrizione macroscopica di un corpo grande, per la quale ultima il concetto di traiettoria continua sembra essere essenziale.

Модель для макроскопического описания и непрерывных измерений в квантовой механике.

Резюме (*). — Исходя из идеи обобщенных наблюдаемых величин, связанных с измерением эффективных значений, как было сформулировано Людвигом, обсуждаются некоторые примеры непрерывных измерений в квантовой механике. Используя фейнмановский формализм интегралов, конструируется распределение функциональной вероятности на системе траектории, которые представляют возможные результаты непрерывных измерений. Отмечаются интересные связи с теорией динамических полугрупп. Примеры относятся к малым системам, но они являются интересными, т.к. могут пролить свет на проблему связи между квантовым и макроскопическим уровнями описания больших тел; для которых идея непрерывных траекторий, по-видимому, является существенной для макроскопического уровня описания.

(*) *Переведено редакцией.*