THE BAROCLINIC RESIDUAL CIRCULATION IN SHALLOW SEAS

I. THE HYDRODYNAMIC MODELS

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Abstract

There are two different modeling approaches which have been proposed and utilized to derive residual currents: the first approach is equivalent to deducing residual currents from current-meter records using filtering techniques or time averages of time-series records to remove tidal variations; in the second approach, filters or time averages over several tidal cycles are applied to the hydrodynamic equations to generate the governing equations for residual circulation. Based on the latter, both the baroclinic dynamic models of residual circulation are proposed, of which one is a two-dimensional transport model and the other is a three-dimensional model with variable eddy viscosity. The two-dimensional transport model is a direct generalization from the barotropic model of residual circulation presented by Nihoul and Randy (1975) and Heaps (1978) to the baroclinic model. In the three-dimensional model with variable eddy viscosity, using a Sturm-Liouville system adopted in the reference [5], the nondimensional problem for residual circulation reduces to the nondimensional problem of the elevation and the expression of residual currents.

It should be pointed out that both the baroclinic models developed in the present paper are confined to describe the Eulerian residual circulation only.

Circulation in continental shelf seas, semi-enclosed shallow seas, gulfs or bays and tidal estuaries is to be driven by tides, storm or wind, density and open boundary forces, with different spatial and temporal scales. Tides, sometimes storm surges, also dominate the circulation in shallow seas, such as the Bohai Sea and the Yellow Sea. However, the smaller longer-term residual currents, of which the order of magnitude is typically smaller than those of tides and storm surges, play a key role in the understanding of the transport phenomena of long-term processes, for example in an ecological system. Therefore, the studies on the dynamics of residual circulation are of considerable interest in both practice and theory. In recent years, many investigations on residual currents have been made e.g. [1-3,7-15,17,19]. Nevertheless, studies in this field are, for the most part, to treat the barotropic two-dimensional vertically-integrated or transport hydrodynamic models. It should be pointed out that the baroclinic effect on residual circulation, or the density-driven component of residual currents, might be necessary, at least not trivial, in such water areas as the northern reach of San Francisco Bay[2], the Bay of Fundy and Gulf of Maine[7] or the Bohai Sea and the Yellow Sea, or generally, the East China Sea[4,16,18]. On the other hand, there is a need to estimate the three-dimensional residual current in some physical processes. For example, the estimation of sediment fluxes requires that the vertical velocity profile at each location be known because the concentration of the sediment in the water column varies with depth. In this paper, both the baroclinic dynamic models of residual circulation are proposed, one of which is a two-dimensional transport model and the other is a three-dimensional model with variable eddy viscosity.

As pointed out by Cheng et al[1)], there are two different modeling approaches that have been proposed and utilized to derive residual currents. The first approach is equivalent to deducing residual currents from current-meter records using filtering techniques or time averages of time-series records to remove tidal variations[9,15]. In the second approach, filters or time averages over several tidal cycles are applied to the hydrodynamic equations to generate the governing equations for residual circulation[8,10]. Both the baroclinic models for residual circulation proposed in the present paper are derived just based on the second approach. In this paper, we put emphasis on Eulerian residual currents.

FORMULATION

Based upon the nondimensional dynamic problem for barotropic shallow seas presented in the reference [5], a nondimensional dynamic problem for baroclinic shallow seas is proposed as follows:

$$\nabla \cdot \vec{V} + \frac{\partial w}{\partial z} = 0,$$

$$\rho\left(k_t \frac{\partial}{\partial t} + k_c \vec{e}_3 \times\right)\vec{V} + \rho k_t k_n \left(\vec{V}\cdot\nabla + w\frac{\partial}{\partial z}\right)\vec{V} = -\nabla(\zeta - k_a\zeta_a - k_T\zeta_T)$$

$$-k_\rho \nabla \int_z^0 \rho' dz + k_v \frac{\partial}{\partial z}\left(\nu \frac{\partial \vec{V}}{\partial z}\right),$$

$$R_0\left(\vec{V}\cdot\nabla + w\frac{\partial}{\partial z}\right)\rho' = \frac{E_k}{P_r}\frac{\partial}{\partial z}\left(\gamma \frac{\partial \rho'}{\partial z}\right); \qquad (1)$$

