Infinite-Dimensional Algebra of General Covariance Group as the Closure of Finite-Dimensional Algebras of Conformal and Linear Groups.

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(ricevuto il 25 Settembre 1973)

The Einstein theory of general relativity rests on the group of general co-ordinate transformations (or general covariance group)

$$x_i = f_i(x_1, x_2, x_3, x_4)\,, \tag{1}$$

where $f_i(x)$ are arbitrary functions of the co-ordinates $x_1, x_2, x_3, x_4 = ict$. The general covariance group is an infinite-parameter group. The aim of the present paper is to call attention to the simple and remarkable fact that the action of the general covariance group can be reduced to alternating actions of its two finite-parameter subgroups: the special linear group $SL_{4,R}$ and the conformal group C_{15}. Both the linear and conformal groups act on the same manifold, that of co-ordinates, but these groups do not commute with each other. Below we prove that the algebra of the general covariance group (1) turns out to be the closure of algebras of the linear and conformal groups, *i.e.* that any generator of (1) is representable as some linear combination of repeated commutators of generators of $SL_{4,R}$ and C_{15}. In this way, the transformation properties (invariance, in particular) of any quantity under the action of the infinite-dimensional general covariance group are determined by its transformation properties under that of the essentially simpler finite-dimensional groups $SL_{4,R}$ and C_{15}. Some perspectives of this new approach to the general covariance group will be sketched in the concluding remarks.

Now we proceed to prove the main statement. To that end we expand the functions $f_i(x)$ (1) into infinite series in powers of co-ordinates. Coefficients of the series serve as parameters of the general covariance group, and its generators can be written as follows:

$${}^{n}L_k^{n_1,n_2,n_3,n_4} = -ix_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}\,\partial_k \qquad (n = n_1+n_2+n_3+n_4\,,\ \partial_k = \partial/\partial x_k)\,. \tag{2}$$

The group $SL_{4,R}$ is formed by all linear transformations $x_i' = a_{ik}x_k$ with the determinant equal to unity, and its generators are

$$SR_{ik} = -i\left[x_i\partial_k - \tfrac{1}{4}\,\delta_{ik}(x\partial)\right]. \tag{3}$$

A set of generators of the conformal group C_{15}, as is known, consists of those of four-dimensional rotations (also entering into (3) as a subalgebra), translations P_i, dilatations D and special conformal transformations K_i:

(4a) $$P_i = -i\partial_i\,,$$

(4b) $$D = -i(x\partial)\,,$$

(4c) $$K_i = -i(x^2\partial_i - 2x_i(x\partial))\,.$$

Consider the closure of the linear- and conformal-group algebras, *i.e.* such minimal algebra which would contain all commutators of the generators (3) and (4) and their linear combinations. We will prove that this algebra coincides with that of the general covariance group (2), *i.e.* the following theorem is valid:

Any generator of the general covariance group $^nL_k^{n_1,n_2,n_3,n_4}$ *is representable as some linear combination of the commutators of generators of the special linear and conformal groups.*

The proof is carried out by mathematical induction. The translation generators (4a) give all generators $^nL_k^{\cdots}$ with $n=0$. The generators of $SL_{4,R}$ together with those of dilatations (4b) constitute the algebra $L_{4,R}$ with generators

(5) $$R_{ik} = -ix_i\partial_k$$

and give all $^nL_k^{\cdots}$ with $n=1$. Generators of the special conformal transformations K_i (4c) are quadratic in x. Let us show that all generators $^nL_k^{\cdots}$ with $n=2$ are contained in the closing algebra. Consider the commutator of the dilatation along the m-th axis R_{mm} with K_p, $m\neq p$,

(6) $$[R_{mm}, K_p] = -2x_m^2\partial_p\,.$$

So, we have the generator $-ix_m^2\partial_p$. Further,

(6′) $$[R_{pm}, -ix_m^2\partial_p] = -2x_mx_p\partial_p + x_m^2\partial_m\,.$$

Comparing eqs. (6) and (6′) with K_m (4c) itself we conclude that the closing algebra includes the generators $-ix_mx_p\partial_p$ $(m\neq p)$ and $-ix_m^2\partial_m$, as well. Finally, the generator $-ix_mx_n\partial_p$ $(m\neq n\neq p)$ arises from the commutator

