Relativistic Reference Systems and Motion of Test Bodies in the Vicinity of the Earth.

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Summary. — Relativistic theory of constructing nonrotating harmonic reference systems (RS) is developed. The theory enables one to produce the celestial RS for solar-system dynamics neglecting the gravitational field of the Galaxy. Particular attention is focused on the barycentric RS (BRS) with the origin at the solar-system barycentre and the geocentric RS (GRS) with the origin at the geocentre. It is assumed therewith that the velocities of bodies are small as compared with the light velocity and the gravitational field is weak everywhere. The specific RS and the gravitational field are described by the metric tensor to be found by Newtonian approximations from the Einstein field equations with given boundary conditions. The BRS coordinates cover all the solar-system space. The GRS coordinates are initially restricted in space by the orbit of the Moon. The relationship between BRS and GRS is established by the asymptotic matching technique. The explicit transformation formulae permit to prolonge the GRS coordinates beyond the lunar orbit to cover actually all the solar-system space. The GRS equations of the Earth satellite motion have been deduced. The relativistic right-hand members of these equations contain Schwarzschild, Lense-Thirring and quadrupole terrestrial perturbations as well as tidal perturbations due to the Sun, the Moon and the major planets. The equations are derived by two different techniques. The first one implies the application of the geodesic principle to the GRS metric. The second one is based on the transformation of the BRS satellite equations of motion into the GRS equations. Both techniques result in the same final expressions.

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1. – Introduction.

Until recently in the relativistic treatment of the Earth satellite motion one took into account as usual only the Schwarzschild and Lense-Thirring perturbations caused by the spherically symmetrical component of the gravitational field of the Earth and its axial rotation, respectively. These perturbations are presented for a variety of satellite orbits for example in (1). At present, consideration should be given to the more refined relativistic effects due to the Earth oblateness (2) and the influence of the Sun and the Moon.

In investigating the solar and lunar action in the framework of the general relativity theory (GRT) the choice of the reference system (RS) becomes of particular importance. Equations of motion of the solar-system bodies in the barycentric RS (BRS) and barycentric metric of the total solar system gravitational field being well known (3-5), the BRS equations of the Earth satellite motion may be formulated without any difficulties. Subtracting the BRS equations of motion of the Earth one gets the equations of the satellite motion in terms of the relative (formally geocentric) coordinates being actually the differences of the BRS coordinates of the satellite and the Earth. Such equations have been employed for example in (6-8). The advantage of these equations is that both the satellite coordinates and the coordinates of the external disturbing bodies are expressed as a function of the same argument, the barycentric dynamical time (TDB). But the main relativistic solar terms occurring in the right-hand members of these equations are of the order $\varepsilon \sim c^{-2}GM_\odot/R \sim 10^{-8}$ with respect to the principal Newtonian geopotential term proportional to $GM_\oplus/r^2$ (R and r being the Earth heliocentric distance and the satellite geocentric distance, respectively).

The relativistic perturbations of the order ε are due to the terms depending on the Earth orbital BRS velocity as well as on the external mass gravitational potentials and their first derivatives. These perturbations have nothing to do with dynamics of the satellite geocentric motion. They are caused by the «bad»

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(4) C. M. Will: in *Experimental Gravitation*, edited by B. Bertotti (Academic Press, New York, N.Y., 1974), p. 1.
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(conflicting with the principle of equivalence) choice of the formally geocentric coordinates in the curved space-time and are fictitious as giving no contribution to the actual evaluation of the satellite geocentric distance and velocity. This is due to the fact that the equations of the light propagation in the formally geocentric coordinates also contain large relativistic terms of the order ε cancelling out the corresponding terms of the equations of the satellite motion. The appearance of the fictitious relativistic perturbations makes the discussion of satellite observations more complicated and may deteriorate the accuracy of the measurable parameters of the Earth gravitational field.

The adequate description of the satellite geocentric motion is achieved in the «good» geocentric RS (GRS). Its origin moves also along the world line of the centre of mass of the Earth but the relativistic influence of the external masses (Sun, Moon, planets) manifests itself in the appropriate equations of motion only in form of the tidal terms proportional to the second derivatives of the gravitational-field potentials. In other words, the construction of the «good» GRS should be compatible with the principle of equivalence.

Ashby and Bertotti (9) were the first to underline the merits of the «good» GRS for the representation of the satellite motion. Satellite dynamical perturbations in this RS enable one to evaluate immediately the correct order of magnitude of the actually measurable effects facilitating the comparison of theoretical and observational data. This was explicitly demonstrated in (10) for the Moon and in (11) for Earth satellites.

In the subsequent paper Ashby and Bertotti (12) (see also (13)) succeeded to construct a «good» GRS introducing the generalized Fermi normal coordinates for the vicinity of the massive self-gravitating Earth. Along with the advantages this approach does not deprive of shortcomings. First of all, this concerns the use of the background space-time metric not representing the solution of the Einstein field equations. Among other things the technique of the generalized Fermi normal coordinates seems to be elaborated so far only for the spherically symmetrical nonrotating masses. The application of this technique to the real Earth whose rotation and oblateness cannot be ignored in the astronomical practice remains not so clear. This approach is discussed also in (14-16).

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Another approach to construct a «good» RS has been proposed in ([15-17]). This approach is based on using 1) post-Newtonian approximations ([3,14,18-23]), 2) multiple expansion formalism for the gravitational fields ([24-28]), 3) asymptotic matching technique for the gravitational-field potentials ([14,29-32]).

Post-Newtonian approximations (PNA) and multipole formalism are employed by us for solving the Einstein equations and constructing BRS and a «good» GRS. The transformation formulae relating these RS are deduced by means of the asymptotic matching technique. Our approach may be directly used in many actual celestial mechanics and astrometry problems dealing with weak gravitating and slow moving bodies possessing arbitrary shape, internal structure and velocity distribution.

The aim of the present paper is to derive the relativistic equations of the satellite motion in the «good» nonrotating harmonic GRS. The principles of constructing such GRS were exposed by us in ([15,16]). The list of designations is given in sect. 2. Section 3 deals with the statement of the problem and the methods used for solving the Einstein equations in harmonic coordinates. Construction of GRS within accuracy needed for the problem in question is performed in sect. 4. The solar-system BRS metric is established in sect. 5. Matching the BRS and GRS metric tensor components is accomplished in sect. 6. The Earth satellite equations of motion are derived in sect. 7 by applying the geodesic principle to the GRS metric. Another approach to obtain the same

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equations of motion is exposed in sect. 8. This approach involves the transformation of the BRS equations of motion into the GRS equations using the matching relations of sect. 6.

The GRS satellite equations of motion obtained here were briefly outlined earlier in (33).

2. – Designations.

Greek indices $\alpha, \beta, \ldots$ take values from 0 to 3, Latin indices $i, k, \ldots$ run values from 1 to 3. Capital Latin letters $A, B, \ldots$ specify the solar-system bodies. Each repeated index implies summation over appropriate values.

The BRS coordinates are denoted as $x^\alpha = (ct, x^i)$. The GRS coordinates are $w^\alpha = (cu, w^i)$. The triplets of spatial components will be denoted as $\boldsymbol{x} = (x^1, x^2, x^3)$ or $\boldsymbol{w} = (w^1, w^2, w^3)$. δ_{ik} is the Kronecker symbol; $\eta_{\alpha\beta} = \eta^{\alpha\beta}$ is the Minkowski tensor ($\eta_{00} = 1$, $\eta_{0i} = 0$, $\eta_{ik} = -\delta_{ik}$); $g_{\alpha\beta}$ is the metric tensor of the curved space-time with $g = \det(g_{\alpha\beta})$; ε_{ikm} is the antisymmetric Levi-Civita symbol ($\varepsilon_{123} = +1$); G is the gravitational constant; c is the light velocity. Greek indices of the tensor components are raised and lowered with the metric tensor $g_{\alpha\beta}$. In expanding tensor quantities in series in powers of small parameters the Greek and Latin indices of any term of the series are raised and lowered with $\eta_{\alpha\beta}$ and δ_{ik}, respectively.

The functions depending only on the GRS coordinates will be marked by ^. Sometimes, to avoid misunderstanding the GRS indices will be also supplied with this symbol.

Comma after the sign of a function with the subsequent Greek or Latin index designates the partial derivative with respect to the corresponding variable. Comma followed by the zero index denotes the partial derivative with respect to the time coordinate (without factor c^{-1}). The dot denotes the total derivative with respect to the time coordinate of the corresponding RS. Thus, unless otherwise specified, the differentiation is meant to be related with the coordinates occurring in the expression of a function in question. For example, for any function $f(t, \boldsymbol{x})$ in BRS one has

$$f_{,\alpha} \equiv \frac{\partial f}{\partial x^\alpha}, \quad f_{,i} \equiv \frac{\partial f}{\partial x^i}, \quad f_{,0} \equiv \frac{\partial f}{\partial t}, \quad \dot{f} \equiv f_{,0} + \dot{x}^k f_{,k}.$$

The similar relations are valid for any function $\hat{g}(u, \boldsymbol{w})$ in GRS

$$\hat{g}_{,\alpha} \equiv \frac{\partial \hat{g}}{\partial w^\alpha}, \quad \hat{g}_{,i} \equiv \frac{\partial \hat{g}}{\partial w^i}, \quad \hat{g}_{,0} \equiv \frac{\partial \hat{g}}{\partial u}, \quad \dot{\hat{g}} \equiv \hat{g}_{,0} + \dot{w}^{\hat{k}} \hat{g}_{,\hat{k}}.$$

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In what follows we shall often have to do with the functions and their derivatives defined along the world line of the geocentre $x^i_E(t)$. We shall assume for such functions that the spatial coordinates of the geocentre are always substituted after performing differentiation.

3. – Problem statement and methods of constructing BRS and GRS.

Assuming that the action of the gravitational field of the Galaxy on dynamics of the solar-system bodies may be ignored we shall consider the solar system as isolated. In the problem in question an Earth satellite may be regarded as a test body moving in the space-time on the geodesic line.

The matter of the attracting masses will be described by means of the energy-momentum tensor of the perfect fluid

$$T^{\alpha\beta} = [\rho(c^2 + \Pi) + p]\, u^\alpha u^\beta - p g^{\alpha\beta} \tag{3.1}$$

with ρ, p, Π and u^α being respectively mass density, pressure, internal energy specific density and four-velocity of the fluid element. The first three quantities are scalars. The pressure and the specific energy are related with the mass density by the equation of state and the first law of thermodynamics.

Reference system and gravitational field are simultaneously described in GRT by the metric tensor $g_{\alpha\beta}$ to be found as the solution of the Einstein field equations. Owing to the Bianchi identities one may impose on the metric tensor components four arbitrary coordinate conditions. We suggest to adopt the harmonic (de Donder) conditions

$$(\sqrt{-g}\, g^{\alpha\beta})_{,\beta} = 0\,. \tag{3.2}$$

These conditions do not fix RS in a unique manner. There remains the arbitrariness of not reducing to the group of Poincaré transformations. In particular, the spatial axes of the harmonic RS may rotate with an arbitrary (but sufficiently small) angular velocity [27] and its origin may move along an arbitrary timelike world line possessing the small first curvature (acceleration) [15,16,27].

The harmonic RS may be uniquely fixed by choosing 1) the world line of the RS origin, 2) the rotation velocity of the spatial axes, 3) the canonical form of the metric tensor [24,26-28]. The specific realization of this procedure is performed in the present paper.

The gravitational field may be conveniently described in terms of the variables $\gamma^{\alpha\beta} = \eta^{\alpha\beta} - \sqrt{-g}\, g^{\alpha\beta}$. Then the Einstein equations in harmonic coordinates take the well-known form [20,34]

$$\eta^{\mu\nu}\gamma^{\alpha\beta}{}_{,\mu\nu} = \frac{16\pi G}{c^4}(-g)(T^{\alpha\beta}+t^{\alpha\beta}) + \chi^{\alpha\beta\mu\nu}{}_{,\mu\nu}, \tag{3.3}$$

$$\chi^{\alpha\beta\mu\nu} \equiv \gamma^{\alpha\beta}\gamma^{\mu\nu} - \gamma^{\alpha\mu}\gamma^{\beta\nu}, \tag{3.4}$$

where $t^{\alpha\beta}$ is the Landau-Lifshitz pseudotensor.

The Earth is assumed to have in GRS the stationary rigid-body rotation with the angular velocity $\hat{\omega}^i_E$. Its external gravitational field is described by the multipole moments characterizing the nonsphericity deviations of its shape and the internal distribution of matter. The Sun, the Moon and the planets will be regarded as the spherically symmetrical nonrotating bodies in heliocentric, selenocentric and planetocentric RS, respectively. These RS may be constructed just as GRS.

Denote the total mass of the solar-system bodies as M, the mass of any planet B as M_B, its mean radius as L_B, its characteristic heliocentric distance as R_B, the minimal separation of body B from its nearest-neighbouring body as D_B and its BRS velocity as v_B. The problem under consideration involves the small parameters as follows: 1) $\varepsilon \sim v_B/c \ll 1$, the slowness of the orbital motion; 2) $\eta \sim (c^{-2}GM/R_B)^{1/2} \ll 1$, the weakness of the gravitational field everywhere outside the bodies; 3) $\eta_B \sim (c^{-2}GM_B/L_B)^{1/2} \ll 1$, the weakness of the gravitational field inside the bodies; 4) $\delta_B \sim L_B/D_B \ll 1$, the quasi-point structure of any body; 5) $\alpha_E \sim 1/300$, the Earth oblateness. The virial relation implies that $\varepsilon \sim \eta$. It should be noted that η and η_B satisfy the relation $\eta \sim \eta_B\delta_B(M/M_B)^{1/2}(D_B/R_B)^{1/2}$. Besides this, one has $\hat{\omega}_E L_E \ll c$ for the Earth and $v_S \ll c$, v_S being the satellite GRS velocity.

The metric tensor $g_{\alpha\beta}$ may be found by solving eqs. (3.3) with the supplementary boundary conditions on $\gamma^{\alpha\beta}$ enabling to specify the harmonic RS [15,16]. We are particularly interested in two systems, BRS and GRS.

The absence of the spatial axis rotation of these RS is meant here in dynamical sense involving that there are no Coriolis and centrifugal forces of inertia in the equations of motion of test particles in these RS [35,36]. For BRS describing the isolated solar system the dynamical and kinematical rotations are

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equivalent and BRS is nonrotating in kinematical sense as well. Remind ([35,36]) that celestial RS is called kinematically nonrotating if its spatial axes preserve constant directions toward the external distant astronomical objects considered as fixed by definition.

In general case, dynamical and kinematical rotations of RS are not equivalent. An illustrative example is furnished by GRS. GRS is attached to the Earth that cannot be regarded as isolated from the gravitational action of Sun, Moon and planets. Hence, away from the Earth the space-time does not pass into the asymptotic flat space. This results in rotation (relativistic precession) of the GRS spatial axes with respect to the BRS axes related with the «fixed» stars. But the GRS equations of satellite motion do not contain Coriolis and centrifugal forces of inertia caused by the relativistic precession. This means that GRS is nonrotating in dynamical sense. The explicit evidences of this fact will be given in sect. 7 and 8.

In the problem in question BRS $(ct, \boldsymbol{x})$ is constructed by the post-Newtonian approximations in the near zone of the solar system. The near zone is restricted by the domain whose radius does not exceed the minimal length of the gravitational waves radiated by the solar system. According to our assumptions this minimal length belongs to the gravitational waves generated by the Earth-Moon system. Proceeding from this it is easy to evaluate the near-zone radius as exceeding the radius of the Pluto orbit by about 100 times.

The BRS metric tensor is determined from the Einstein equations (3.3) with the energy-momentum tensor of all solar-system bodies. Solving eqs. (3.3) is achieved by iterations relative to small parameters ε, η, η_B with the complementary boundary condition prescribing to match the functions $\gamma^{\alpha\beta}(t, \boldsymbol{x})$ on the near-zone boundary with the outgoing solar system gravitational radiation. The BRS metric tensor is invariant under Lorentz transformations permitting to consider BRS as the generalization of the inertial RS of Newtonian mechanics.