$z = \chi\zeta$:

$$w = \left(k_r \frac{\partial}{\partial t} + \chi \vec{V}\cdot\nabla\right)\zeta,$$

$$\nu \frac{\partial \vec{V}}{\partial z} = k_\tau \vec{\tau}_a,$$

$$\nu \frac{\partial \rho'}{\partial z} = k_t \Gamma_a;$$

$z = -h$:

$$\vec{V} = w = 0,$$

$$\left(\nu \frac{\partial \vec{V}}{\partial z} = k_b \vec{\tau}_b\right);$$

along the shore boundary C_1:

$$\vec{n}_0 \cdot \int_{-h}^{\chi\zeta} \vec{V} dz = 0,$$

1) Cheng, R. T. and V. Casulli, 1982. On Lagrangian residual currents with application in south San Francisco Bay, California. (personal communication)

along the open boundary C_2:

$$\vec{n}_0 \cdot \int_{-h}^{\chi\zeta} \vec{V} dz = Q$$

or

$$\zeta = \mathscr{S};$$

where $\chi=\zeta_0/D$, $k_i=S_t E_u^{-1}$, $k_n=S_t^{-1}$, $k_c=R_o^{-1}E_u^{-1}$, $k_a=\zeta_{a0}/\zeta_0$, $k_T=\zeta_{T0}/\zeta_0$, $k_\rho=\varepsilon/\chi$, $k_v=E_k R_o^{-1}E_u^{-1}$, $k_r=\chi S_t$, $k_\tau=\frac{\tau_{a0}D}{\rho_0\nu_0 V_0}$

$\varepsilon=\rho_0'/\rho_0$, $k_b=\frac{\tau_{b0}D}{\rho_0\nu_0 V_0}$, $k_\gamma=\frac{\Gamma_{a0}D}{\rho_0'\gamma_0}$;

the Rossby number $R_o=\frac{V_0}{fL}$, the Ekman number $E_k=\frac{\nu_0}{fD^2}$,

the Eular number $E_u=\frac{g\zeta_0}{V_0^2}$, the Strouhal number $S_t=\frac{L}{V_0 T}$,

the Prandtl number $P_r=\frac{\nu_0}{\gamma_0}$,

in the above, $\nabla=\vec{e}_1\frac{\partial}{\partial x}+\vec{e}_2\frac{\partial}{\partial y}$, (x,y,z) constitute an f-coordinates at the right-hand side, the plane (x,y) coincides with the undisturbed water surface, z is positive upward, $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ denote three unit vectors of coordinates; t denotes time; $\vec{V}=\vec{e}_1 u+\vec{e}_2 v$, u and v are the components of the current in x and y direction respectively and w denotes the vertical component of current; ζ is the elevation measured from the undisturbed sea surface; the density $\rho=1+\varepsilon\rho'$, ρ' is the variation of density, ρ_0 the reference constant density; f is the Coriolis parameter; g is the gravitational acceleration; ζ_a and ζ_T represent the effects of atmosphere pressure and tide-generating force, respectively; $\vec{\tau}_a=\vec{e}_1\tau_{ax}+\vec{e}_2\tau_{ay}$, τ_{ax}, τ_{ay} denote the components of wind stress at the sea surface in the x, y directions respectively; Γ_a denotes the density flux through the sea surface, h denotes the depth; $\vec{\tau}_b=\vec{e}_1\tau_{bx}+\vec{e}_2\tau_{by}$ denotes the bottom friction; $\vec{n}_0=\vec{e}_1\cos\alpha_x+\vec{e}_2\cos\alpha_y$, $\cos\alpha_x$ and $\cos\alpha_y$ are the direction-cosines of the boundary normal; Q and $\mathscr{S}$ denote the normal volume transport and the water elevation along the opon boundary, respectively; ν and γ denote the eddy viscosity and diffusion coefficient respectively; D and L denote the vertical and horizontal scale respectively; T denotes the time scale; V_0 denotes the characteristic horizontal current velocity, ζ_0 denotes the characteristic amplitude of the elevation, ρ_0' is the characteristic quantity of ρ'; ν_0 and γ_0 denote the characteristic eddy viscosity and diffusion coefficient, respectively; τ_{a0}, τ_{b0}, ζ_{a0}, ζ_{T0} and Γ_{a0} denote the characteristic quantities of $\vec{\tau}_a$, $\vec{\tau}_b$, ζ_a, ζ_T and Γ_a respectively.

A scale analysis is made and we take the typical scales in Bohai Sea dynamics as the characteristic quantities: $D\sim 2\times 10^3$, $L\sim 4\times 10^7$, $g\sim 10^3$, $f\sim 10^{-4}$, $\chi\sim 10^{-1}$, $\tau_{c0}\sim 10$ (for storm surges) or $O(\tau_{a0})\leqslant 1$ (for longer-term residual currents), $\varepsilon\sim 5\times 10^{-3}$ (in summer) (c.g.s.). $\partial\rho'/\partial t=0$ will be explained later.