(6″) $$[R_{mp}, -ix_nx_p\partial_p] = -x_mx_n\partial_p\,.$$

Hence, all the generators $^nL_k^{\cdots}$ with $n=2$ are exhausted. Commuting the generators quadratic in x with each other, we arrive at those cubic in x:

(7a) $$[-ix_m^2\partial_n, -ix_n^2\partial_n] = -2x_m^2x_n\partial_n\,,$$

(7b) $$[-ix_mx_n\partial_n, -ix_n^2\partial_n] = -x_mx_n^2\partial_n\,,$$

i.e. we have found the generators $-ix_m^2x_n\partial_n$, $-ix_mx_n^2\partial_n$ $(m\neq n)$. The commutator of the latter generator with R_{nm} (5) is the following:

(8a) $$[R_{nm}, -ix_mx_n^2\partial_n] = -x_n^3\partial_n + x_n^2x_m\partial_m\,.$$

and with (7a) taken into account we obtain the generator of further importance

$$(8b)\qquad -ix_n^3\partial_n .$$

Other generators of third power in x will not be required for our proof by induction. We have shown above that the theorem holds for $n=0, 1, 2$. Suppose that the one is valid for some n, *i.e.* all generators ${}^nL_k^{n_1,n_2,n_3,n_4}$ can be represented as linear combinations of repeated commutators of the generators of linear and conformal groups. Prove that then the theorem will be valid for $n+1$, as well.

All the differentiation indices k of ${}^nL_k^{n_1,n_2,n_3,n_4}$ are on the same status, so it suffices to consider $k=1$. Then, if $n_1\neq 0$, $n_1\neq 3$,

$$(9)\qquad {}^{n+1}L_1^{n_1,n_2,n_3,n_4}=\frac{i}{n_1-3}\left[-ix_1^2\partial_1,\ {}^nL_1^{n_1-1,n_2,n_3,n_4}\right].$$

Consider now the cases $n_1=0$ and $n_1=3$. In these cases, if at least one ot n_2, n_3, n_4 is larger than zero, for instance $n_2>0$, then we have

$$(10)\qquad {}^{n+1}L_1^{n_1,n_2,n_3,n_4}=\frac{i}{n_1-1}\left[-ix_1x_2\partial_1,\ {}^nL_1^{n_1,n_2-1,n_3,n_4}\right].$$

Finally, for $n_1=3$, $n_2=n_3=n_4=0$ the corresponding generator is given by (8). So, the theorem is proven.

One can see the validity of this theorem for spaces of any dimension, the generators of the general covariance group being representable as linear combinations of repeated commutators of generators of the corresponding special linear and conformal groups.

Elsewhere we will show that the Einstein equations of general relativity follow from the invariance under the conformal and linear groups, and this is quite natural within the framework of the approach developed. In deep analogy with the fact that pions are connected with nonlinear realizations of the dynamical chiral $SU_2\times SU_2$ symmetry (see, *e.g.*, ref. (1)), gravity field proves to be connected with common nonlinear realizations of the dynamical conformal and affine symmetries (*).

Note also, that the presented approach raises hopes that unitary representations for the infinite-dimensional algebra of the general covariance group can be constructed.

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The author thanks sincerely F. A. BEREZIN, A. B. BORISOV, D. V. VOLKOV and V. TYBOR for useful discussions.

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(1) S. WEINBERG: *Proceedings of the 1970 Brandeis Summer Institute in Theoretical Physics*, edited by S. DESER (Cambridge, 1970), p. 287.

(*) The nonlinear realizations of space-time symmetries are discussed in ref. (2-4).

(2) D. V. VOLKOV: *Particles and Nuclei*, **4**, 3 (1973).

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