GRS $(cu, \boldsymbol{w})$ is constructed in the vicinity of the world line of the geocentre. It covers the spatial domain surrounding the Earth and lying inside the lunar orbit. The spatial sizes of GRS are confined so far to this limit. This is necessary for matching BRS and GRS to deduce the transformations between them. As will be demonstrated below the spatial GRS axes may be actually prolonged beyond the Pluto orbit. The functions $\hat{\gamma}^{\alpha\beta}(u, \boldsymbol{w})$ are determined in GRS from eqs. (3.3) by post-Newtonian approximations and multipole formalism for the external gravitational-field expansions. The right-hand members of eqs. (3.3) include now only the energy-momentum tensor due to the Earth. The iterations are performed therewith with respect to the small parameters r/D_E, η_E, η and ε. The boundary condition to be imposed on $\hat{\gamma}^{\alpha\beta}(u, \boldsymbol{w})$ in GRS demands that away from the geocentre these functions be matched with the functions representing the tidal gravitational action of Sun, Moon and planets. In this sense GRS generalizes the quasi-inertial RS of Newtonian mechanics.

4. – Geocentric reference system.

The GRS metric tensor has the form [15,16]

$$\mathrm{d}s^2 = \hat{g}_{\alpha\beta}(u, \boldsymbol{w})\,\mathrm{d}w^\alpha \mathrm{d}w^\beta , \tag{4.1}$$

$$\hat{g}_{00}(u, \boldsymbol{w}) = 1 + c^{-2} \underset{(2)}{\hat{g}}_{00}(u, \boldsymbol{w}) + c^{-4} \underset{(4)}{\hat{g}}_{00}(u, \boldsymbol{w}) + O(c^{-5}) , \tag{4.2}$$

$$\hat{g}_{0i}(u, \boldsymbol{w}) = c^{-3} \underset{(3)}{\hat{g}}_{0i} + O(c^{-5}) , \tag{4.3}$$

$$\hat{g}_{ik}(u, \boldsymbol{w}) = -\delta_{ik} + c^{-2} \underset{(2)}{\hat{g}}_{ik} + O(c^{-4}) \tag{4.4}$$

with

$$\underset{(2)}{\hat{g}}_{00}(u, \boldsymbol{w}) = -2\hat{U}_{\mathrm{E}}(u, \boldsymbol{w}) - 2Q_i^{(\mathrm{E})} w^i - 3Q_{ik}^{(\mathrm{E})} w^i w^k - 5Q_{ikm}^{(\mathrm{E})} w^i w^k w^m + O(\hat{r}^4) , \tag{4.5}$$

$$\underset{(2)}{\hat{g}}_{ik}(u, \boldsymbol{w}) = \delta_{ik} \underset{(2)}{\hat{g}}_{00}(u, \boldsymbol{w}) , \tag{4.6}$$

$$\underset{(3)}{\hat{g}}_{0i}(u, \boldsymbol{w}) = 4\hat{U}_{\mathrm{E}}^i(u, \boldsymbol{w}) + 4\varepsilon_{ikm} C_{kn}^{(\mathrm{E})} w^m w^n + O(\hat{r}^3) , \tag{4.7}$$

$$\underset{(4)}{\hat{g}}_{00}(u, \boldsymbol{w}) = 2\hat{U}_{\mathrm{E}}^2(u, \boldsymbol{w}) + 2\hat{U}_{\mathrm{E}}(u, \boldsymbol{w}) Q_i^{(\mathrm{E})} w^i + 6\hat{U}_{\mathrm{E}}(u, \boldsymbol{w}) Q_{ik}^{(\mathrm{E})} w^i w^k + O(\hat{U}_{\mathrm{E}} \hat{r}^3) , \tag{4.8}$$

$$\hat{U}_{\mathrm{E}}(u, \boldsymbol{w}) = G \int_{(\mathrm{E})} \frac{\rho^*(u, \boldsymbol{w}')}{|\boldsymbol{w} - \boldsymbol{w}'|} \mathrm{d}^3 w' + O(\eta_{\mathrm{E}}^2) , \tag{4.9}$$

$$\hat{U}_{\mathrm{E}}^i(u, \boldsymbol{w}) = G \int_{(\mathrm{E})} \frac{\rho^*(u, \boldsymbol{w}')}{|\boldsymbol{w} - \boldsymbol{w}'|} \nu^i(u, \boldsymbol{w}')\, \mathrm{d}^3 w' , \tag{4.10}$$

$$\begin{cases} \rho^*(u, \boldsymbol{w}) = \rho(u, \boldsymbol{w})\, u^0(u, \boldsymbol{w}) \sqrt{-\hat{g}} , \\ \nu^i(u, \boldsymbol{w}) = \dfrac{\mathrm{d}w^i}{\mathrm{d}u} = \varepsilon_{ikm} \hat{\omega}_{\mathrm{E}}^k w^m , \end{cases} \tag{4.11}$$

where the quantities $Q_i^{(\mathrm{E})}$, $Q_{ik}^{(\mathrm{E})}$, $Q_{ikm}^{(\mathrm{E})}$, $C_{ik}^{(\mathrm{E})}$ are functions of the time u on the world line of the GRS origin.

Along with remainder terms indicated explicitly in (4.2)-(4.9) we have neglected 1) in $\hat{g}_{00}(u, \boldsymbol{w})$ the quadric terms with respect to $Q_i^{(\mathrm{E})}$, $Q_{ik}^{(\mathrm{E})}$, $Q_{ikm}^{(\mathrm{E})}, \dots$ and to the time derivatives of the first order as well as the linear terms with the time derivatives of the second order, 2) in $\hat{g}_{0i}(u, \boldsymbol{w})$ the terms with the time derivatives of $Q_i^{(\mathrm{E})}$, $Q_{ik}^{(\mathrm{E})}$, $Q_{ikm}^{(\mathrm{E})}, \dots$ resulting from the fulfilment of harmonic conditions (3.2). This neglection is justified here because our aim is to derive the equations of satellite motion taking into account the influence of the external masses only in linear approximation with respect to the spatial coordinates of the satellite.

The potentials $\hat{U}_{\mathrm{E}}$, $\hat{U}_{\mathrm{E}}^i$ occurring in expressions (4.5)-(4.10) characterize the Earth own gravitational field of electric and magnetic type, respectively. They result from solving the inhomogeneous equations (3.3) with the right-hand members including the energy-momentum tensor to the Earth alone.

The functions $Q_i^{(\mathrm{E})}$, $Q_{ik}^{(\mathrm{E})}$, $Q_{ikm}^{(\mathrm{E})}$, $C_{ik}^{(\mathrm{E})}, \dots$ of expressions (4.5)-(4.8) depend only on the time u and are determined by solving the homogeneous equations (3.3). The function $Q_i^{(\mathrm{E})}$ is characteristic of the first curvature of the world line of the GRS origin and is numerically equal to the acceleration of the GRS origin with respect to the RS freely falling ($Q_i = 0$) in the gravitational field of the external masses. The functions $Q_{ik}^{(\mathrm{E})}$, $Q_{ikm}^{(\mathrm{E})}$, $C_{ik}^{(\mathrm{E})}, \dots$ are referred to as the external multipole moments. They are symmetric and trace-free with respect to any pair of indices and behave as tensors under linear transformations of the spatial coordinates. The functions $Q_{ik}^{(\mathrm{E})}$, $Q_{ikm}^{(\mathrm{E})}, \dots$ and $C_{ik}^{(\mathrm{E})}, \dots$ characterize respectively the gravielectric and gravimagnetic tidal field caused by Sun, Moon and planets. The explicit expressions for the external multipole moments will be given in sect. **6**.

Outside the Earth the potentials $\hat{U}_{\mathrm{E}}$ and $\hat{U}_{\mathrm{E}}^i$ are expanded in GRS in series of the multipole harmonics of the geopotential

$$\hat{U}_{\mathrm{E}}(u, \boldsymbol{w}) = \frac{G\hat{M}_{\mathrm{E}}}{\hat{r}} + \frac{1}{2\hat{r}^3} G\hat{I}_{\mathrm{E}}^{ik}\left(-\delta_{ik} + \frac{3}{\hat{r}^2} w^i w^k\right) + O\left(\alpha_{\mathrm{E}} \frac{L_{\mathrm{E}}^3}{\hat{r}^3}\right), \tag{4.12}$$

$$\hat{U}_{\mathrm{E}}^i(u, \boldsymbol{w}) = G\varepsilon_{ijk}\,\hat{\omega}_{\mathrm{E}}^j\,\hat{I}_{\mathrm{E}}^{km}\frac{w^m}{\hat{r}^3} + O\left(\alpha_{\mathrm{E}}\frac{L_{\mathrm{E}}^3}{\hat{r}^3}\right) \tag{4.13}$$

with $\hat{r} = (w^k w^k)^{1/2}$. The (constant) rest mass of the Earth $\hat{M}_{\mathrm{E}}$ and its moments of inertia are defined by the relations

$$\hat{M}_{\mathrm{E}} = \int_{(\mathrm{E})} \rho^*(u, \boldsymbol{w}')\,\mathrm{d}^3 w' + O(\eta_{\mathrm{E}}^2), \tag{4.14}$$

$$\hat{I}_{\mathrm{E}}^{ik} = \int_{(\mathrm{E})} \rho^*(u, \boldsymbol{w}')\,w'^i w'^k\,\mathrm{d}^3 w' + O(\eta_{\mathrm{E}}^2). \tag{4.15}$$

Relativistic corrections due to the own gravitational field, pressure and specific energy of the Earth are not included in the definitions of potential $\hat{U}_{\rm E}$, mass $\hat{M}_{\rm E}$ and moments of inertia $\hat{I}^{ik}_{\rm E}$ since their contribution is negligibly small and does not affect the results of satellite observations. In fact, satellite laser observations being among the most high-precision ones enable to determine the geocentric gravitational constant $G\hat{M}_{\rm E}$ within the relative accuracy of 10^{-7} whereas the relativistic corrections to the Newtonian value of $G\hat{M}_{\rm E}$ are of the order $O(\eta^2_{\rm E}) \sim 10^{-9}$.

The GRS coordinates of the centre of mass of the Earth $w^i_{\rm E}(u)$ are defined by the formula

$$\hat{M}_{\rm E}\, w^i_{\rm E} \equiv \hat{I}^i_{\rm E} = \int_{\rm (E)} \rho^*(u, \boldsymbol{w}')\, w'^i\, {\rm d}^3 w' + O(\eta^2_{\rm E})\,. \tag{4.16}$$

By the construction of GRS the following relations are valid for any moment of the time u

$$\hat{M}_{\rm E}\, w^i_{\rm E} \equiv \hat{I}^i_{\rm E} = 0\,, \tag{4.17}$$

$$\hat{M}_{\rm E}\, \dot{w}^i_{\rm E} \equiv \hat{P}^i_{\rm E} = \int_{\rm (E)} \rho^*(u, \boldsymbol{w}')\, \nu'^i(u, \boldsymbol{w}')\, {\rm d}^3 w' + O(\eta^2_{\rm E}) = 0\,. \tag{4.18}$$

This means the coincidence of the world lines of the GRS origin ($w^i = 0$) and the centre of mass of the Earth. In virtue of relations (4.17), (4.18) the dipole terms do not occur in expressions (4.12), (4.13).

The conditions $\hat{I}^i_{\rm E} = 0$ and $\hat{P}^i_{\rm E} = 0$ for any moment of time may be fulfilled provided that the second derivatives of $w^i_{\rm E}(u)$ vanish identically. As demonstrated in [15,16] the condition $\ddot{w}^i_{\rm E} = 0$ is fulfilled due to specifically choosing the acceleration $Q^{\rm (E)}_i$ of the world line of the geocentre:

$$Q^{\rm (E)}_i = -\frac{15}{2} \hat{M}^{-1}_{\rm E}\, \hat{I}^{km}_{\rm E}\, Q^{\rm (E)}_{ikm} + O\left(\alpha_{\rm E} \frac{L^3_{\rm E}}{R^3_{\rm E}}\right). \tag{4.19}$$

From this it follows that the centre of mass of the Earth does not move on the geodesic world line for which identically $Q_i = 0$.

The component $\hat{g}_{0i}$ of the metric tensor contains no term of the type $c^{-1}\varepsilon_{ijk}\Omega^j w^k$, Ω^j having the dimension of angular velocity. The absence of such terms implies that the GRS spatial axes are dynamically nonrotating.

To conclude this section it should be noted that 1) the integrals in (4.9), (4.10), (4.14)-(4.18) are calculated over the Earth volume on the hypersurface of the constant time u; 2) differentiating the Earth mass and its moments of inertia with respect to u yields

$$\dot{\hat{M}}_{\rm E} = 0\,, \tag{4.20}$$

$$\dot{\hat{I}}^{ik}_{\rm E} = \varepsilon_{ijm}\,\hat{\omega}^j_{\rm E}\,\hat{I}^{km}_{\rm E} + \varepsilon_{kjm}\,\hat{\omega}^j_{\rm E}\,\hat{I}^{im}_{\rm E}\,. \tag{4.21}$$

These formulae are used below in differentiating the potentials $\hat{U}_{\rm E}$ and $\hat{U}^i_{\rm E}$ with respect to the time u.

5. – Solar-system BRS.

The BRS metric tensor may be described as follows ([18,3,5]):

$$ds^2 = g_{\alpha\beta}(t, \boldsymbol{x})\,dx^\alpha\,dx^\beta\,, \tag{5.1}$$

$$g_{00}(t, \boldsymbol{x}) = 1 + c^{-2}\underset{(2)}{g}_{00}(t, \boldsymbol{x}) + c^{-4}\underset{(4)}{g}_{00}(t, \boldsymbol{x}) + O(c^{-5})\,, \tag{5.2}$$

$$g_{0i}(t, \boldsymbol{x}) = c^{-3}\underset{(3)}{g}_{0i} + O(c^{-5})\,, \tag{5.3}$$

$$g_{ik}(t, \boldsymbol{x}) = -\,\delta_{ik} + c^{-2}\underset{(2)}{g}_{ik} + O(c^{-4})\,, \tag{5.4}$$

$$\underset{(2)}{g}_{00}(t, \boldsymbol{x}) = -\,2U(t, \boldsymbol{x})\,, \tag{5.5}$$

$$\underset{(2)}{g}_{ik}(t, \boldsymbol{x}) = -\,2\delta_{ik}\,U(t, \boldsymbol{x})\,, \tag{5.6}$$

$$\underset{(3)}{g}_{0i}(t, \boldsymbol{x}) = 4U^i(t, \boldsymbol{x})\,, \tag{5.7}$$

$$\underset{(4)}{g}_{00}(t, \boldsymbol{x}) = -\,2U^2(t, \boldsymbol{x}) - 2W(t, \boldsymbol{x})\,, \tag{5.8}$$

$$U(t, \boldsymbol{x}) = U_{\rm E}(t, \boldsymbol{x}) + \overline{U}(t, \boldsymbol{x})\,, \tag{5.9}$$

$$U^i(t, \boldsymbol{x}) = U^i_{\rm E}(t, \boldsymbol{x}) + \overline{U}^i(t, \boldsymbol{x})\,, \tag{5.10}$$

$$W(t, \boldsymbol{x}) = W_{\rm E}(t, \boldsymbol{x}) + \overline{W}(t, \boldsymbol{x})\,, \tag{5.11}$$

$$U_{\rm E}(t, \boldsymbol{x}) = G\int\limits_{(\rm E)} \frac{\rho^*(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\,d^3x' + O(\eta^2_{\rm E})\,, \tag{5.12}$$

$$(5.13)\qquad U^i_{\rm E}(t,\boldsymbol{x}) = G\int\limits_{(\rm E)} \frac{\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} v^i(t,\boldsymbol{x}')\,{\rm d}^3x' ,$$

$$(5.14)\qquad W_{\rm E}(t,\boldsymbol{x}) = -\frac{3}{2}v^2_{\rm E}\,U_{\rm E}(t,\boldsymbol{x}) + \\ + 3v^i_{\rm E}\,U^i_{\rm E}(t,\boldsymbol{x}) - G\int\limits_{(\rm E)} \frac{\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\overline{U}(t,\boldsymbol{x}')\,{\rm d}^3x' - \frac{1}{2}\chi_{{\rm E},00}(t,\boldsymbol{x}) ,$$

$$(5.15)\qquad \chi_{\rm E}(t,\boldsymbol{x}) = -G\int\limits_{(\rm E)}\rho^*(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\,{\rm d}^3x' ,$$

$$(5.16)\qquad \rho^*(t,\boldsymbol{x}) = \rho(t,\boldsymbol{x})\,u^0(t,\boldsymbol{x})\sqrt{-g} ,\qquad v^i = \frac{{\rm d}x^i}{{\rm d}t} ,$$

$$(5.17)\qquad \overline{U}(t,\boldsymbol{x}) = \sum_{\rm A\neq E}\frac{GM_{\rm A}}{r_{\rm A}} ,$$

$$(5.18)\qquad \overline{U}^i(t,\boldsymbol{x}) = \sum_{\rm A\neq E}\frac{GM_{\rm A}}{r_{\rm A}}v^i_{\rm A} ,$$

$$(5.19)\qquad \overline{W}(t,\boldsymbol{x}) = \frac{3}{2}\sum_{\rm A\neq E}\frac{GM_{\rm A}}{r_{\rm A}}v^2_{\rm A} - \sum_{\rm A\neq E}\sum_{\rm B\neq A}\frac{G^2M_{\rm A}M_{\rm B}}{r_{\rm A}r_{\rm AB}} + \frac{1}{2}\sum_{\rm A\neq E}GM_{\rm A}r_{{\rm A},00} .$$

The quantities occurring in (5.17)-(5.19) are $r_{\rm A} = (r^k_{\rm A}r^k_{\rm A})^{1/2}$; $r_{\rm AB} = (r^k_{\rm AB}r^k_{\rm AB})^{1/2}$; $r^k_{\rm A} = x^k - x^k_{\rm A}(t)$; $r^k_{\rm AB} = x^k_{\rm A}(t) - x^k_{\rm B}(t)$. $v^k_{\rm E}$ is the BRS velocity of the centre of mass of the Earth. $x^k_{\rm A}$ and $v^k_{\rm A} = {\rm d}x^k_{\rm A}/{\rm d}t$ are respectively BRS coordinates and velocity components of the body A. $M_{\rm A}$ is the (constant) rest mass of the body A defined by the relation

$$(5.20)\qquad M_{\rm A} = \int\limits_{(\rm A)}\rho^*(t,\boldsymbol{x})\,{\rm d}^3x + O(\eta^2_{\rm A}) .$$

It is to be noted that the integrals of (5.12)-(5.15) and (5.20) are taken over the volume of the appropriate body on the hypersurface of the constant time t not coinciding with the hypersurface of the constant time u.