In view of the fact that the effect of the eddy viscosity on the motion in shallow seas are essential, we set $k_v = 1$; and as mentioned above, tides dominate the motion, so $k_r = 1$ and $k_i = 1$. Thus, we have $k_i k_n = k_n = S_t^{-1} = \chi$ and $E_k R_o^{-1} = E_u = S_t = \chi^{-1}$, and then the non-dimensional problem (1) is reduced to the following

$$\nabla \cdot \vec{V} + \frac{\partial w}{\partial z} = 0,$$

$$\rho\left(\frac{\partial}{\partial t} + k_c \vec{e}_3 \times\right)\vec{V} + \chi\left(\vec{V} \cdot \nabla + w\frac{\partial}{\partial z}\right)\vec{V} = -\nabla\,(\zeta - k_a\zeta_a - k_T\zeta_T)$$

$$-k_\rho \nabla \int_z^0 \rho' dz + \frac{\partial}{\partial z}\left(\nu \frac{\partial \vec{V}}{\partial z}\right),$$

$$\left(\vec{V} \cdot \nabla + w\frac{\partial}{\partial z}\right)\rho' = \frac{1}{\chi P_r}\frac{\partial}{\partial z}\left(\gamma \frac{\partial \rho'}{\partial z}\right); \qquad (2)$$

$z = \chi\zeta$:

$$w = \left(\frac{\partial}{\partial t} + \chi \vec{V} \cdot \nabla\right)\zeta,$$

$$\nu \frac{\partial \vec{V}}{\partial z} = k_\tau \vec{\tau}_a,$$

$$\gamma \frac{\partial \rho'}{\partial z} = k_\gamma \Gamma_a;$$

$z = -h$

$$\vec{V} = w = 0,$$

$$\left(\nu \frac{\partial \vec{V}}{\partial z} = k_b \vec{\tau}_b\right);$$

along the shore boundary C_1:

$$\vec{n}_0 \cdot \int_{-h}^{\chi\zeta} \vec{V} dz = 0,$$

along the open boundary C_2:

$$\vec{n}_0 \cdot \int_{-h}^{\chi\zeta} \vec{V} dz = Q$$

or

$$\zeta = \mathscr{S};$$

where $T = \frac{L}{\sqrt{gD}}$, $V_0 = \chi\sqrt{gD}$, $\nu_0 = \frac{D^2}{T}$ $\left(\text{or } \nu_0/D^2 = \frac{1}{T}\right)$, $k_c = fT$.

The scale analysis is continued as follows: The significant influence of Coriolis force on the dynamics in such shallow seas as the Bohai Sea is derived from $k_c = fT = f\frac{L}{\sqrt{gD}} =$

$\frac{4}{\sqrt{2}}\sim 1$. Whereas $k_\tau=\frac{\tau_{a0}/\rho_0}{(\nu_0/D^2)\,V_0D}=\frac{\tau_{a0}}{\rho_0}\frac{k_e}{fD\chi\sqrt{gD}}=1$ for storm surges while $O(k_\tau)<1$ as the seasonal mean winds, $O(\bar{\tau}_{a0})\leqslant 1$ are taken, the tides are severely affected by storm surges but almost not by the seasonal mean winds. However, since the wind-driven current $V_0=\frac{\bar{\tau}_{a0}}{\rho_0}\frac{1}{(\nu_0/D^2)D}\sim 10$ as the seasonal mean wind of order of magnitude of $\bar{\tau}_{a0}\sim 1$ is taken, it is expected that the residual circulation is influenced by the seasonal mean wind. It is derived from $O(k_\rho)=O\left(\frac{\varepsilon}{\chi}\right)=5\times 10^{-3}$ that the baroclinic force has the trivial effect on tides and storm surges, namely, there is the barotropic dynamics for tides and storm surges. Therefore the solutions of tides and/or storm surges can be obtained by using the dynamic problem (2) with $\rho'=0$. It should be emphasized to point out that the density effect on the residual circulation might be really essential in view of the density-driven current $O(V_0)=O(\chi\sqrt{gD})=O(\varepsilon k_\rho^{-1}\sqrt{gD})=O(\varepsilon\sqrt{gD})=10$. Thus it can be seen that the temporal scale of baroclinity, or ρ', is the same as that of residual currents and, of course, much larger than that of tides and storm surges. Since the very slowly time-varying residual current field can be treated as a steady-state one[10], we suppose reasonably ρ' to be independent of time and $\partial\rho'/\partial t$ dropped. The terms with $(\chi\varepsilon)$ have been neglected for $O(\chi)<1$ and $O(\varepsilon)<1$.