Outside the Earth the functions $U_{\rm E}$, $U^i_{\rm E}$, $W_{\rm E}$ and $\chi_{\rm E}$ may be presented in the form

$$(5.21)\qquad U_{\rm E}(t,\boldsymbol{x}) = \frac{GM_{\rm E}}{r} + GI^k_{\rm E}\frac{r^k}{r^3} + \frac{1}{2r^3}GI^{km}_{\rm E}\left(-\delta_{km} + \frac{3}{r^2}r^kr^m\right) + O\left(\alpha_{\rm E}\frac{L^3_{\rm E}}{r^3}\right) ,$$

$$(5.22)\qquad U^i_{\rm E}(t,\boldsymbol{x}) = \frac{GM_{\rm E}}{r}v^i_{\rm E} + \frac{1}{2r^3}GI^{km}_{\rm E}\left(-\delta_{km} + \frac{3}{r^2}r^kr^m\right)v^i_{\rm E} + \\ + G\varepsilon_{ijk}\,\hat{\omega}^j_{\rm E}\,I^{km}_{\rm E}\frac{r^m}{r^3} + O\left(\alpha_{\rm E}\frac{L^3_{\rm E}}{r^3}\right) ,$$

$$W_{\rm E}(t,\boldsymbol{x}) = \frac{3}{2}\frac{GM_{\rm E}}{r}v_{\rm E}^2 + \frac{3}{4r^3}GI_{\rm E}^{km}\left(-\delta_{km}+\frac{3}{r^2}r^k r^m\right)v_{\rm E}^2 - \frac{GM_{\rm E}}{r}\overline{U}(\boldsymbol{x}_{\rm E}) + \tag{5.23}$$

$$+ 3G\varepsilon_{kjn}\hat{\omega}_{\rm E}^j v_{\rm E}^k I_{\rm E}^{mn}\frac{r^m}{r^3} - \frac{1}{2}\chi_{{\rm E},00}(t,\boldsymbol{x}) + O\left(\alpha_{\rm E}\frac{L_{\rm E}^2}{r^2}\overline{U}\right) + O\left(\alpha_{\rm E}\frac{L_{\rm E}^3}{r^3}v_{\rm E}^2\right),$$

$$\chi_{\rm E}(t,\boldsymbol{x}) = -GM_{\rm E}r - \frac{1}{2r}GI_{\rm E}^{km}\left(\delta_{km} - \frac{1}{r^2}r^k r^m\right) + O\left(\frac{L_{\rm E}^3}{r^3}\right) \tag{5.24}$$

with $r = (r^k r^k)^{1/2}$; $r^k = x^k - x_{\rm E}^k(t)$; $x_{\rm E}^k$ and $v_{\rm E}^k = {\rm d}x_{\rm E}^k/{\rm d}t$ being BRS coordinates and velocity components of the geocentre.

The mass of the Earth and its moments of inertia are defined in BRS by means of formulae

$$M_{\rm E} = \int_{(\rm E)} \rho^*(t,\boldsymbol{x})\,{\rm d}^3x + O(\eta_{\rm E}^2), \tag{5.25}$$

$$I_{\rm E}^i = \int_{(\rm E)} \rho^*(t,\boldsymbol{x})\,r^i\,{\rm d}^3x + O(\eta_{\rm E}^2), \tag{5.26}$$

$$I_{\rm E}^{ik} = \int_{(\rm E)} \rho^*(t,\boldsymbol{x})\,r^i r^k\,{\rm d}^3x + O(\eta_{\rm E}^2), \tag{5.27}$$

where the integrals are evaluated over the Earth volume on the hypersurface $t = {\rm const}$.

It is suitable to make here four methodological remarks.

1. In deriving (5.22)-(5.24) we have used the relation

$$v^i(t,\boldsymbol{x}) = v_{\rm E}^i(t) + \nu^i(u,\boldsymbol{w}) + O(c^{-2}) = v_{\rm E}^i(t) + \varepsilon_{ikm}\hat{\omega}_{\rm E}^k r^m + O(c^{-2}) \tag{5.28}$$

resulting immediately from the BRS to GRS relationship formulae (6.2) and (6.3).

2. The time t derivatives of the moments of inertia $I_{\rm E}^{ik}$ are calculated on the basis of relation

$$\dot{I}_{\rm E}^{ik} = \varepsilon_{ijm}\hat{\omega}_{\rm E}^j I_{\rm E}^{km} + \varepsilon_{kjm}\hat{\omega}_{\rm E}^j I_{\rm E}^{im} + O(c^{-2}) \tag{5.29}$$

which is used, for example, in calculating $\chi_{{\rm E},00}$ in (5.23). The second derivatives of $I_{\rm E}^{ik}$ will be ignored. This is in accordance with adopted restriction to ignore time derivatives in the GRS expression for $\hat{g}_{00}(u,\boldsymbol{w})$.

3. The BRS expansion (5.21) of the geopotential $U_{\rm E}$ contains explicitly the dipole moment of inertia of the Earth $I_{\rm E}^i$. The centre of mass of the Earth has been defined by formula (4.17) making the GRS Earth dipole moment $\hat{I}_{\rm E}^i$ vanish.

Generally speaking, there is no reason to assume the equality of $I^i_{\rm E}$ and $\hat{I}^i_{\rm E}$. Therefore, one may anticipate that $I^i_{\rm E}$ is not equal to zero. In fact, it will be shown in the next section that $I^i_{\rm E}$ is of the relativistic order of smallness. To be rigorous, the terms with $I^i_{\rm E}$ should be included in formulae (5.22)-(5.24). But this would be of no practical importance because the potentials $U^i_{\rm E}$, $W_{\rm E}$, $\chi_{\rm E}$ are used only in the relativistic members of the equations of motion.

4. The BRS origin moves along the world line of the solar-system barycentre. This involves the representation of the integral of the centre of mass in the form (neglecting the angular velocities of the bodies and their moments of inertia)

$$\sum_{\rm A} M_{\rm A} x^i_{\rm A}\left(1+\frac{1}{2}c^{-2}v^2_{\rm A}-\frac{1}{2}c^{-2}\sum_{\rm B\neq A}\frac{GM_{\rm B}}{r_{\rm AB}}\right)=0\,.$$

Just as in GRS the BRS metric tensor contains no term of the type $c^{-1}\varepsilon_{ijk}\Omega^j x^k$. This means that the BRS spatial axes are nonrotating in dynamical sense. As pointed out in sect. 3, the absence of dynamical rotation in BRS is equivalent to the absence of kinematical rotation. This is due to our neglecting the gravitational field of the Galaxy.

6. – Matching and relationship formulae between BRS and GRS.

6·1. *Relationship formulae.* – Asymptotic matching of the metric tensors $g_{\alpha\beta}(t,\boldsymbol{x})$ and $\hat{g}_{\alpha\beta}(u,\boldsymbol{w})$ of BRS and GRS is performed in their overlap region by means of the transformation

$$g_{\alpha\beta}(t,\boldsymbol{x})=c^2\hat{g}_{00}(u,\boldsymbol{w})\frac{\partial u}{\partial x^\alpha}\frac{\partial u}{\partial x^\beta}+c\hat{g}_{0i}(u,\boldsymbol{w})\frac{\partial u}{\partial x^\alpha}\frac{\partial w^i}{\partial x^\beta}+ \\ +c\hat{g}_{0i}(u,\boldsymbol{w})\frac{\partial w^i}{\partial x^\alpha}\frac{\partial u}{\partial x^\beta}+\hat{g}_{ik}(u,\boldsymbol{w})\frac{\partial w^i}{\partial x^\alpha}\frac{\partial w^k}{\partial x^\beta}\,. \tag{6.1}$$

The transformation from BRS to GRS is looked in the form

$$u=t-c^{-2}[S(t)+v^k_{\rm E}r^k]+ \\ +c^{-4}\left[B(t)-\frac{1}{2}v^2_{\rm E}v^k_{\rm E}r^k+B^k(t)r^k+B^{km}(t)r^kr^m\right]+O(c^{-4}r^3)+O(c^{-5})\,, \tag{6.2}$$

$$w^i=r^i+c^{-2}\left\{\left[\frac{1}{2}v^i_{\rm E}v^k_{\rm E}+F^{ik}(t)+D^{ik}(t)\right]r^k+D^{ikm}(t)r^kr^m\right\}+O(c^{-4})\,. \tag{6.3}$$

The functions $S(t)$, $B(t)$, $B^k(t)$, $F^{ik}(t) = -F^{ki}(t)$, $D^{ik}(t) = D^{ki}(t)$, $B^{km}(t) = B^{mk}(t)$, $D^{ikm}(t) = D^{imk}(t)$ are yet unknown and will be determined later by matching procedure. This procedure enables also to get the second-order ordinary differential equations for the functions $x^i_{\mathrm{E}}(t)$, *i.e.* to determine the BRS equations of motion of the centre of mass of the Earth.

We shall match the metric tensors expressed by formulae (4.1)-(4.10) and (5.1)-(5.19). In doing so, the sequence of operations will be as follows:

1) calculating the derivatives $\partial u/\partial x^\alpha$, $\partial w^i/\partial x^\alpha$ and substituting the results into the right-hand member of relation (6.1);

2) expressing the components $\hat{g}_{\alpha\beta}(u, \boldsymbol{w})$ in terms of the BRS coordinates;

3) expanding the functions $\overline{U}$, $\overline{U}^i$, $\overline{W}$ in Taylor series in the vicinity of the point $x^i_{\mathrm{E}}(t)$;

4) comparing the coefficients of equal powers of c^{-1} and r^k in both sides of eq. (6.1) and determining the functions $a^i_{\mathrm{E}}(t) = \mathrm{d}v^i_{\mathrm{E}}/\mathrm{d}t$, S, B, B^k, B^{km}, F^{ik}, D^{ik}, D^{ikm} and the external multipole moments $Q^{(\mathrm{E})}_{ik}$, $Q^{(\mathrm{E})}_{ikm}$, $C^{(\mathrm{E})}_{ik}$, ... The function $Q^{(\mathrm{E})}_i$ does not result from the matching procedure and is given independently by (4.19).

6'2. *Transformation of the coordinate bases.* – The first point of the matching procedure is equivalent to find the transformation law between the BRS coordinate base $e_\alpha = \partial/\partial x^\alpha$ and the GRS base $e_{\hat{\alpha}} = \partial/\partial w^{\hat{\alpha}}$. In accordance with (6.2) and (6.3) one has

$$e_\alpha = \Lambda^{\hat{\beta}}_\alpha e_{\hat{\beta}} \tag{6.4}$$

with

$$\Lambda^{\hat{0}}_0 = \frac{\partial u}{\partial t} = 1 - c^{-2}(\dot{S} - v^2_{\mathrm{E}} + a^k_{\mathrm{E}} r^k) + c^{-4}\left[\frac{1}{2}(v^2_{\mathrm{E}})^2 + \dot{B} - B^k v^k_{\mathrm{E}} + \dot{B}^k r^k - \right.$$
$$\left. - 2B^{km} v^m_{\mathrm{E}} r^k - \frac{1}{2} v^2_{\mathrm{E}} a^k_{\mathrm{E}} r^k - a^m_{\mathrm{E}} v^m_{\mathrm{E}} v^k_{\mathrm{E}} r^k\right] + O(c^{-4} r^2) + O(c^{-5}), \tag{6.5}$$

$$\Lambda^{\hat{i}}_0 = c^{-1}\frac{\partial w^i}{\partial t} = -c^{-1} v^i_{\mathrm{E}} + c^{-3}\left[-\left(\frac{1}{2} v^i_{\mathrm{E}} v^k_{\mathrm{E}} + F^{ik} + D^{ik}\right) v^k_{\mathrm{E}} + \right.$$
$$\left. + \left(\frac{1}{2} v^i_{\mathrm{E}} a^k_{\mathrm{E}} + \frac{1}{2} a^i_{\mathrm{E}} v^k_{\mathrm{E}} + \dot{F}^{ik} + \dot{D}^{ik} - 2D^{ikm} v^m_{\mathrm{E}}\right) r^k + \dot{D}^{ikm} r^k r^m\right] + O(c^{-5}), \tag{6.6}$$

$$\Lambda^{\hat{0}}_i = c\frac{\partial u}{\partial x^i} = -c^{-1} v^i_{\mathrm{E}} + c^{-3}\left(-\frac{1}{2} v^2_{\mathrm{E}} v^i_{\mathrm{E}} + B^i + 2B^{ik} r^k\right) + O(c^{-3} r^2) + O(c^{-4}), \tag{6.7}$$

$$\Lambda^{\hat{i}}_k = \frac{\partial w^i}{\partial x^k} = \delta_{ik} + c^{-2}\left(\frac{1}{2} v^i_{\mathrm{E}} v^k_{\mathrm{E}} + F^{ik} + D^{ik} + 2D^{ikm} r^m\right) + O(c^{-4}). \tag{6.8}$$

Using formulae (6.22), (6.26) and (6.27) given below it is easy to calculate the determinant of the transformation matrix

$$\det(\Lambda_{\hat\beta}^{\hat\alpha}) = 1 + c^{-2}2[\overline{U}(\boldsymbol{x}_{\mathrm{E}}) + a_{\mathrm{E}}^{k} r^{k}] + O(c^{-4}).$$

This determinant vanishes at the distance $r^* \sim c^2/|a_{\mathrm{E}}| \sim 7.5 \cdot 10^{20}$ cm from the centre of mass of the Earth. From this it follows that in spite of initial constructing GRS in the region lying inside the lunar orbit it is possible to smoothly (without intersecting) prolongate the GRS spatial coordinate axes for a much larger distance.

6·3. *Transformations of potentials.* – To accomplish the second step of the matching procedure it is necessary to find the transformation laws for the potentials $\hat{U}_{\mathrm{E}}$ and $\hat{U}_{\mathrm{E}}^{i}$ in converting from GRS to BRS. The potentials $\hat{U}_{\mathrm{E}}$ and $\hat{U}_{\mathrm{E}}^{i}$ are integral quantities defined on the hypersurface of the constant time u. Therefore, to find the relevant transformation laws one has to take into account that the integration in BRS and GRS is performed over the different hypersurfaces of the constant time coordinate. Thus, the transformation of the integrands should include the point transformation combined with the Lie transfer from one hypersurface to another. This transfer is produced along the integral curves of the vector field of the Earth matter four-velocity.

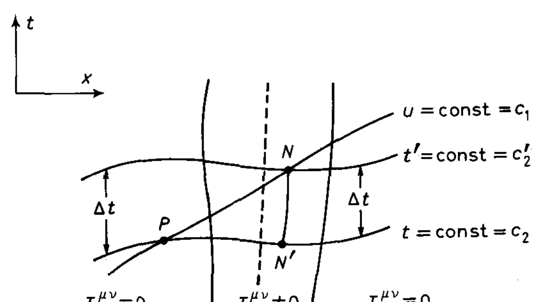

Fig. 1. – The space-time region in the vicinity of the world tube of the Earth ($T_{\mathrm{E}}^{\mu\nu} \neq 0$) is presented. The GRS and BRS solutions of the Einstein field equations for some fixed moments of time scales u and t relate respectively to the hypersurfaces $u = \text{const} = c_1$ and $t = \text{const} = c_2 \neq c_1$. The hypersurface $t' = \text{const} = c_2' \neq c_2$ is shifted along the time lines $(t, \boldsymbol{x} = \text{const})$ from the hypersurface $t = c_2$ for the coordinate distance $\Delta x^0 = c\Delta t$. The matching point P for the GRS and BRS metric tensors lies at the intersection of the hypersurfaces $u = c_1$ and $t = c_2$. It has the BRS coordinates $x^\alpha(P) = (ct, \boldsymbol{x})$ and the GRS coordinates $w^\alpha(P) = (cu, \boldsymbol{w})$. The point N lying at the intersection of the hypersurfaces $u = c_1$ and $t' = c_2'$ has the BRS coordinates $x^\alpha(N) = (ct', \boldsymbol{x}')$ and the GRS coordinates $w^\alpha(N) = (cu, \boldsymbol{w}')$. The point N' lies at the intersection of the hypersurface $t = c_2$ and the emanating from N integral curve of the vector field of the fluid four-velocity u^α. This point has the BRS coordinates $x^\alpha(N') = (ct, \boldsymbol{x}')$. The dashed line denotes the worldline of the geocentre.

First of all, let us find the transformation law of the spatial coordinates of any point of the Earth provided that the matching point P is fixed. Let the GRS and BRS coordinates of this point be $w^\alpha(P) = (cu, \boldsymbol{w})$ and $x^\alpha(P) = (ct, \boldsymbol{x})$, respectively. These sets of coordinates are related by transformations (6.2) and (6.3). The matching point P belongs to both hypersurfaces of the constant time, *i.e.* $u = \text{const} = c_1$ and $t = \text{const} = c_2 \neq c_1$ (see fig. 1).

Consider now another point N lying inside the world tube of the Earth body and belonging to the hypersurface $u = c_1$. Let the GRS and BRS coordinates of N be $w^\alpha(N) = (cu, \boldsymbol{w}')$ and $x^\alpha(N) = (ct', \boldsymbol{x}')$, respectively. Considered in BRS the point N lies on the hypersurface $t' = \text{const} = c_2'$ which does not coincide with the hypersurface $t = c_2$.

To determine the coordinate time interval Δt between the hypersurfaces $t = c_2$ and $t = c_2'$ we construct through N the integral curve of the vector field u^α to be parametrized by means of the coordinate time t. This integral curve intersects the hypersurface $t = c_2$ at the point N' with the BRS coordinates $x^\alpha(N') = (ct, \boldsymbol{x}')$. The relationship between the GRS and BRS coordinate times at the point N has the form

$$u = t' - c^{-2}[S(t') + v_{\mathrm{E}}^k(t')\, r'^k] + O(c^{-4}) . \tag{6.9}$$

The analogous relationship at the point P is given by (6.2). Subtracting these two relations yields the required coordinate time interval between the hypersurfaces $t = c_2$ and $t = c_2'$:

$$\Delta t = t' - t = c^{-2} v_{\mathrm{E}}^k(t)(x'^k - x^k) + O(c^{-4}) . \tag{6.10}$$

The relationship of the BRS spatial coordinates of the fluid elements at the points N and N' is determined by the formula

$$x'^i(t') = x'^i(t) + v'^i(t)\,\Delta t + O((\Delta t)^2) = x'^i(t) + c^{-2} v'^i v_{\mathrm{E}}^k(x'^k - x^k) + O(c^{-4}) \tag{6.11}$$

with $v'^i = \mathrm{d}x'^i/\mathrm{d}t$.

The relationship between the BRS spatial coordinates of the geocentre is established in a similar manner:

$$x_{\mathrm{E}}^i(t') = x_{\mathrm{E}}^i(t) + v_{\mathrm{E}}^i(t)\,\Delta t + O((\Delta t)^2) = x_{\mathrm{E}}^i(t) + c^{-2} v_{\mathrm{E}}^i v_{\mathrm{E}}^k(x'^k - x^k) + O(c^{-4}) . \tag{6.12}$$

With the aid of (6.3), (6.11), (6.12) it is easy to obtain the relationship between the BRS spatial coordinates of the point N' and the GRS spatial coordinates of the point N:

$$w'^i = r'^i + c^{-2}\left[\left(\frac{1}{2} v_{\mathrm{E}}^i v_{\mathrm{E}}^k + F^{ik} + D^{ik}\right) r'^k + D^{ikm} r'^k r'^m\right] + c^{-2}(v'^i - v_{\mathrm{E}}^i)\, v_{\mathrm{E}}^k (r'^k - r^k) + O(c^{-4}) . \tag{6.13}$$

It is to be noted that this relation differs from (6.3) by the complementary term.

Now one may derive the transformation relating the GRS coordinate distance between P and N with the BRS coordinate distance between P and N'. Subtracting (6.13) from (6.3) and using (6.26), (6.27) one obtains

$$(6.14)\qquad \frac{1}{|\boldsymbol{w}-\boldsymbol{w}'|}=\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\left\{1+c^{-2}\left[\frac{1}{2}(v_{\mathrm{E}}^{k}n^{k})^{2}-(v'^{k}n^{k})(v_{\mathrm{E}}^{m}n^{m})-\overline{U}(\boldsymbol{x}_{\mathrm{E}})-\right.\right.$$
$$\left.\left.-\frac{1}{2}a_{\mathrm{E}}^{k}(r'^{k}-r^{k})-a_{\mathrm{E}}^{k}r^{k}\right]\right\}+O(c^{-4})$$

with $n^{k}=(x^{k}-x'^{k})/|\boldsymbol{x}-\boldsymbol{x}'|$. The product of the rest mass density by the coordinate elementary volume does not change both under the coordinate transformation and the just considered Lie transfer [16]. Hence

$$(6.15)\qquad \rho^{*}(t,\boldsymbol{x})\,\mathrm{d}^{3}x=\rho^{*}(u,\boldsymbol{w})\,\mathrm{d}^{3}w\,,$$

$$(6.16)\qquad u^{\alpha}(\rho^{*}\,\mathrm{d}^{3}x)_{,\alpha}=u^{\hat{\alpha}}(\rho^{*}\,\mathrm{d}^{3}w)_{,\hat{\alpha}}=0\,.$$

From (6.15), (6.16) it follows that the product $\rho^{*}(u,\boldsymbol{w})\,\mathrm{d}^{3}w$ at the point N is equal to the product $\rho^{*}(t,\boldsymbol{x})\,\mathrm{d}^{3}x$ at the point N'. Substituting relations (6.14)-(6.16) into definition (4.9) for the potential $\hat{U}_{\mathrm{E}}$ results in the transformation law for the potential

$$(6.17)\qquad \hat{U}_{\mathrm{E}}(u,\boldsymbol{w})=U_{\mathrm{E}}(t,\boldsymbol{x})\left[1+c^{-2}\left(\frac{1}{2}v_{\mathrm{E}}^{2}-\overline{U}(\boldsymbol{x}_{\mathrm{E}})-a_{\mathrm{E}}^{k}r^{k}\right)\right]+$$
$$+c^{-2}\left[\frac{1}{2}v_{\mathrm{E}}^{k}v_{\mathrm{E}}^{m}\chi_{\mathrm{E},km}(t,\boldsymbol{x})+v_{\mathrm{E}}^{k}\chi_{\mathrm{E},0k}(t,\boldsymbol{x})-\frac{1}{2}a_{\mathrm{E}}^{k}\chi_{\mathrm{E},k}(t,\boldsymbol{x})-v_{\mathrm{E}}^{k}U_{\mathrm{E}}^{k}(t,\boldsymbol{x})\right]+O(c^{-4})\,.$$

The transformation law for the potential $\hat{U}_{\mathrm{E}}^{i}$ is needed only in the Newtonian approximation. Its derivation demands again relations (6.14)-(6.16) as well as formula (5.28) for the transformation of the three-velocity of the element of fluid. This latter formula may be obtained by taking the total time derivative of both members of relation (6.13). Proceeding from definition (4.10) of the potential $\hat{U}_{\mathrm{E}}^{i}$ one gets

$$(6.18)\qquad \hat{U}_{\mathrm{E}}^{i}(u,\boldsymbol{w})=U_{\mathrm{E}}^{i}(t,\boldsymbol{x})-v_{\mathrm{E}}^{i}U_{\mathrm{E}}(t,\boldsymbol{x})+O(c^{-2})\,.$$

It may be noted that the transformation laws for the potentials under Lorentz transformations were obtained earlier by Will [4].

Relations (6.13)-(6.16) enable one to derive the transformation laws for the mass of the Earth and its moments of inertia. One has

$$M_{\rm E} = \hat{M}_{\rm E}\,, \tag{6.19}$$

$$I^i_{\rm E} = -\,c^{-2}\,\varepsilon_{ikm}\,\hat\omega^k_{\rm E}\,v^n_{\rm E}\,\hat I^{mn}_{\rm E} - c^{-2}\,a^k_{\rm E}\,\hat I^{ik}_{\rm E} + \frac{1}{2}\,c^{-2}\,a^i_{\rm E}\,\hat I^{kk}_{\rm E} + O(c^{-4})\,, \tag{6.20}$$

$$I^{ik}_{\rm E} = \hat I^{ik}_{\rm E} - c^{-2}\left(\frac{1}{2}v^i_{\rm E}\,v^m_{\rm E} + F^{im} + D^{im}\right)\hat I^{km}_{\rm E} - c^{-2}\left(\frac{1}{2}v^k_{\rm E}\,v^m_{\rm E} + F^{km} + D^{km}\right)\cdot$$
$$\cdot\,\hat I^{im}_{\rm E} + c^{-2}\,\hat\omega^j_{\rm E}(v^n_{\rm E}\,w^n)(\varepsilon_{ijm}\,\hat I^{km}_{\rm E} + \varepsilon_{kjm}\,\hat I^{im}_{\rm E}) + O(c^{-2}L^3_{\rm E}) + O(c^{-4})\,. \tag{6.21}$$

In deriving the equations of satellite motion we shall neglect everywhere the products of the Earth moments of inertia $\hat I^{ik}_{\rm E}$ by the terms due to the gravitational action of the external masses. For example, we shall use actually in the right-hand member of (6.20) only the first term.

It is of interest that expression (6.21) relating the BRS and GRS moments of inertia of the Earth involves the coordinates of the matching point P (see fig. 1). At first sight it might seem strange. Considering that the BRS moments of inertia are functions of t and the GRS moments of inertia are functions of u and taking into account that the relationship (6.2) between t and u depends on the spatial coordinates of the matching point this becomes clear.

Let us describe the matching procedure in detail.

6·4. *Matching* $\underset{(2)}{g}_{00}(t, \boldsymbol{x})$ *and* $\hat g_{\alpha\beta}(u, \boldsymbol{w})$. – This is the first step in the iteration procedure to determine the unknown coefficients of (6.2), (6.3) and the external multipole moments. At this step of matching it is possible to determine the function $\dot S(t)$, the external multipole moments $Q^{(\rm E)}_{ik}$, $Q^{(\rm E)}_{ikm}$, ... in the Newtonian approximation and the Newtonian equations of the translatory motion of the centre of mass of the Earth. One has

$$\dot S(t) = \frac{1}{2}v^2_{\rm E} + \overline{U}(\boldsymbol{x}_{\rm E})\,, \tag{6.22}$$

$$a^i_{\rm E}(t) = \overline{U}_{,i}(\boldsymbol{x}_{\rm E}) - Q^{(\rm E)}_i + O(c^{-2})\,, \tag{6.23}$$

$$Q^{(\rm E)}_{ik} = \frac{1}{3}\,\overline{U}_{,ik}(\boldsymbol{x}_{\rm E}) + O(c^{-2})\,, \tag{6.24}$$

$$Q^{(\rm E)}_{ikm} = \frac{1}{15}\,\overline{U}_{,ikm}(\boldsymbol{x}_{\rm E}) + O(c^{-2})\,. \tag{6.25}$$

The function $\dot S(t)$ gives the differential equation relating the BRS time scale (TDB) and the GRS time scale (TDT) at the geocentre.

6·5. *Matching* $\underset{(2)}{g}_{ik}(t, \boldsymbol{x})$ *and* $\hat{g}_{\alpha\beta}(u, \boldsymbol{w})$. – This matching enables one to determine explicitly D^{ik} and D^{ikm}. Using the values (6.24), (6.25) one obtains

$$D^{ik}(t) = \delta_{ik}\, \overline{U}\,(\boldsymbol{x}_{\mathrm{E}}) \,, \tag{6.26}$$

$$D^{ikm}(t) = \frac{1}{2}(\delta_{ik}\, a^m_{\mathrm{E}} + \delta_{im}\, a^k_{\mathrm{E}} - \delta_{km}\, a^i_{\mathrm{E}}) \,. \tag{6.27}$$

The function D^{ik} describes the gravitational spatial contraction of GRS with respect to BRS. It should be particularly emphasized that the relationship (6.3) between GRS and BRS spatial coordinates is presented in the post-Newtonian approximation by the finite sum of linear and quadric terms.

6·6. *Matching* $\underset{(3)}{g}_{0i}(t, \boldsymbol{x})$ *and* $\hat{g}_{\alpha\beta}(u, \boldsymbol{w})$. – This matching is performed by using the already known values (6.23), (6.24), (6.26), (6.27). This allows one to find the external multipole moments $C^{(\mathrm{E})}_{ik}$ and the coefficients B^i, B^{ik}, $\dot{F}^{ik}$ of transformations (6.2), (6.3). There results

$$B^i(t) = 4\overline{U}^i(\boldsymbol{x}_{\mathrm{E}}) - 3v^i_{\mathrm{E}}\, \overline{U}\,(\boldsymbol{x}_{\mathrm{E}}) \,, \tag{6.28}$$

$$B^{ik}(t) = \overline{U}^i_{,k}(\boldsymbol{x}_{\mathrm{E}}) + \overline{U}^k_{,i}(\boldsymbol{x}_{\mathrm{E}}) - \frac{1}{2}[v^i_{\mathrm{E}}\, \overline{U}_{,k}(\boldsymbol{x}_{\mathrm{E}}) + v^k_{\mathrm{E}}\, \overline{U}_{,i}(\boldsymbol{x}_{\mathrm{E}})] + \frac{1}{2}\delta_{ik}\,[v^m_{\mathrm{E}}\, \overline{U}_{,m}(\boldsymbol{x}_{\mathrm{E}}) - \overline{U}^m_{,m}(\boldsymbol{x}_{\mathrm{E}})] \,, \tag{6.29}$$

$$\dot{F}^{ik}(t) = \frac{3}{2}[v^i_{\mathrm{E}}\, \overline{U}_{,k}(\boldsymbol{x}_{\mathrm{E}}) - v^k_{\mathrm{E}}\, \overline{U}_{,i}(\boldsymbol{x}_{\mathrm{E}})] - 2[\overline{U}^i_{,k}(\boldsymbol{x}_{\mathrm{E}}) - \overline{U}^k_{,i}(\boldsymbol{x}_{\mathrm{E}})] \,, \tag{6.30}$$

$$\varepsilon_{ijk}\, C_{jm} = \frac{1}{3}\overline{U}^i_{,km}(\boldsymbol{x}_{\mathrm{E}}) - \frac{1}{3}\overline{U}^k_{,im}(\boldsymbol{x}_{\mathrm{E}}) - \frac{1}{3}v^i_{\mathrm{E}}\, \overline{U}_{,km}(\boldsymbol{x}_{\mathrm{E}}) + \frac{1}{3}v^k_{\mathrm{E}}\, \overline{U}_{,im}(\boldsymbol{x}_{\mathrm{E}}) + \frac{1}{6}\delta_{im}\, \dot{a}^k_{\mathrm{E}} - \frac{1}{6}\delta_{km}\, \dot{a}^i_{\mathrm{E}} \,. \tag{6.31}$$

Due to the smallness of $Q^{(\mathrm{E})}_i$ the corresponding terms have been ignored in (6.29)-(6.31).