As pointed out by N. Heaps[8], studies in this field have, for the most part, employed the two-dimensional vertically-integrated, or transport, hydrodynamic equations. An attempt will be made in the next section to generalize simply the two-dimensional problem residual currents from the barotropic to the baroclinic. Here, the nondimensional volume-transport problem is derived from (2) as follows:

$$\frac{\partial\zeta}{\partial t}+\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0,$$

$$\frac{\partial\mathscr{U}}{\partial t}+\chi\left\{\frac{\partial}{\partial x}\left(\frac{U^2}{h+\chi\zeta}\right)+\frac{\partial}{\partial y}\left(\frac{UV}{h+\chi\zeta}\right)\right\}-k_e\mathscr{V}=-(h+\chi\zeta)\frac{\partial}{\partial x}(\zeta-k_a\zeta_a-k_T\zeta_T)$$

$$-k_\rho\int_{-h}^{\chi\zeta}\frac{\partial}{\partial x}\int_z^0\rho'dz'dz+k_\tau\tau_{ax}-k_b\tau_{bx},$$

$$\frac{\partial\mathscr{V}}{\partial t}+\chi\left\{\frac{\partial}{\partial x}\left(\frac{UV}{h+\chi\zeta}\right)+\frac{\partial}{\partial y}\left(\frac{V^2}{h+\chi\zeta}\right)\right\}+k_e\mathscr{U}=-(h+\chi\zeta)\frac{\partial}{\partial y}(\zeta-k_a\zeta_a-k_T\zeta_T)$$

$$-k_\rho\int_{-h}^{\chi\zeta}\frac{\partial}{\partial y}\int_z^0\rho'dz'dz+k_\tau\tau_{ay}-k_b\tau_{by},$$

$$\frac{\partial\zeta}{\partial t}+\frac{\partial\mathscr{U}'}{\partial x}+\frac{\partial\mathscr{V}'}{\partial y}=\frac{k_r\Gamma_a}{\chi P_r},\tag{3}$$

where

$$\vec{e}_1 U + \vec{e}_2 V = \int_{-h}^{\chi\zeta} \vec{V} dz, \quad \text{the volume-transport,}$$

$$\vec{e}_1 \mathscr{U} + \vec{e}_2 \mathscr{V} = \int_{-h}^{\chi\zeta} \rho \vec{V} dz = \int_{-h}^{\chi\zeta} (1+\varepsilon\rho') \vec{V} dz = (\vec{e}_1 U + \vec{e}_2 V)$$

$$+\varepsilon(\vec{e}_1 \mathscr{U}' + \vec{e}_2 \mathscr{V}'), \quad \text{the mass-transport,}$$

$$\vec{e}_1 \mathscr{U}' + \vec{e}_2 \mathscr{V}' = \int_{-h}^{\chi\zeta} \rho' \vec{V} dz .$$

The scales of (U, V) and $(\mathscr{U}, \mathscr{V})$ are $(V_0 D)$ and $(\rho_0 V_0 D)$ respectively; the scale of $(\mathscr{U}', \mathscr{V}')$ is $(\rho_0' V_0 D)$.

The bottom stress is generally assumed to be a quadratic law or a linear law; here we choose the former only;

$$\vec{\tau}_b = \mathscr{K} \frac{(U^2+V^2)^{\frac{1}{2}}}{(h+\chi\zeta)^2} (\vec{e}_1 U + \vec{e}_2 V), \tag{4}$$

where

$$\mathscr{K} = \frac{\rho_0 V_0^2}{\tau_{b0}} k,$$

k is a frictional coefficient of order 10^{-3}.

In the equations of motion, the horizontal dispersive stresses derived due to the velocity shear have been neglected because these terms are small compared with the bottom friction for shallow seas.

A TWO-DIMENSIONAL TRANSPORT MODEL

Noting that the residual motion is a nonlinear phenomenon and χ is a small nonlinear parameter, it is expected that the longer-term residual currents have the order of magnitude of χ [6]. We suppose that the total motion (U, V, ζ) is separated into a mainly tidal part $(\tilde{U}, \tilde{V}, \tilde{\zeta})$ and a residual part $(\bar{U}, \bar{V}, \bar{\zeta})$, where

$$(U, V, \zeta) = (\tilde{U}, \tilde{V}, \tilde{\zeta}) + \chi(\bar{U}, \bar{V}, \bar{\zeta}), \tag{5}$$

$(\bar{U}, \bar{V}, \bar{\zeta})$ to be independent of t.