The function $\dot{F}^{ik}$ has the physical meaning of the angular velocity of rotation (relativistic precession) of the GRS spatial axes with respect to the BRS spatial axes. The first term of this function in the right-hand member of (6.30) represents the de Sitter geodesic precession, while the second term describes the gravimagnetic precession caused by the motion of the external masses in BRS. It should be reminded that the GRS spatial axes are nonrotating in dynamical sense.

6·7. *Matching* $\underset{(4)}{g}_{00}(t, \boldsymbol{x})$ *and* $\hat{g}_{\alpha\beta}(u, \boldsymbol{w})$. – All previous values are used here to find the function $\dot{B}(t)$ and the relativistic post-Newtonian corrections to the equations of motion of the geocentre and to the multipole moments $Q^{(\mathrm{E})}_{ik}$. The explicit expression for $\dot{B}(t)$ is not needed for the subsequent calculations. This expression may be found in ([16]). The equations of motion of the Earth centre of mass and the moments $Q^{(\mathrm{E})}_{ik}$ taking into account the relativistic corrections are as follows:

$$a^i_{\mathrm{E}}(t) = \overline{U}_{,i}(\boldsymbol{x}_{\mathrm{E}}) - Q^{(\mathrm{E})}_i + c^{-2}\,\overline{G}^{\,i}(t) + O(c^{-4})\,, \tag{6.32}$$

$$\overline{G}^{\,i}(t) = -4\overline{U}(\boldsymbol{x}_{\mathrm{E}})\,\overline{U}_{,i}(\boldsymbol{x}_{\mathrm{E}}) - v^i_{\mathrm{E}}v^k_{\mathrm{E}}\,\overline{U}_{,k}(\boldsymbol{x}_{\mathrm{E}}) + v^2_{\mathrm{E}}\,\overline{U}_{,i}(\boldsymbol{x}_{\mathrm{E}}) - 3v^i_{\mathrm{E}}\,\dot{\overline{U}}(\boldsymbol{x}_{\mathrm{E}}) + 4\dot{\overline{U}}{}^i(\boldsymbol{x}_{\mathrm{E}}) - 4v^k_{\mathrm{E}}\,\overline{U}^k_{,i}(\boldsymbol{x}_{\mathrm{E}}) + \overline{W}_{,i}(\boldsymbol{x}_{\mathrm{E}})\,, \tag{6.33}$$

$$\begin{aligned} Q^{(\mathrm{E})}_{ik} = {} & \frac{1}{3}\overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) + c^{-2}\Big[\frac{1}{3}F^{im}\,\overline{U}_{,km}(\boldsymbol{x}_{\mathrm{E}}) + \frac{1}{3}F^{km}\,\overline{U}_{,im}(\boldsymbol{x}_{\mathrm{E}}) - \\ & - \frac{4}{3}v^m_{\mathrm{E}}\,\overline{U}^m_{,ik}(\boldsymbol{x}_{\mathrm{E}}) + \frac{2}{3}v^2_{\mathrm{E}}\,\overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) - \frac{2}{3}\overline{U}(\boldsymbol{x}_{\mathrm{E}})\,\overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) - \frac{1}{6}v^m_{\mathrm{E}}v^i_{\mathrm{E}}\,\overline{U}_{,km}(\boldsymbol{x}_{\mathrm{E}}) - \\ & - \frac{1}{6}v^m_{\mathrm{E}}v^k_{\mathrm{E}}\,\overline{U}_{,im}(\boldsymbol{x}_{\mathrm{E}}) + \frac{1}{3}\overline{W}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) + \frac{1}{3}\delta_{ik}\,\ddot{\overline{U}}(\boldsymbol{x}_{\mathrm{E}}) + \frac{2}{3}\dot{\overline{U}}{}^i_{,k}(\boldsymbol{x}_{\mathrm{E}}) + \\ & + \frac{2}{3}\dot{\overline{U}}{}^k_{,i}(\boldsymbol{x}_{\mathrm{E}}) - a^i_{\mathrm{E}}a^k_{\mathrm{E}} - \frac{1}{3}v^i_{\mathrm{E}}\dot{a}^k_{\mathrm{E}} - \frac{1}{3}v^k_{\mathrm{E}}\dot{a}^i_{\mathrm{E}}\Big] + O(c^{-4})\,. \end{aligned} \tag{6.34}$$

The matching procedure is completed by obtaining these formulae. It is possible to proceed further and to get the relativistic corrections to the moments $Q^{(\mathrm{E})}_{ikm}$, $C^{(\mathrm{E})}_{ik}$, ... but this is unnecessary for the problem in question. The available relations permit to derive the relativistic equations of the Earth satellite motion with the accuracy far exceeding the present practical requirements.

To conclude this section it may be noted that the external multipole moments $Q^{(\mathrm{E})}_{ik}$, $Q^{(\mathrm{E})}_{ikm}$, $C^{(\mathrm{E})}_{ik}$ occurring in relations (6.25), (6.31), (6.34) depend on time t. In the expression for the GRS metric tensor (4.5)-(4.8) the same moments depend on time u. The relationship (6.2) between the time scales t and u for the external multipole moments should be taken on the world line of the centre of mass of the Earth $x^i = x^i_{\mathrm{E}}(t)$.

7. – Equations of motion of an Earth satellite. Geocentric approach.

An Earth satellite considered as a test body moves on the geodesic world line. The geocentric approach to derive the satellite equations of motion is to apply the geodesic principle immediately to the GRS metric.

Adopting the coordinate time u to parametrize the geodesic line the equations of geodesic motion will be

$$(7.1)\qquad \ddot{w}^i + c^2\hat{\Gamma}^i_{00} + 2c\hat{\Gamma}^i_{0k}\dot{w}^k + \hat{\Gamma}^i_{km}\dot{w}^k\dot{w}^m - c\hat{\Gamma}^0_{00}\dot{w}^i - 2\hat{\Gamma}^0_{0k}\dot{w}^k\dot{w}^i - c^{-1}\hat{\Gamma}^0_{km}\dot{w}^k\dot{w}^m\dot{w}^i = 0\,.$$

Expressing the Christoffel symbols in terms of the metric tensor components (4.1)-(4.4) and substituting them into (7.1) one has

$$(7.2)\qquad \ddot{w}^i = -\frac{1}{2}\underset{(2)}{\hat{g}}_{00,i} + c^{-2}\Big[-\frac{1}{2}\underset{(4)}{\hat{g}}_{00,i} - \frac{1}{2}\underset{(2)}{\hat{g}}_{ik}\underset{(2)}{\hat{g}}_{00,k} + \underset{(3)}{\hat{g}}_{0i,0} + \frac{1}{2}\underset{(2)}{\hat{g}}_{00,0}\dot{w}^i + \underset{(2)}{\hat{g}}_{ik,0}\dot{w}^k + (\underset{(3)}{\hat{g}}_{0i,k} - \underset{(3)}{\hat{g}}_{0k,i})\dot{w}^k + \Big(\underset{(2)}{\hat{g}}_{ik,m} - \frac{1}{2}\underset{(2)}{\hat{g}}_{km,i}\Big)\dot{w}^k\dot{w}^m + \underset{(2)}{\hat{g}}_{00,k}\dot{w}^k\dot{w}^i\Big] + O(c^{-4})\,.$$

Substituting the explicit expressions (4.5)-(4.8), (4.12), (4.13), (4.19), (6.25), (6.31) and (6.34) into eq. (7.2) we obtain the relativistic GRS equations of the Earth satellite motion in the form

$$(7.3)\qquad \ddot{w}^i = F^i_0 + F^i_1 + F^i_2 + F^i_3 + c^{-2}\sum_{n=1}^{6}\Phi^i_n$$

with

$$(7.4)\qquad F^i_0 = -\frac{G\hat{M}_E}{\hat{r}^3}w^i\,,$$

$$(7.5)\qquad F^i_1 = \frac{3}{2}\frac{G}{\hat{r}^5}\Big(\hat{I}^{kk}_E w^i + 2\hat{I}^{ik}_E w^k - \frac{5}{\hat{r}^2}\hat{I}^{km}_E w^k w^m w^i\Big)\,,$$

$$(7.6)\qquad F^i_2 = \overline{U}_{,ik}(\boldsymbol{x}_E)\, w^k + \frac{1}{2}\overline{U}_{,ikm}(\boldsymbol{x}_E)\, w^k w^m\,,$$

$$(7.7)\qquad F^i_3 = Q^{(E)}_i = -\frac{1}{2}\hat{M}^{-1}_E\hat{I}^{km}_E\overline{U}_{,ikm}(\boldsymbol{x}_E)\,,$$

$$(7.8)\qquad \Phi^i_1 = \frac{G\hat{M}_E}{\hat{r}^3}\Big[\Big(4\frac{G\hat{M}_E}{\hat{r}} - \dot{w}^k\dot{w}^k\Big)w^i + 4w^k\dot{w}^k\dot{w}^i\Big]\,,$$

$$(7.9)\qquad \Phi^i_2 = \frac{4}{\hat{r}^3}G\,\hat{\omega}^j_E\hat{I}^{mn}_E\Big[\varepsilon_{ijn}\Big(\delta_{km} - \frac{3}{\hat{r}^2}w^k w^m\Big) - \varepsilon_{kjn}\Big(\delta_{im} - \frac{3}{\hat{r}^2}w^i w^m\Big)\Big]\dot{w}^k - \frac{9}{\hat{r}^5}G\,\varepsilon_{kjn}\hat{\omega}^j_E\hat{I}^{mn}_E w^k w^m\dot{w}^i\,,$$

$$(7.10)\quad \Phi_3^i = 4\frac{G^2\hat{M}_E}{\hat{r}^6}\left(-2\hat{I}_E^{kk}w^i - 3\hat{I}_E^{ik}w^k + \frac{9}{\hat{r}^2}\hat{I}_E^{km}w^k w^m w^i\right) + \frac{3}{2}\frac{G}{\hat{r}^5}(\dot{w}^n\dot{w}^n)\left(\hat{I}_E^{kk}w^i + 2\hat{I}_E^{ik}w^k - \frac{5}{\hat{r}^2}\hat{I}_E^{km}w^k w^m w^i\right) + 6\frac{G}{\hat{r}^5}\dot{w}^i\dot{w}^n\left(-\hat{I}_E^{kk}w^n - 2\hat{I}_E^{kn}w^k + \frac{5}{\hat{r}^2}\hat{I}_E^{km}w^k w^m w^n\right),$$

$$(7.11)\quad \Phi_4^i = -4\frac{G\hat{M}_E}{\hat{r}}\overline{U}_{,ik}(\boldsymbol{x}_E)w^k + 2\frac{G\hat{M}_E}{\hat{r}^3}\overline{U}_{,km}(\boldsymbol{x}_E)w^k w^m w^i,$$

$$(7.12)\quad \Phi_5^i = w^k[-4\dot{w}^i\dot{w}^m\overline{U}_{,km}(\boldsymbol{x}_E) + \dot{w}^m\dot{w}^m\overline{U}_{,ik}(\boldsymbol{x}_E) + 4\dot{w}^m\overline{U}^i_{,km}(\boldsymbol{x}_E) - 4\dot{w}^m\overline{U}^m_{,ik}(\boldsymbol{x}_E) - 4v_E^i\dot{w}^m\overline{U}_{,km}(\boldsymbol{x}_E) + 4v_E^m\dot{w}^m\overline{U}_{,ik}(\boldsymbol{x}_E) + 2\delta_{ik}\dot{a}_E^m\dot{w}^m - 2\dot{a}_E^i\dot{w}^k],$$

$$(7.13)\quad \Phi_6^i = w^k\Big[F^{im}\overline{U}_{,km}(\boldsymbol{x}_E) + F^{km}\overline{U}_{,im}(\boldsymbol{x}_E) - 4v_E^m\overline{U}^m_{,ik}(\boldsymbol{x}_E) + 2v_E^2\overline{U}_{,ik}(\boldsymbol{x}_E) - 2\overline{U}(\boldsymbol{x}_E)\overline{U}_{,ik}(\boldsymbol{x}_E) - \frac{1}{2}v_E^i v_E^m\overline{U}_{,km}(\boldsymbol{x}_E) - \frac{1}{2}v_E^k v_E^m\overline{U}_{,im}(\boldsymbol{x}_E) + \overline{W}_{,ik}(\boldsymbol{x}_E) + \delta_{ik}\ddot{\overline{U}}(\boldsymbol{x}_E) + 2\dot{\overline{U}}^i_{,k}(\boldsymbol{x}_E) + 2\dot{\overline{U}}^k_{,i}(\boldsymbol{x}_E) - 3a_E^i a_E^k - v_E^i\dot{a}_E^k - v_E^k\dot{a}_E^i\Big].$$

The physical meaning of the separate terms is as follows: F_0^i is the spherically symmetrical component of the Newtonian attraction of the Earth; F_1^i is the Newtonian perturbation due to the quadrupole harmonics of the geopotential; F_2^i is the Newtonian tidal perturbation of the gravielectric type due to the Sun, the Moon and the planets; F_3^i is the Newtonian perturbation resulting from the term $Q_i^{(E)}$ which is included in the $\underset{(2)}{\hat{g}}_{00}(u, \boldsymbol{w})$ and expressed according to formula (4.19) (note that although F_3^i is of Newtonian origin its physical meaning is clearly and unambiguously formulated in the relativistic language as deviation of the worldline of the geocentre from the geodesic line); Φ_1^i is the relativistic Schwarzschild perturbation due to the spherically symmetrical component of the Earth gravitational field; Φ_2^i is the relativistic Lense-Thirring gravimagnetic perturbation caused by the Earth rotation; Φ_3^i is the relativistic Earth quadrupole perturbation generated by the Earth quadrupole moments; Φ_4^i is the relativistic perturbation caused by the nonlinear coupling of the Earth attraction and the gravielectric tidal field of Sun, Moon and planets; Φ_5^i is the relativistic gravimagnetic tidal perturbation due to the external masses; Φ_6^i is the relativistic correction to the force F_2^i.

It should be particularly noted that the derived equations of the satellite motion contain no Coriolis or centrifugal terms. This results from the principles of constructing GRS as the dynamically nonrotating RS. From this it follows that the present laser and Doppler satellite observations cannot give any information concerning the magnitude of the GRS relativistic precession with respect to BRS. In principle this is possible since the relativistic precession F^{ik} enters into the expression of Φ_6^i. But the influence of such terms on the satellite orbital motion is negligibly small. Basically, the GRS dynamical perturbations of the satellite orbit enable one to evaluate only the magnitude of the classical precession of the satellite orbital plane caused by the Earth oblateness and the Newtonian tidal action of the external masses. Thus the magnitude of the relativistic precession of the satellite orbit and of the GRS spatial axes with respect to BRS may be directly derived from observations by subtracting from the total precession of the satellite orbit in BRS the classical precession determined by satellite laser observations. In application to the Moon this technique has been used for experimentally determining the value of the geodesic precession [37].

From methodological point of view it is useful to derive the GRS satellite equations of motion (7.3)-(7.13) in another manner, *i.e.* by transforming the equations of the geodesic motion from BRS to GRS.

8. – Equations of motion of an Earth satellite. Barycentric approach.

8·1. BRS *equations of satellite motion in relative (formally geocentric) coordinates.* – In distinction to the geocentric approach the barycentric approach implies first to derive the BRS equations and then to transform them into the GRS equations using the relationship (6.2), (6.3) and the transformations of the Earth moments of inertia (6.19)-(6.21). This approach is of interest enabling to gain clear insight into the procedure of eliminating from the BRS satellite equations large unobservable terms caused by the inadequate choice of RS («bad» RS for describing the satellite geocentric motion).

The BRS equations of the geodesic motion are identical with eqs. (7.1), (7.2) replacing the derivatives $\mathrm{d}w^i/\mathrm{d}u$ by $\mathrm{d}x^i/\mathrm{d}t$ and the components $\hat{g}_{\alpha\beta}(u, \boldsymbol{w})$ by $g_{\alpha\beta}(t, \boldsymbol{x})$. Thus, the BRS equations of motion of an Earth satellite have the form

$$\ddot{x}^i = U_{,i} + c^{-2} G^i + O(c^{-4}), \tag{8.1}$$

$$G^i = -4U\,U_{,i} - \dot{x}^i \dot{x}^k U_{,k} + \dot{x}^k \dot{x}^k U_{,i} - 3\dot{x}^i \dot{U} + 4\dot{U}^i - 4\dot{x}^k U^k_{,i} + W_{,i}. \tag{8.2}$$

The first step of calculations is to change in these expressions from the BRS spatial coordinates to the relative (formally geocentric) coordinates (t, r^i) by means of the formulae

$$r^i = x^i - x^i_{\mathrm{E}}(t). \tag{8.3}$$

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[37] B. Bertotti, I. Ciufolini and P. L. Bender: *Phys. Rev. Lett.*, 58, 1062 (1987).

Then the potentials $\overline{U}$, $\overline{U}^i$, $\overline{W}$ are expanded in series in powers of r^i in the vicinity of the point $x^i_{\rm E}(t)$. Using eqs. (6.32), (6.33) for the acceleration of the geocentre $a^i_{\rm E} = \ddot{x}^i_{\rm E}$ one obtains

$$\ddot{r}^i = U_{{\rm E},i} + Q^{({\rm E})}_i + \overline{U}_{,ik}(\boldsymbol{x}_{\rm E})\, r^k + \frac{1}{2}\overline{U}_{,ikm}(\boldsymbol{x}_{\rm E})\, r^k r^m + c^{-2}(G^i - \overline{G}^i) + O(r^3) + O(c^{-4}). \tag{8.4}$$

Within the required accuracy the function $G^i - \overline{G}^i$ has the form

$$\begin{aligned} G^i - \overline{G}^i = &-4U_{\rm E}\, U_{{\rm E},i} - 3\dot{U}_{\rm E}\,\dot{x}^i - U_{{\rm E},k}\,\dot{x}^k\dot{x}^i + U_{{\rm E},i}\,\dot{x}^k\dot{x}^k + 4\dot{U}^i_{\rm E} - \\ &- 4U^k_{{\rm E},i}\,\dot{x}^k + W_{{\rm E},i} - 4U_{\rm E}\,\overline{U}_{,i}(\boldsymbol{x}_{\rm E}) - 4U_{{\rm E},i}\,\overline{U}(\boldsymbol{x}_{\rm E}) - 3\overline{U}_{,0}(\boldsymbol{x}_{\rm E})\,\dot{r}^i - \\ &- 4\overline{U}_{,k}(\boldsymbol{x}_{\rm E})(\dot{x}^k\dot{x}^i - v^k_{\rm E}v^i_{\rm E}) + \overline{U}_{,i}(\boldsymbol{x}_{\rm E})(\dot{x}^k\dot{x}^k - v^2_{\rm E}) + 4\overline{U}^i_{,k}(\boldsymbol{x}_{\rm E})\,\dot{r}^k - 4\overline{U}^k_{,i}(\boldsymbol{x}_{\rm E})\,\dot{r}^k + \\ &+ [-4U_{\rm E}\,\overline{U}_{,ik}(\boldsymbol{x}_{\rm E}) - 4U_{{\rm E},i}\,\overline{U}_{,k}(\boldsymbol{x}_{\rm E}) - 4\overline{U}(\boldsymbol{x}_{\rm E})\,\overline{U}_{,ik}(\boldsymbol{x}_{\rm E}) - 4\overline{U}_{,i}(\boldsymbol{x}_{\rm E})\,\overline{U}_{,k}(\boldsymbol{x}_{\rm E}) + \\ &+ \overline{W}_{,ik}(\boldsymbol{x}_{\rm E}) - 3\overline{U}_{,0k}(\boldsymbol{x}_{\rm E})\,\dot{x}^i - 4\overline{U}_{,km}(\boldsymbol{x}_{\rm E})\,\dot{x}^m\dot{x}^i + \overline{U}_{,ik}(\boldsymbol{x}_{\rm E})\,\dot{x}^m\dot{x}^m + \\ &+ 4\overline{U}^i_{,0k}(\boldsymbol{x}_{\rm E}) + 4\overline{U}^i_{,km}(\boldsymbol{x}_{\rm E})\,\dot{x}^m - 4\overline{U}^m_{,ik}(\boldsymbol{x}_{\rm E})\,\dot{x}^m]\,r^k - 2U_{{\rm E},i}\,\overline{U}_{,km}(\boldsymbol{x}_{\rm E})\,r^k r^m. \end{aligned} \tag{8.5}$$

To express explicitly this function one may use expansions (5.21)-(5.24) for $U_{\rm E}$, $U^i_{\rm E}$, $W_{\rm E}$ and expressions (5.17)-(5.19) for $\overline{U}$, $\overline{U}^i$, $\overline{W}$. The most cumbersome procedure therewith is to differentiate the functions $W_{\rm E}$ and $\overline{W}$. The appropriate derivatives are

$$\begin{aligned} W_{{\rm E},i} = &\frac{1}{2}\frac{GM_{\rm E}}{r}\left(-a^i_{\rm E} + \frac{1}{r^2}a^k_{\rm E} r^k r^i\right) + \frac{GM_{\rm E}}{r^3}\left\{\left[\overline{U}(\boldsymbol{x}_{\rm E}) - 2v^2_{\rm E} + \right.\right. \\ &\left.\left.+ \frac{3}{2r^2}(r^k v^k_{\rm E})^2\right] r^i - v^k_{\rm E} r^k v^i_{\rm E}\right\} + 3\frac{G}{r^5}v^2_{\rm E}\left(I^{kk}_{\rm E} r^i + 2I^{ik}_{\rm E} r^k - \frac{5}{r^2}I^{km}_{\rm E} r^k r^m r^i\right) + \\ &+ \frac{3}{2}\frac{G}{r^5}v^i_{\rm E} v^k_{\rm E}\left(I^{mm}_{\rm E} r^k + 2I^{km}_{\rm E} r^m - \frac{5}{r^2}I^{mn}_{\rm E} r^m r^n r^k\right) + \\ &+ \frac{3}{2}\frac{G}{r^5}v^k_{\rm E} v^m_{\rm E}\left(I^{km}_{\rm E} r^i + 2I^{ik}_{\rm E} r^m - \frac{10}{r^2}I^{kn}_{\rm E} r^m r^n r^i\right) - \\ &- \frac{15}{4}\frac{G}{r^7}(r^k v^k_{\rm E})^2\left(I^{mm}_{\rm E} r^i + 2I^{im}_{\rm E} r^m - \frac{7}{r^2}I^{mn}_{\rm E} r^m r^n r^i\right) + \\ &+ \frac{G}{r^3}\hat{\omega}^j_{\rm E}\left[4\varepsilon_{kjn} v^k_{\rm E} I^{in}_{\rm E} + \varepsilon_{ijn} v^m_{\rm E} I^{mn}_{\rm E} - \frac{3}{r^2}\varepsilon_{kjn} I^{mn}_{\rm E}(v^i_{\rm E} r^k r^m + v^m_{\rm E} r^k r^i + 4v^k_{\rm E} r^m r^i) - \right. \\ &\left. - \frac{3}{r^2}\varepsilon_{kjn} I^{in}_{\rm E} r^k r^m v^m_{\rm E} - \frac{3}{r^2}\varepsilon_{ijn} I^{kn}_{\rm E} r^k r^m v^m_{\rm E} + \frac{15}{r^4}\varepsilon_{kjn} I^{mn}_{\rm E} r^k r^m r^i r^s v^s_{\rm E}\right], \end{aligned} \tag{8.6}$$

$$(8.7)\qquad \overline{W}_{,ik}(\boldsymbol{x}_{\rm E}) = \sum_{{\rm A}\neq{\rm E}} \frac{GM_{\rm A}}{r^3_{\rm EA}} \left\{\left[\sum_{{\rm B}\neq{\rm A}} \frac{GM_{\rm B}}{r_{\rm AB}} - 2v^2_{\rm A} + \frac{1}{2} r^m_{\rm EA}\, a^m_{\rm A} + \frac{3}{2r^2_{\rm EA}} (r^m_{\rm EA}\, v^m_{\rm A})^2\right]\delta_{ik} + \right.$$

$$+ \frac{1}{2}(r^k_{\rm EA}\, a^i_{\rm A} + r^i_{\rm EA}\, a^k_{\rm A}) - v^i_{\rm A}\, v^k_{\rm A} + \frac{3}{r^2_{\rm EA}} (r^k_{\rm EA}\, v^i_{\rm A} + r^i_{\rm EA}\, v^k_{\rm A})(r^m_{\rm EA}\, v^m_{\rm A}) +$$

$$\left. + \frac{3}{r^2_{\rm EA}} r^i_{\rm EA}\, r^k_{\rm EA} \left[2v^2_{\rm A} - \sum_{{\rm B}\neq{\rm A}} \frac{GM_{\rm B}}{r_{\rm AB}} - \frac{1}{2} r^m_{\rm EA}\, a^m_{\rm A} - \frac{5}{2r^2_{\rm EA}} (r^m_{\rm EA}\, v^m_{\rm A})^2\right]\right\}.$$

Substituting expressions (5.17), (5.18), (5.21), (5.22), (8.6), (8.7) into (8.5) results in

$$(8.8)\qquad G^i - \overline{G}^i = \sum_{n=1}^{5} (\varphi^i_n + g^i_n)$$

with

$$(8.9)\qquad \varphi^i_1 = \frac{GM_{\rm E}}{r^3}\left(4\frac{GM_{\rm E}}{r} r^i - \dot r^k \dot r^k r^i + 4r^k \dot r^k \dot r^i\right),$$

$$(8.10)\qquad g^i_1 = \frac{GM_{\rm E}}{r^3}\left\{\left[2\dot r^k v^k_{\rm E} + v^2_{\rm E} + \frac{3}{2r^2}(r^k v^k_{\rm E})^2\right] r^i + r^k v^k_{\rm E}\, \dot r^i\right\},$$

$$(8.11)\qquad \varphi^i_2 = 4\frac{G}{r^3}\hat\omega^j_{\rm E} I^{mn}_{\rm E}\left[\varepsilon_{ijn}\left(\delta_{km} - \frac{3}{r^2} r^k r^m\right) - \right.$$

$$\left. - \varepsilon_{kjn}\left(\delta_{im} - \frac{3}{r^2} r^i r^m\right)\right]\dot r^k - 9\frac{G}{r^5}\varepsilon_{kjn}\hat\omega^j_{\rm E}\dot r^i I^{mn}_{\rm E} r^k r^m,$$

$$(8.12)\qquad g^i_2 = \frac{G}{r^3}\hat\omega^j_{\rm E}\left[\varepsilon_{ijn} v^m_{\rm E} I^{mn}_{\rm E} - \frac{3}{r^2}\varepsilon_{kjn} v^m_{\rm E} I^{mn}_{\rm E} r^k r^i + \right.$$

$$\left. + \frac{3}{r^2}(r^s v^s_{\rm E})\left(-\varepsilon_{kjn} I^{in}_{\rm E} r^k - \varepsilon_{ijn} I^{kn}_{\rm E} r^k + \frac{5}{r^2}\varepsilon_{kjn} I^{mn}_{\rm E} r^k r^m r^i\right)\right],$$

$$(8.13)\qquad \varphi^i_3 = 4\frac{G^2 M_{\rm E}}{r^6}\left(-2I^{kk}_{\rm E} r^i - 3I^{ik}_{\rm E} r^k + \frac{9}{r^2} I^{km}_{\rm E} r^k r^m r^i\right) +$$

$$+ \frac{3}{2}\frac{G}{r^5}(\dot r^n \dot r^n)\left(I^{kk}_{\rm E} r^i + 2I^{ik}_{\rm E} r^k - \frac{5}{r^2} I^{km}_{\rm E} r^k r^m r^i\right) +$$

$$+ 6\frac{G}{r^5}\dot r^i \dot r^n\left(-I^{kk}_{\rm E} r^n - 2I^{kn}_{\rm E} r^k + \frac{5}{r^2} I^{km}_{\rm E} r^k r^m r^n\right),$$

$$(8.14)\quad g_3^i = -3\frac{G}{r^5}v_E^k\dot r^k\left(I_E^{mm}r^i + 2I_E^{im}r^m - \frac{5}{r^2}I_E^{mn}r^m r^n r^i\right) - \\ -\frac{3}{2}\frac{G}{r^5}v_E^2\left(I_E^{kk}r^i + 2I_E^{ik}r^k - \frac{5}{r^2}I_E^{mn}r^m r^n r^i\right) + \\ +\frac{3}{2}\frac{G}{r^5}\dot r^i v_E^k\left(-I_E^{mm}r^k - 2I_E^{km}r^m + \frac{5}{r^2}I_E^{mn}r^m r^n r^k\right) + \\ +\frac{3}{2}\frac{G}{r^5}v_E^k v_E^m\left(I_E^{km}r^i + 2I_E^{ik}r^m - \frac{10}{r^2}I_E^{kn}r^m r^n r^i\right) + \\ +\frac{15}{4}\frac{G}{r^7}(r^k v_E^k)^2\left(-I_E^{mm}r^i - 2I_E^{im}r^m + \frac{7}{r^2}I_E^{mn}r^m r^n r^i\right),$$

$$(8.15)\quad \varphi_4^i = 2\frac{GM_E}{r}\sum_{A\neq E}\frac{GM_A}{r_{EA}^3}\left[r^i - \frac{6}{r_{EA}^2}r_{EA}^i r_{EA}^k r^k + \frac{3}{r^2 r_{EA}^2}(r^k r_{EA}^k)^2 r^i\right],$$

$$(8.16)\quad g_4^i = 5\frac{GM_E}{r^3}r^i\overline{U}(\boldsymbol{x}_E) + \frac{1}{2}\frac{GM_E}{r}\left(-a_E^i + \frac{9}{r^2}r^i r^k a_E^k\right),$$

$$(8.17)\quad \varphi_5^i = [-4\overline{U}_{,km}(\boldsymbol{x}_E)(\dot r^i + v_E^i)\dot r^m - 3\dot a_E^k(\dot r^i + v_E^i) - \overline{U}_{,km}(\boldsymbol{x}_E)(\dot r^i + v_E^i)v_E^m + \\ +\overline{U}_{,ik}(\boldsymbol{x}_E)(\dot r^m + v_E^m)(\dot r^m + v_E^m) + 4\dot{\overline{U}}{}^i_{,k}(\boldsymbol{x}_E) - 4\overline{U}{}^m_{,ik}(\boldsymbol{x}_E)(\dot r^m + v_E^m) + \\ +4\overline{U}{}^i_{,km}(\boldsymbol{x}_E)\dot r^m + \overline{W}_{,ik}(\boldsymbol{x}_E) - 4\overline{U}(\boldsymbol{x}_E)\overline{U}_{,ik}(\boldsymbol{x}_E) - 4a_E^i a_E^k]r^k,$$

$$(8.18)\quad g_5^i = 4\overline{U}{}^i_{,k}(\boldsymbol{x}_E)\dot r^k - 4\overline{U}{}^k_{,i}(\boldsymbol{x}_E)\dot r^k + a_E^i(\dot r^k\dot r^k + 2\dot r^k v_E^k) - \\ - a_E^k(4\dot r^k\dot r^i + 4\dot r^k v_E^i + \dot r^i v_E^k) - 3\dot{\overline{U}}(\boldsymbol{x}_E)\dot r^i.$$

The relativistic terms of the function (8.8) consist of ten groups.

The first group φ_1^i (8.9) depending only on the Earth mass M_E represents the Schwarzschild terms due to the spherically symmetrical component of the gravitational field of the Earth.

The second group g_1^i (8.10) depends on M_E and the velocity of the geocentre v_E^i. These terms arise from the BRS orbital motion of the Earth. For close satellites of the Earth these terms are the most significant. But being of kinematical origin these terms are expected to disappear in converting to GRS.

The terms φ_2^i (8.11) depending on the angular velocity $\hat\omega_E^i$ are the Lense-Thirring terms generated by the axial rotation of the Earth.

The BRS orbital motion of the Earth leads to the spin-orbital terms g_2^i (8.12) depending on $\hat\omega_E^i$ and v_E^i. These terms should disappear in converting to GRS.

The second-order moments of inertia of the Earth are responsible for two groups of quadrupole terms. The terms φ^i_3 (8.13) depend only on I^{ik}_{E} and the terms g^i_3 (8.14) depend both on I^{ik}_{E} and v^i_{E}. By analogy with g^i_1 and g^i_2 it should be expected that the terms g^i_3 vanish in the GRS equations.

The terms φ^i_4 (8.15) depending on masses M_{E} and M_{A} describe the physically meaningful nonlinear coupling of the gravitational fields of the Earth and the external masses.