An operator "$\wedge$" may take the form of a time average:

$$(\hat{U}, \hat{V}, \hat{\zeta}) = \frac{T}{T_0} \int_{-T_0/2T}^{T_0/2T} (U, V, \zeta) dt, \tag{6}$$

which filters out the tidal motion and storm surges to a greater or lesser extent according to the length of the averaging period, T_0, and its commensurability with tidal periods[8,10]. Noting $(\hat{U}, \hat{V}, \hat{\zeta}) = (\hat{\tilde{U}}, \hat{\tilde{V}}, \hat{\tilde{\zeta}}) + \chi(\hat{\bar{U}}, \hat{\bar{V}}, \hat{\bar{\zeta}}) = \chi(\bar{U}, \bar{V}, \bar{\zeta})$, the nondimensional problem (3) will be translated into the problem for the residual currents.

Noting $\vec{\tau}_a = \vec{\tilde{\tau}}_a + \delta_1 \vec{\bar{\tau}}_a \left(\delta_1 = \frac{\bar{\tau}_{a0}}{\tau_{a0}}\right.$ and $\left. O(\delta_1) < 1\right)$, $\zeta_T = \tilde{\zeta}_T$, $\zeta_a = \tilde{\zeta}_a + \delta_2 \bar{\zeta}_a$ ($\delta_2 = \bar{\zeta}_{a0}/\zeta_{a0}$ and $O(\delta_2) < 1$) and

$$\chi \widehat{\left\{\frac{\partial}{\partial x}\left(\frac{U^2}{h+\chi\zeta}\right)+\frac{\partial}{\partial y}\left(\frac{UV}{h+\chi\zeta}\right)\right\}} = \chi\left\{\frac{\partial}{\partial x}\left(\frac{\widehat{\tilde{U}^2}}{h}\right)+\frac{\partial}{\partial y}\left(\frac{\widehat{\tilde{U}\tilde{V}}}{h}\right)\right\} + O(\chi^2),$$

$$\chi \widehat{\left\{\frac{\partial}{\partial x}\left(\frac{UV}{h+\chi\zeta}\right)+\frac{\partial}{\partial y}\left(\frac{V^2}{h+\chi\zeta}\right)\right\}} = \chi\left\{\frac{\partial}{\partial x}\left(\frac{\widehat{\tilde{U}\tilde{V}}}{h}\right)+\frac{\partial}{\partial y}\left(\frac{\widehat{\tilde{V}^2}}{h}\right)\right\} + O(\chi^2),$$

$$\widehat{-\chi_\rho \int_{-h}^{\chi\zeta} \nabla \int_z^0 \rho' dz' dz} = -\frac{\varepsilon}{\chi}\int_{-h}^0 \nabla \int_z^0 \rho' dz' dz + O(\chi\varepsilon),$$

$$-\widehat{(h+\chi\zeta)\nabla(\zeta - k_a\zeta_a - k_T\zeta_T)} = -\chi h \nabla\left(\bar{\zeta} - k_a \frac{\delta_2}{\chi}\bar{\zeta}_a\right)$$

$$-\widehat{\chi\tilde{\zeta}\nabla\tilde{\zeta}} + \widehat{\chi\tilde{\zeta}\nabla k_a\tilde{\zeta}_a} + \widehat{\chi\tilde{\zeta}\nabla k_T\tilde{\zeta}_T} + O(\chi^2),$$

$$-k_b \widehat{\vec{\tau}_b} \propto -\chi k_b \mathscr{K} (\vec{e}_1 \bar{U} - \vec{e}_2 \bar{V}) \frac{\widehat{(\tilde{U}^2+\tilde{V}^2)^{\frac{1}{2}}}}{h^2} \text{ [1,4]}$$

and neglecting $O(\chi^2)$, we have the problem for the residual currents

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0,$$

$$-k_c \overline{\mathscr{V}} = -h\frac{\partial}{\partial x}\left(\bar{\zeta} - k_a \frac{\delta_2}{\chi}\bar{\zeta}_a\right) + \tau'_x + \left(k_\tau \frac{\delta_1}{\chi}\right)\bar{\tau}_{ax} - (\varepsilon/\chi^2)\int_{-h}^0 \frac{\partial}{\partial x}\int_z^0 \rho' dz' dz - \mathscr{R}\bar{U},$$

$$k_c \overline{\mathscr{U}} = -h\frac{\partial}{\partial y}\left(\bar{\zeta} - k_a \frac{\delta_2}{\chi}\bar{\zeta}_a\right) + \tau'_y + \left(k_\tau \frac{\delta_1}{\chi}\right)\bar{\tau}_{ay} - (\varepsilon/\chi^2)\int_{-h}^0 \frac{\partial}{\partial y}\int_z^0 \rho' dz' dz - \mathscr{R}\bar{V}, \quad (7)$$