The terms g^i_4 (8.16) depend on the superposition of the Earth mass M_{E} and the external mass potential $U(\boldsymbol{x}_{\mathrm{E}})$ with its first derivatives $U_{,k}(\boldsymbol{x}_{\mathrm{E}})$. These terms have no physical meaning and stem from the inadequate choice of RS. Converting to GRS should annul these terms.

The terms φ^i_5 (8.17) being proportional to the satellite coordinates and the second derivatives of the external mass potentials describe the tidal perturbations due to the Sun, the Moon and the planets. In converting to GRS these terms may change a little but their form will retain.

Of particular interest are the terms g^i_5 (8.18) depending on the first derivatives of the external mass potentials and the satellite velocity components $\dot{r}^i$. These terms contain explicitly the Coriolis force caused by the geodesic precession of the GRS spatial axes with respect to BRS. Due to this precession the perigee and the node of any satellite of the Earth including the Moon move with respect to the BRS spatial axes at a rate of 1.91″ per century ([37]). In fact, the terms g^i_5 may be rewritten explicitly as follows:

$$(8.19)\qquad g^i_5 = \sum_{\mathrm{A}\neq\mathrm{E}} \frac{GM_{\mathrm{A}}}{r^3_{\mathrm{EA}}}[4(v^i_{\mathrm{E}} - v^i_{\mathrm{A}})\, r^k_{\mathrm{EA}}\, \dot{r}^k - 2r^i_{\mathrm{EA}}(v^k_{\mathrm{E}} - v^k_{\mathrm{A}})\, \dot{r}^k + \\ + 4r^k_{\mathrm{EA}}(v^k_{\mathrm{E}} - v^k_{\mathrm{A}})\, \dot{r}^i + 2r^i_{\mathrm{EA}}\, v^k_{\mathrm{A}}\, \dot{r}^k + r^k_{\mathrm{EA}}\, v^k_{\mathrm{A}}\, \dot{r}^i - r^i_{\mathrm{EA}}\, \dot{r}^k \dot{r}^k + 4r^k_{\mathrm{EA}}\, \dot{r}^k \dot{r}^i],$$

where the first two terms may be presented in the vector form as

$$(8.20)\qquad \boldsymbol{g}_5 = \sum_{\mathrm{A}\neq\mathrm{E}} \frac{GM_{\mathrm{A}}}{r^3_{\mathrm{EA}}}\{3[\dot{\boldsymbol{r}} \times (\dot{\boldsymbol{r}}_{\mathrm{EA}} \times \boldsymbol{r}_{\mathrm{EA}})] + (\boldsymbol{r}_{\mathrm{EA}}\dot{\boldsymbol{r}})\, \dot{\boldsymbol{r}}_{\mathrm{EA}} + (\dot{\boldsymbol{r}}_{\mathrm{EA}}\dot{\boldsymbol{r}})\, \boldsymbol{r}_{\mathrm{EA}} + \ldots\}\,.$$

The second and the third terms here determine the perturbations depending on the orbital elements of the satellite. The first term in form of the double vector product gives the Coriolis terms describing the effect of geodesic precession. Being dependent on the first derivatives of the potentials $\overline{U}$ and $\overline{U}^i$ the terms g^i_5 will disappear in converting to GRS.

8·2. *Transformation of the* BRS *satellite equations of motion to* GRS. – For the barycentric approach it is sufficient to use only the terms $O(c^{-2})$ in the time transformation (6.2). For the geocentric approach this transformation was required completely including $O(c^{-4})$ terms. Among other things this implies that in the barycentric approach one needs to know only the components (4.5)-(4.7) of the GRS metric tensor which is sufficient to establish transformations (6.2), (6.3) within the required accuracy. Hence, the barycentric approach to derive the GRS satellite equations is in a sense more economical than the geocentric approach.

The derivation of the GRS satellite equation on the basis of eqs. (8.4), (8.8)-(8.18) involves three steps: 1) converting the acceleration $\ddot{r}^i$ in the left-hand side of eq. (8.4) to the acceleration $\mathrm{d}^2w^i/\mathrm{d}u^2$ by means of transformations (6.2), (6.3), 2) changing in eqs. (8.4), (8.8)-(8.18) from the spatial relative coordinates r^k to the GRS spatial coordinates w^k by means of (6.3), 3) transforming the Earth mass and its moments of inertia from BRS to GRS according to formulae (6.19)-(6.21).

The first step encounters no difficulties. Differentiating twice expressions (6.2), (6.3) with respect to t and substituting the results into relation

$$\frac{\mathrm{d}^2w^i}{\mathrm{d}u^2}=\frac{1}{\dot{u}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{w}^i}{\dot{u}}\right)=\frac{\ddot{w}^i}{\dot{u}^2}-\frac{\dot{w}^i}{\dot{u}^3}\ddot{u} \tag{8.21}$$

with dot denoting the differentiation relative to t one obtains

$$\begin{aligned}\frac{\mathrm{d}^2w^i}{\mathrm{d}u^2}=\ddot{r}^i+c^{-2}\Big[&2(\dot{S}+v_{\mathrm{E}}^k\dot{w}^k+a_{\mathrm{E}}^k w^k)\ddot{w}^i+\\&+\left(\frac{1}{2}v_{\mathrm{E}}^i v_{\mathrm{E}}^k+F^{ik}+D^{ik}+v_{\mathrm{E}}^k\dot{w}^i+2D^{ikm}w^m\right)\ddot{w}^k+(\ddot{S}+2a_{\mathrm{E}}^k\dot{w}^k+\dot{a}_{\mathrm{E}}^k w^k)\dot{w}^i+\\&+(v_{\mathrm{E}}^k a_{\mathrm{E}}^i+v_{\mathrm{E}}^i a_{\mathrm{E}}^k+2\dot{F}^{ik}+2\dot{D}^{ik}+2D^{ikm}\dot{w}^m)\dot{w}^k+\\&+\left(\frac{1}{2}v_{\mathrm{E}}^k\dot{a}_{\mathrm{E}}^i+a_{\mathrm{E}}^k a_{\mathrm{E}}^i+\frac{1}{2}v_{\mathrm{E}}^i\dot{a}_{\mathrm{E}}^k+\ddot{F}^{ik}+\ddot{D}^{ik}+4\dot{D}^{ikm}\dot{w}^m\right)w^k+\ddot{D}^{ikm}w^k w^m\Big].\end{aligned} \tag{8.22}$$

At the second step it is necessary to transform the Newtonian right-hand member of eq. (8.4) involving the potential $U_{\mathrm{E}}(t,\boldsymbol{x})$. Denoting

$$U_{\mathrm{E}}(t,\boldsymbol{w})=\frac{GM_{\mathrm{E}}}{\hat{r}}+GI_{\mathrm{E}}^k\frac{w^k}{\hat{r}^3}+\frac{1}{2\hat{r}^3}GI_{\mathrm{E}}^{km}\left(-\delta_{km}+\frac{3}{\hat{r}^2}w^k w^m\right)+O\left(\alpha_{\mathrm{E}}\frac{L_{\mathrm{E}}^3}{\hat{r}^3}\right) \tag{8.23}$$

the expression (5.21) for $U_{\mathrm{E}}(t,\boldsymbol{x})$ may be rewritten in the form

$$\begin{aligned}U_{\mathrm{E}}(t,\boldsymbol{x})=U_{\mathrm{E}}(t,\boldsymbol{w})-c^{-2}\cdot&\\\cdot\left[\left(\frac{1}{2}v_{\mathrm{E}}^k v_{\mathrm{E}}^m+F^{km}+D^{km}\right)w^m+D^{kmn}w^m w^n\right]&\frac{\partial U_{\mathrm{E}}(t,\boldsymbol{w})}{\partial w^k}+O(c^{-4}).\end{aligned} \tag{8.24}$$

The potential $U_{\mathrm{E}}(t,\boldsymbol{w})$ contains the BRS mass and moments of inertia of the

Earth and for this reason differs from the potential $\hat{U}_{\mathrm{E}}(u, \boldsymbol{w})$. Differentiating this relation with respect to the spatial coordinates

$$\frac{\partial U_{\mathrm{E}}(t,\boldsymbol{x})}{\partial x^i} = \frac{\partial U_{\mathrm{E}}(t,\boldsymbol{x})}{\partial u}\frac{\partial u}{\partial x^i} + \frac{\partial U_{\mathrm{E}}(t,\boldsymbol{x})}{\partial w^k}\frac{\partial w^k}{\partial x^i} \tag{8.25}$$

and substituting into eq. (8.4) one obtains

$$\begin{aligned}\ddot{r}^i = {} & \frac{\partial U_{\mathrm{E}}(t,\boldsymbol{w})}{\partial w^i} + Q_i^{(\mathrm{E})} + \overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}})\, w^k + \frac{1}{2}\overline{U}_{,ikm}(\boldsymbol{x}_{\mathrm{E}})\, w^k w^m + \\ & + c^{-2}(G^i - \overline{G}^i) - c^{-2}\left[\left(\frac{1}{2}v_{\mathrm{E}}^k v_{\mathrm{E}}^m + F^{km} + D^{km}\right) w^m + D^{kmn} w^m w^n\right] \cdot \\ & \cdot \left[\frac{\partial^2 U_{\mathrm{E}}(t,\boldsymbol{w})}{\partial w^i \partial w^k} + \overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) + \overline{U}_{,ijk}(\boldsymbol{x}_{\mathrm{E}})\, w^j\right] - c^{-2} v_{\mathrm{E}}^i \frac{\partial U_{\mathrm{E}}(t,\boldsymbol{w})}{\partial u}.\end{aligned} \tag{8.26}$$

The last step to derive the GRS satellite equations of motion is to transform the Earth mass and its moments of inertia occurring in expression (8.23) for $U_{\mathrm{E}}(t, \boldsymbol{w})$. Using (6.19)-(6.21) we obtain

$$\begin{aligned}U_{\mathrm{E}}(t,\boldsymbol{w}) = {} & \hat{U}_{\mathrm{E}}(u,\boldsymbol{w}) + c^{-2}\frac{G}{\hat{r}^3}\hat{I}_{\mathrm{E}}^{mn}\left[\frac{1}{2}\left(v_{\mathrm{E}}^m - \frac{3}{\hat{r}^2} v_{\mathrm{E}}^k w^k w^m\right) v_{\mathrm{E}}^n + \right. \\ & \left. + \varepsilon_{kjn}\,\hat{\omega}_{\mathrm{E}}^j w^k \left(- v_{\mathrm{E}}^m + \frac{3}{\hat{r}^2} v_{\mathrm{E}}^s w^s w^m\right) - \frac{3}{\hat{r}^2} F^{km} w^k w^n\right].\end{aligned} \tag{8.27}$$

Now the difference of the first and the last terms entering into eq. (8.26) will be

$$\begin{aligned}& \frac{\partial U_{\mathrm{E}}(t,\boldsymbol{w})}{\partial w^i} - c^{-2} v_{\mathrm{E}}^i \frac{\partial U_{\mathrm{E}}(t,\boldsymbol{w})}{\partial u} = \\ & = \hat{U}_{\mathrm{E},i} + c^{-2}\frac{G}{\hat{r}^3} I_{\mathrm{E}}^{mn}\left[\frac{3}{2\hat{r}^2}\left(- v_{\mathrm{E}}^m w^i - v_{\mathrm{E}}^i w^m - \delta_{im} v_{\mathrm{E}}^k w^k + \frac{5}{\hat{r}^2} v_{\mathrm{E}}^k w^k w^m w^i\right) v_{\mathrm{E}}^n + \right. \\ & \left. + \varepsilon_{kjn}\,\hat{\omega}_{\mathrm{E}}^j\left(- \delta_{ik} + \frac{3}{\hat{r}^2} w^i w^k\right) v_{\mathrm{E}}^m + \frac{3}{\hat{r}^2}\varepsilon_{kjn}\,\hat{\omega}_{\mathrm{E}}^j v_{\mathrm{E}}^s w^s\left(\delta_{ik} w^m + \delta_{im} w^k - \frac{5}{\hat{r}^2} w^i w^k w^m\right)\right].\end{aligned} \tag{8.28}$$

Combining expressions (8.22), (8.26), (8.28) one obtains the GRS satellite equations of motion in the form

$$\begin{aligned}\frac{\mathrm{d}^2 w^i}{\mathrm{d}u^2} = {} & \hat{U}_{\mathrm{E},i} + Q_i^{(\mathrm{E})} + \overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}})\, w^k + \frac{1}{2}\overline{U}_{,ikm}(\boldsymbol{x}_{\mathrm{E}})\, w^k w^m + \\ & + c^{-2}\Phi^i + O(\hat{r}^3) + O(c^{-4}),\end{aligned} \tag{8.29}$$

where the relativistic right-hand member is determined by

$$(8.30)\quad \Phi^i = G^i - \overline{G}^i + 2(\dot{S} + v^k_{\rm E}\dot{w}^k + a^k_{\rm E} w^k)\Big[\hat{U}_{{\rm E},i} + \overline{U}_{,im}(\boldsymbol{x}_{\rm E})\, w^m + \frac{1}{2}\overline{U}_{,imn}(\boldsymbol{x}_{\rm E})\, w^m w^n\Big] + \Big(\frac{1}{2}v^i_{\rm E} v^k_{\rm E} + F^{ik} + D^{ik} + v^k_{\rm E}\dot{w}^i + 2D^{ijk} w^j\Big)\cdot [\hat{U}_{{\rm E},k} + \overline{U}_{,km}(\boldsymbol{x}_{\rm E})\, w^m] + (\ddot{S} + 2a^k_{\rm E}\dot{w}^k + \dot{a}^k_{\rm E} w^k)\dot{w}^i + (v^k_{\rm E} a^i_{\rm E} + v^i_{\rm E} a^k_{\rm E} + 2\dot{F}^{ik} + 2\dot{D}^{ik} + 2D^{ikm}\dot{w}^m)\dot{w}^k + \Big(\frac{1}{2}v^k_{\rm E}\dot{a}^i_{\rm E} + a^k_{\rm E} a^i_{\rm E} + \frac{1}{2}v^i_{\rm E}\dot{a}^k_{\rm E} + \ddot{F}^{ik} + \ddot{D}^{ik} + 4\dot{D}^{ikm}\dot{w}^m\Big) w^k + \ddot{D}^{ikm} w^k w^m - \Big[\Big(\frac{1}{2}v^k_{\rm E} v^m_{\rm E} + F^{km} + D^{km}\Big) w^m + D^{kmn} w^m w^n\Big][\hat{U}_{{\rm E},ik} + \overline{U}_{,ik}(\boldsymbol{x}_{\rm E})] + \frac{3}{2}\frac{G}{\hat{r}^5}\hat{I}^{mn}_{\rm E}\Big(-v^m_{\rm E} w^i - v^i_{\rm E} w^m - \delta_{im} v^k_{\rm E} w^k + \frac{5}{\hat{r}^2} v^k_{\rm E} w^k w^m w^i\Big) v^n_{\rm E} + 3\frac{G}{\hat{r}^5}\varepsilon_{kjn}\hat{\omega}^j_{\rm E} v^s_{\rm E} w^s \hat{I}^{mn}_{\rm E}\Big(\delta_{ik} w^m + \delta_{im} w^k - \frac{5}{\hat{r}^2} w^i w^k w^m\Big) + \frac{G}{\hat{r}^3}\varepsilon_{kjn}\hat{\omega}^j_{\rm E} v^m_{\rm E}\hat{I}^{mn}_{\rm E}\Big(-\delta_{ik} + \frac{3}{\hat{r}^2} w^i w^k\Big).$$