$$\frac{\partial \overline{\mathscr{U}}'}{\partial x} + \frac{\partial \overline{\mathscr{V}}'}{\partial y} = \frac{k_\gamma}{\chi^2 P_r}\Gamma_a;$$

where $\mathscr{R} = \frac{\mathscr{C}}{h}$, $\mathscr{C} = k_b \mathscr{K} \frac{\widehat{(\tilde{U}^2+\tilde{V}^2)^{\frac{1}{2}}}}{h}$

$$\tau'_x = -\left\{\widehat{\tilde{\zeta}\frac{\partial\tilde{\zeta}}{\partial x}} - \widehat{\tilde{\zeta}\frac{\partial}{\partial x}(k_a\tilde{\zeta}_a)} - \widehat{\tilde{\zeta}\frac{\partial}{\partial x}(k_T\tilde{\zeta}_T)} + \frac{\partial}{\partial x}\left(\frac{\widehat{\tilde{U}^2}}{h}\right) + \frac{\partial}{\partial y}\left(\frac{\widehat{\tilde{U}\tilde{V}}}{h}\right)\right\},$$

$$\tau'_y = -\left\{\widehat{\tilde{\zeta}\frac{\partial\tilde{\zeta}}{\partial y}} - \widehat{\tilde{\zeta}\frac{\partial}{\partial y}(k_a\tilde{\zeta}_a)} - \widehat{\tilde{\zeta}\frac{\partial}{\partial y}(k_T\tilde{\zeta}_T)} + \frac{\partial}{\partial x}\left(\frac{\widehat{\tilde{U}\tilde{V}}}{h}\right) + \frac{\partial}{\partial y}\left(\frac{\widehat{\tilde{V}^2}}{h}\right)\right\},$$

(τ'_x, τ'_y) have been called "tidal stresses"[10].

In shallow seas, the effect of bottom stress on the motion is essential, or $O(\mathscr{R}) = O(\mathscr{C})$ $O(k_b \mathscr{K}) = 1$; thus, $k \sim 0.5 \times 10^{-3}$, which is a correct order of magnitude of k.

Introducing the stream function Ψ (8)

$$\frac{\partial \Psi}{\partial x}=\bar{V}, \quad \frac{\partial \Psi}{\partial y}=-\bar{U}, \tag{8}$$

and eliminating $\bar{\zeta}$, the problem (7) reduces to the equation satisfied by Ψ as follows

$$\nabla \cdot \left(\frac{\mathscr{C}}{h} \nabla \Psi\right)-\frac{\mathscr{C}}{h^2} \nabla h \cdot \nabla \Psi-\frac{k_c}{h}\nabla h \cdot \vec{e}_3 \times \nabla \Psi = \vec{e}_3 \cdot \nabla \times \left(\vec{\tau}'+k_\tau \frac{\delta_1}{\chi}\overline{\vec{\tau}_a}\right)$$

$$+\frac{1}{h} \nabla h \cdot \vec{e}_3 \times \left(\vec{\tau}'+k_\tau \frac{\delta_1}{\chi}\overline{\vec{\tau}_a}\right)-\mathscr{T}\Gamma_a+\mathscr{D}, \tag{9}$$

where $\vec{\tau}'=\vec{e}_1\tau'_x+\vec{e}_2\tau'_y$, $\quad \mathscr{T}=\left(\frac{k_c k_\gamma}{P_r}\right)\frac{\varepsilon}{\chi^2}$,

$$\mathscr{D}=-\left(\frac{\varepsilon}{\chi^2}\right) h \left\{\frac{\partial}{\partial x}\left(\frac{1}{h}\int_{-h}^{0}\int_{z}^{0}\frac{\partial \rho'}{\partial y}dz'dz\right)-\frac{\partial}{\partial y}\left(\frac{1}{h}\int_{-h}^{0}\int_{z}^{0}\frac{\partial \rho'}{\partial x}dz'dz\right)\right\},$$

The lateral boundary conditions of the problem (2) reduces to (10) and (11):

$$C_1: \quad \vec{n}_0 \cdot (\vec{e}_1\bar{U}+\vec{e}_2\bar{V})=0; \tag{10}$$

$$C_2: \quad \vec{n}_0 \cdot (\vec{e}_1\bar{U}+\vec{e}_2\bar{V})=\bar{Q} \tag{11}$$

or

$$\bar{\zeta}=\overline{\mathscr{S}};$$

here, we have assumed $(Q, \mathscr{S})=(\tilde{Q}, \tilde{\mathscr{S}})+\chi(\bar{Q}, \bar{\mathscr{S}})$.

The equation (9) shows that (i) the "tidal stress" is an essential force of order of magnitude of 1, (ii) the wind stress has the order of magnitude of δ_1/κ in view of $O(k_\tau)=1$, and (iii) both the thermohalinic forces are the same in order of magnitude, $O(\varepsilon/\kappa^2)$, since $P_r=1$ and $O(k_r)=1$, $O(k_c)=1$.

The problem thus reduces to a single elliptic equation for the stream function of residual currents (9) and the boundary conditions (10) and (11).

A THREE-DIMENSIONAL MODEL

To study the three-dimensional residual circulation over seasonal time scales, the dynamic problem (2) must be directly time-averaged to remove variations of tidal period or shorter such as storm surges. As mentioned above, we may express each dependent variable in terms of a slowly-varying or steady-state residual part and a tidal part

$$(\vec{V}, w, \zeta)=(\tilde{\vec{V}}, \tilde{w}, \tilde{\zeta})+\chi(\bar{\vec{V}}, \bar{w}, \bar{\zeta}), \tag{12}$$

$(\bar{\vec{V}}, \bar{\zeta})$ assumed to be independent of t in this paper.

An operator "$\wedge$" may take the form of a time average:

$$(\hat{\vec{V}}, \hat{w}, \hat{\zeta})=\frac{T}{T_0}\int_{-T_0/2T}^{T_0/2T}(\vec{V}, w, \zeta)dt \tag{13}$$

with $(\hat{\tilde{\vec{V}}}, \hat{\tilde{w}}, \hat{\tilde{\zeta}})=0$ and $(\hat{\bar{\vec{V}}}, \hat{\bar{w}}, \hat{\bar{\zeta}})=(\bar{\vec{V}}, \bar{w}, \bar{\zeta})$. Thus, setting $\rho=1$ and dropping the diffusion equation, the time-averaged nondimensional problem (2) become

$$\nabla \overline{\vec{V}} + \frac{\partial \bar{w}}{\partial z} = 0,$$

$$\frac{\partial}{\partial z}\left(\nu \frac{\partial \overline{\vec{V}}}{\partial z}\right) - k_c \vec{e}_3 \times \overline{\vec{V}} = \nabla \bar{\zeta} + \vec{\Pi};$$

$z = 0$:

$$\bar{w} = \Theta, \tag{14}$$

$$\nu \frac{\partial \overline{\vec{V}}}{\partial z} = \vec{\tau},$$

$z = -h$:

$$\overline{\vec{V}} = \bar{w} = 0;$$

along the shore boundary C_1:

$$\vec{n}_0 \cdot \left(\int_{-h}^{0} \overline{\vec{V}}\, dz + \vec{q}\right) = 0;$$

along the open boundary C_2:

$$\vec{n}_0 \cdot \left(\int_{-h}^{0} \overline{\vec{V}}\, dz + \vec{q}\right) = \bar{Q},$$

or

$$\bar{\zeta} = \overline{\mathscr{P}};$$

where
$$\vec{\Pi} = -\nabla\left(k_a \frac{\delta_2}{\chi} \bar{\zeta}_a\right) + \frac{\varepsilon}{\chi_2} \nabla \int_z^0 \rho' dz + \left(\widehat{\tilde{\vec{V}} \cdot \nabla \tilde{\vec{V}}} + \widehat{\tilde{w} \frac{\partial \tilde{\vec{V}}}{\partial z}}\right),$$

$$\Theta = \nabla \cdot \overline{(\tilde{\vec{V}} \tilde{\zeta})}, \qquad \vec{\tau} = k_\tau \frac{\delta_1}{\chi} \bar{\vec{\tau}}_a + \widehat{\left[-\frac{\partial}{\partial z}\left(\nu \frac{\partial \tilde{\vec{V}}}{\partial z}\right) \tilde{\zeta}\right]},$$

$$\vec{q} = \widehat{\tilde{\zeta} \tilde{\vec{V}}}.$$

The nondimensional problem (14) is in the same form as the nondimensional problem (2) derived in the reference [5]. Thus, a Sturm–Liouville system proposed in the reference [5] can be used here and is written as follows

$$\left[\frac{\partial}{\partial z}\left(\nu \frac{\partial}{\partial z}\right) + \lambda\right] F(z) = 0,$$

$$\left(\frac{\partial F}{\partial z}\right)_{z=0} = 0, \tag{15}$$

$$(F)_{z=-h} = 0.$$

Suppose $F = F(x, y, z)$ and λ being independent of z, and for each (x, y), let the ascending eigenvalues and corresponding eigenfunctions derived from (15) be denoted by