This expression enables one to see the contribution of each term of transformations (6.2), (6.3) into the right-hand member of the GRS satellite equations. Substituting (6.22), (6.26), (6.27) and (6.30) into this expression one has

$$(8.31)\qquad \Phi^i = G^i - \overline{G}^i - \sum_{n=1}^{5} g^i_n + \Delta\varphi^i_5,$$

where the functions g^i_n are determined by relations (8.10), (8.12), (8.14), (8.16), (8.18) and

$$(8.32)\quad \Delta\varphi^i_5 = \Big[2\dot{a}^m_{\rm E}\dot{w}^m\delta_{ik} + \ddot{\overline{U}}(\boldsymbol{x}_{\rm E})\delta_{ik} + 2v^i_{\rm E}\dot{a}^k_{\rm E} + a^i_{\rm E} a^k_{\rm E} - v^k_{\rm E}\dot{a}^i_{\rm E} + 3\dot{a}^k_{\rm E}\dot{w}^i - 2\dot{a}^i_{\rm E}\dot{w}^k - 2\dot{\overline{U}}^i_{,k}(\boldsymbol{x}_{\rm E}) + 2\dot{\overline{U}}^k_{,i}(\boldsymbol{x}_{\rm E}) - \frac{1}{2}v^k_{\rm E} v^m_{\rm E}\overline{U}_{,im}(\boldsymbol{x}_{\rm E}) + \frac{1}{2}v^i_{\rm E}\dot{v}^m_{\rm E}\overline{U}_{,km}(\boldsymbol{x}_{\rm E}) + v^2_{\rm E}\overline{U}_{,ik}(\boldsymbol{x}_{\rm E}) + 2v^m_{\rm E}\dot{w}^m\overline{U}_{,ik}(\boldsymbol{x}_{\rm E}) + v^m_{\rm E}\dot{w}^i\overline{U}_{,km}(\boldsymbol{x}_{\rm E}) + F^{km}\overline{U}_{,im}(\boldsymbol{x}_{\rm E}) + F^{im}\overline{U}_{,km}(\boldsymbol{x}_{\rm E}) + 2\dot{\overline{U}}(\boldsymbol{x}_{\rm E})\overline{U}_{,ik}(\boldsymbol{x}_{\rm E})\Big] w^k + O(\hat{r}^2).$$

Comparing (8.31) and (8.8) we see that all the terms g^i_n ($n=1, \ldots, 5$) cancel out confirming the reasoning of subsect. 8·1. As a result the relativistic right-hand member of the GRS satellite equations takes the simple form

$$\Phi^i = \sum_{n=1}^{5} \varphi^i_n + \Delta\varphi^i_5 = \sum_{n=1}^{6} \Phi^i_n \tag{8.33}$$

with

$$\Phi^i_n = \varphi^i_n \qquad (n=1, 2, 3, 4)\,, \tag{8.34}$$

$$\begin{aligned}\Phi^i_5 + \Phi^i_6 = \varphi^i_5 + \Delta\varphi^i_5 = \Big[& 4\dot{w}^m v^m_{\mathrm{E}} \overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) - 4\dot{w}^m \overline{U}^m_{,ik}(\boldsymbol{x}_{\mathrm{E}}) - \\ & - 4\dot{w}^m v^i_{\mathrm{E}} \overline{U}_{,km}(\boldsymbol{x}_{\mathrm{E}}) + 4\dot{w}^m \overline{U}^i_{,km}(\boldsymbol{x}_{\mathrm{E}}) + 2\dot{a}^m_{\mathrm{E}} \dot{w}^m \delta_{ik} - 2\dot{a}^i_{\mathrm{E}} \dot{w}^k - 4\dot{w}^m \dot{w}^i \overline{U}_{,km}(\boldsymbol{x}_{\mathrm{E}}) + \\ & + \dot{w}^m \dot{w}^m \overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) + F^{im} \overline{U}_{,km}(\boldsymbol{x}_{\mathrm{E}}) + F^{km} \overline{U}_{,im}(\boldsymbol{x}_{\mathrm{E}}) + \ddot{\overline{U}}(\boldsymbol{x}_{\mathrm{E}})\, \delta_{ik} - \\ & - 3a^i_{\mathrm{E}} a^k_{\mathrm{E}} - 2\overline{U}(\boldsymbol{x}_{\mathrm{E}}) \overline{U}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) + \overline{W}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) + 2\dot{\overline{U}}{}^i_{,k}(\boldsymbol{x}_{\mathrm{E}}) + 2\dot{\overline{U}}{}^k_{,i}(\boldsymbol{x}_{\mathrm{E}}) - \\ & - v^k_{\mathrm{E}} \dot{a}^i_{\mathrm{E}} - v^i_{\mathrm{E}} \dot{a}^k_{\mathrm{E}} - 4v^m_{\mathrm{E}} \overline{U}^m_{,ik}(\boldsymbol{x}_{\mathrm{E}}) - \frac{1}{2} v^i_{\mathrm{E}} v^m_{\mathrm{E}} \overline{U}_{,km}(\boldsymbol{x}_{\mathrm{E}}) - \\ & - \frac{1}{2} v^k_{\mathrm{E}} v^m_{\mathrm{E}} \overline{U}_{,im}(\boldsymbol{x}_{\mathrm{E}}) + 2v^2_{\mathrm{E}} \dot{\overline{U}}_{,ik}(\boldsymbol{x}_{\mathrm{E}}) \Big] w^k + O(\hat{r}^2)\,. \end{aligned} \tag{8.35}$$

Thus, we have obtained again eqs. (7.3) with the values (7.4)-(7.13). It remains to give Φ^i_5 and Φ^i_6 in explicit manner, *i.e.*

$$\begin{aligned}\Phi^i_5 = \sum_{\mathrm{A}\neq\mathrm{E}} \frac{GM_{\mathrm{A}}}{r^3_{\mathrm{EA}}} w^k \Big\{ \Big[& - 6\dot{w}^m (v^m_{\mathrm{E}} - v^m_{\mathrm{A}}) + \\ & + \frac{6}{r^2_{\mathrm{EA}}} \dot{w}^m r^m_{\mathrm{EA}} r^n_{\mathrm{EA}} (v^n_{\mathrm{E}} - v^n_{\mathrm{A}}) - \dot{w}^m \dot{w}^m \Big] \delta_{ik} + 6\dot{w}^k (v^i_{\mathrm{E}} - v^i_{\mathrm{A}}) - \\ & - \frac{6}{r^2_{\mathrm{EA}}} \dot{w}^k r^i_{\mathrm{EA}} r^m_{\mathrm{EA}} (v^m_{\mathrm{E}} - v^m_{\mathrm{A}}) + \frac{12}{r^2_{\mathrm{EA}}} r^i_{\mathrm{EA}} r^k_{\mathrm{EA}} (v^m_{\mathrm{E}} - v^m_{\mathrm{A}}) \dot{w}^m - \\ & - \frac{12}{r^2_{\mathrm{EA}}} r^k_{\mathrm{EA}} r^m_{\mathrm{EA}} (v^i_{\mathrm{E}} - v^i_{\mathrm{A}}) \dot{w}^m + 4\dot{w}^i \dot{w}^k - \\ & - \frac{12}{r^2_{\mathrm{EA}}} r^k_{\mathrm{EA}} r^m_{\mathrm{EA}} \dot{w}^m \dot{w}^i + \frac{3}{r^2_{\mathrm{EA}}} r^i_{\mathrm{EA}} r^k_{\mathrm{EA}} \dot{w}^m \dot{w}^m \Big\}, \end{aligned} \tag{8.36}$$

(8.37) $$\Phi_6^i = \sum_{A\neq E} \frac{GM_A}{r_{EA}^3} w^k \Big\{\Big[- r_{EA}^m a_E^m + \frac{3}{2} r_{EA}^m a_A^m - 3(v_E^m - v_A^m)(v_E^m - v_A^m) + $$

$$+ \frac{3}{r_{EA}^2}(r_{EA}^m v_E^m)^2 - \frac{6}{r_{EA}^2}(r_{EA}^m v_E^m)(r_{EA}^n v_A^n) + \frac{9}{2r_{EA}^2}(r_{EA}^m v_A^m)^2 + \frac{G(M_E + 2M_A)}{r_{EA}} +$$

$$+ \sum_{B\neq A,E} RGM_B\left(\frac{1}{r_{AB}} + \frac{2}{r_{EB}}\right)\Big]\delta_{ik} + \frac{3}{r_{EA}^2} r_{EA}^m (r_{EA}^k F^{im} + r_{EA}^i F^{km}) + 3(v_E^i - v_A^i)(v_E^k - v_A^k)$$

$$+ \frac{3}{r_{EA}^2} r_{EA}^i r_{EA}^m \left(-\frac{3}{2} v_E^k v_E^m + v_E^k v_A^m + 2 v_E^m v_A^k - v_A^k v_A^m \right) +$$

$$+ \frac{6}{r_{EA}^2} r_{EA}^i r_{EA}^k (v_E^m - v_A^m)(v_E^m - v_A^m) + \frac{3}{r_{EA}^2} r_{EA}^k r_{EA}^m \Big(-\frac{3}{2} v_E^i v_E^m + v_E^i v_A^m +$$

$$+ 2 v_E^m v_A^i - v_A^i v_A^m \Big) - \frac{15}{2r_{EA}^4} r_{EA}^i r_{EA}^k (r_{EA}^m v_A^m)^2 - \frac{3}{2} r_{EA}^k a_A^i - \frac{3}{2} r_{EA}^i a_A^k -$$

$$- \frac{3}{2r_{EA}^2} r_{EA}^i r_{EA}^k r_{EA}^m a_A^m - \frac{3}{r_{EA}^3} G(M_E + 3M_A) r_{EA}^i r_{EA}^k -$$

$$- 3 r_{EA}^i \sum_{B\neq A,E} GM_B \left(\frac{r_{EA}^k}{r_{EA}^2 r_{AB}} + \frac{2 r_{EA}^k}{r_{EA}^2 r_{EB}} + \frac{r_{EB}^k}{r_{EB}^3} \right)\Big\}.$$

The sum of Φ_5^i and Φ_6^i results in (8.35).

It may be noted that assuming the Earth to be an oblate spheroid rotating with the constant angular velocity $\hat{\omega}$ around the polar axis and adopting the instant equatorial RS $w^1 = x$, $w^2 = y$, $w^3 = z$ one has

(8.38) $$\hat{I}_E^{11} = \hat{I}_E^{22} = \frac{1}{2}C, \quad \hat{I}_E^{33} = A - \frac{1}{2}C, \quad Q = G(A - C),$$

(8.39) $$\hat{\omega}_E^1 = \hat{\omega}_E^2 = 0, \quad \hat{\omega}_E^3 = \hat{\omega}.$$

A and C are the principal moments of inertia of the Earth, Q is proportional to the Earth oblateness. Denoting by $\boldsymbol{s} = (0, 0, 1)$ the unit vector along the polar axis the expressions for F_1^i, F_3^i, Φ_2^i, Φ_3^i take a simpler form

(8.40) $$\boldsymbol{F}_1 = \frac{3Q}{\hat{r}^5}\left[\frac{1}{2}\left(1 - \frac{5z^2}{\hat{r}^2}\right)\boldsymbol{w} + z\boldsymbol{s}\right],$$

(8.41) $$\boldsymbol{F}_3 = -\frac{1}{2}(G\hat{M}_E)^{-1} Q \,\mathrm{grad}\, \overline{U}_{,33}(\boldsymbol{x}_E),$$

(8.42) $$\boldsymbol{\Phi}_2 = \frac{2G}{\hat{r}^3} C\hat{\omega}\left[\dot{\boldsymbol{w}} \times \boldsymbol{s} + \frac{3z}{\hat{r}^2}(\boldsymbol{w} \times \dot{\boldsymbol{w}})\right],$$

$$\Phi_3 = \frac{Q}{\hat{r}^5}\left\{\left[4\left(-2+9\,\frac{z^2}{\hat{r}^2}\right)\frac{G\hat{M}_{\mathrm{E}}}{\hat{r}} + \frac{3}{2}\left(1-\frac{5z^2}{\hat{r}^2}\right)\dot{\boldsymbol{w}}^2\right]\boldsymbol{w} + 3\left(-\frac{4G\hat{M}_{\mathrm{E}}}{\hat{r}} + \dot{\boldsymbol{w}}^2\right)z\boldsymbol{s} - G\left[\left(1-\frac{5z^2}{\hat{r}^2}\right)(\boldsymbol{w}\,\dot{\boldsymbol{w}}) + 2z\dot{z}\right]\dot{\boldsymbol{w}}\right\}. \tag{8.43}$$

9. – Conclusion.

This paper has pursued two objectives, *i.e.* 1) to construct the harmonic, dynamically nonrotating reference system for any body of the solar system and 2) to derive the equations of motion of test particles in the vicinity of the given body using this RS. To be specific, the elaborated technique is applied to the Earth and its satellites. The main results of the paper are the GRS metric (4.1)-(4.8) with the transformation laws (6.2), (6.3) to the BRS coordinates and eqs. (7.3) or (8.29) for the Earth satellite motion. Needless to say, for practical purposes the Newtonian perturbation F^i_1 due to the nonsphericity of the Earth should be taken more accurately. The Newtonian perturbation F^i_3 defined by (7.7) or (8.41) is of particular interest. This perturbing acceleration does not depend on satellite orbit and is of the order $F^i_3 \sim GM_\odot L^2_{\mathrm{E}}\alpha_{\mathrm{E}}/R^4_{\mathrm{E}} \sim 3.7\cdot 10^{-12}\,\mathrm{cm/s^2}$. For LAGEOS the relativistic accelerations $c^{-2}\Phi^i_n$ $(n=1,\dots,6)$ range by the order from $1.0\cdot 10^{-7}\,\mathrm{cm/s^2}$ $(n=1)$ to $1.7\cdot 10^{-14}\,\mathrm{cm/s^2}$ $(n=4)$.

The methods considered above can be applied to derive the relativistic equations of motion of the Moon or a satellite of any planet.

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● RIASSUNTO (*)

Si sviluppa una teoria relativistica per costruire un sistema di riferimento armonico non rotante RS. La teoria permette di produrre l'RS celeste per la dinamica del sistema solare trascurando il campo gravitazionale della galassia. Si presta una particolare attenzione all'RS baricentrico (BRS) con l'origine nel baricentro del sistema solare e all'RS geocentrico (GRS) con l'origine nel geocentro. Si presume con ciò che le velocità dei corpi siano ridotte rispetto alla velocità della luce e il campo gravitazionale sia debole dovunque. L'RS specifico e il campo gravitazionale sono descritti dal tensore metrico che si trova mediante approssimazioni newtoniane dalle equazioni del campo di Einstein con date condizioni di confine. Le coordinate BRS coprono tutto lo spazio del sistema solare. Le coordinate GRS sono inizialmente limitate nello spazio dall'orbita della Luna. Si determina la relazione tra BRS e GRS con la tecnica di adattamento asintotico. Le formule di trasformazione esplicite permettono di prolungare le coordinate GRS oltre

l'orbita lunare per coprire addirittura tutto lo spazio del sistema solare. Sono state derivate le equazioni GRS del moto del satellite della Terra. I membri destri relativistici di queste equazioni contengono perturbazioni terrestri di Schwarzschild, Lense-Thirring e del quadrupolo, nonché perturbazioni di marea dovute al Sole, alla Luna e ai pianeti maggiori. Sono dedotte le equazioni con due diverse tecniche. La prima implica l'applicazione del principio geodesico alla metrica GRS. La seconda è basata sulla trasformazione delle equazioni di moto del satellite BRS in equazioni GRS. Entrambe le tecniche risultano nelle stesse espressioni finali.

(*) *Traduzione a cura della Redazione.*

Релятивистские системы отсчета и движение пробных тел в окрестности Земли.

Резюме (*). — Развивается релятивистская теория для конструирования невращающихся гармонических систем отсчета. Предложенная теория позволяет полчить небесную систему отсчета для солнечной системы, пренебрегая гравитационным полем Галактики. Особое внимание уделяется барицентрической системе отсчета с началом в барицентре солнечной системы и геоцентрической системе отсчета с началом в геоцентре. Предполагается, что скорости тел малы по сравнению со скоростью света и грваитационное поле является слабым. Специальная система отсчета и гравитационное поле описываются с помощью метрического тензора и получаются с помощью ньютоновых приближений из уравнений поля Эйнштейна с заданными граничными условиями. Координаты барицентрической системы отсчета покрывают все пространство солнечной системы. Координаты геоцентрической системы отсчета первоначально ограничены в пространстве орбитой Луны. Устанавливается связь между барицентрической и геоцентрической системами отсчета, используя технику асимптотического согласования. Формулы преобразования позволяют пролонгировать координаты геоцентрической системы отсчета за пределы лунной орбины и покрыть все пространство солнечной системы. Выводятся уравнения движения спутников Земли в геоцентрической системе отсчета. Релятивистские члены в правых частях этих уравнений содержат возмущения Шварцшильда, Ленца-Тирринга и квадрупольные земпные возмущения, а также приливные возмущения, обусловленные Солнцем, Луной и большими планетами. Уравнения выводятся с помощью двух различных способов. В первом подходе применяется геодезический принцип к метрике геоцентрической системы отсчета. Второй способ основан на преобразовании уравнений движений спутников в барицентрической системе отсчета в уравнения движения в геоцентрической системе отсчета. Оба подхода дают одинаковый конечный результат.

(*) *Переведено редакцией.*