$$\lambda=\lambda_n, \qquad F=F_n \qquad (n=1,\ 2,\ \dots).$$

Thus, we have

$$\vec{\overline{V}}=\sum_{n=1}^{\infty} G_n\vec{\mathscr{U}}_n F_n+\frac{\vec{\tau}+(h+z)}{\nu_0} \qquad (\nu_0=\nu_{z=0}), \tag{16}$$

$$\overline{w}=-\sum_{n=1}^{\infty}\nabla\cdot(G_n\vec{\mathscr{U}}_n\mathscr{H}_n)-\nabla\cdot\frac{\vec{\tau}(h+z)^2}{2\nu_0}; \tag{17}$$

where

$$\vec{\mathscr{U}}_n=(\beta_n^2+k_c^2)^{-1}\{(-H_n\beta_n+H_nk_c\vec{e}_3\times)\nabla\overline{\zeta}+(\beta_n-k_c\vec{e}_3\times)\vec{\tau}_n-(\beta_n-k_c\vec{e}_3\times)\vec{\Phi}_n\},$$

$$G_n=\left(\int_{-h}^{0}F_n^2dz\right)^{-1}, \qquad \mathscr{H}_n=\int_{-h}^{z}F_ndz, \qquad H_n=\int_{-h}^{0}F_ndz,$$

$$\beta_n=\lambda_n, \qquad \vec{\Phi}_n=\int_{-h}^{0}\vec{\Pi}F_ndz,$$

$$\vec{\tau}_n=\frac{\vec{\tau}}{\nu_0}\left\{\int_{-h}^{0}F_n\frac{\partial\nu}{\partial z}dz+(\times k_c\vec{e}_3)\left(hH_n-\int_{-h}^{0}\mathscr{H}_ndz\right)\right\},$$

and

$$\left[\mathscr{A}\nabla^2+\left(\frac{\partial\mathscr{A}}{\partial x}-\frac{\partial\mathscr{B}}{\partial y}\right)\frac{\partial}{\partial x}+\left(\frac{\partial\mathscr{A}}{\partial y}+\frac{\partial\mathscr{B}}{\partial x}\right)\frac{\partial}{\partial y}\right]\overline{\zeta}=\Theta+\mathscr{F}+\nabla\cdot\frac{h\vec{\tau}}{2\nu_0}; \tag{18}$$

along the shore boundary C_1:

$$Q_{n0}=-q_{n0}-\frac{h^2}{2\nu_0}\tau_{n0};$$

along the open boundary C_2:

$$Q_{n0}=\overline{Q}-q_{n0}-\frac{h^2}{2\nu_0}\tau_{n0},$$

or

$$\overline{\zeta}=\overline{\mathscr{S}};$$

where

$$\mathscr{F}=-\sum_{n=1}^{\infty}\{\vec{e}_3\cdot\nabla\times(B_n\vec{\Phi}_n)+\nabla\cdot(A_n\vec{\Phi}_n)-\vec{e}_3\cdot\nabla\times(B_n\vec{\tau}_n)-\nabla\cdot(A_n\vec{\tau}_n)\},$$

$$Q_{n0}=-\mathscr{A}\frac{\partial\overline{\zeta}}{\partial n_0}+\mathscr{B}\frac{\partial\overline{\zeta}}{\partial s_0}-\sum_{n=1}^{\infty}\left\{A_n\Phi_{nn_0}-B_n\Phi_{ns_0}-A_n\tau_{nn_0}+B_n\tau_{ns_0}\right\},$$

$$A_n=\frac{H_nG_n\beta_n}{\beta_n^2+k_c^2}, \qquad B_n=\frac{H_nG_nk_c}{\beta_n^2+k_c^2}, \qquad \mathscr{A}=\sum_{n=1}^{\infty}A_nH_n, \qquad \mathscr{B}=\sum_{n=1}^{\infty}B_nH_n,$$

$$\frac{\partial\overline{\zeta}}{\partial n_0}=\frac{\partial\overline{\zeta}}{\partial x}\cos\alpha_x+\frac{\partial\overline{\zeta}}{\partial y}\cos\alpha_y, \qquad \frac{\partial\overline{\zeta}}{\partial s_0}=-\frac{\partial\overline{\zeta}}{\partial y}\cos\alpha_x+\frac{\partial\overline{\zeta}}{\partial x}\cos\alpha_y,$$

$$\Phi_{nn_0}=\Phi_{nx}\cos\alpha_x+\Phi_{ny}\cos\alpha_y, \qquad \Phi_{ns_0}=-\Phi_{ny}\cos\alpha_x+\Phi_{nx}\cos\alpha_y,$$

$$\tau_{nn_0}=\tau_{nx}\cos\alpha_x+\tau_{ny}\cos\alpha_y, \qquad \tau_{ns_0}=-\tau_{ny}\cos\alpha_x+\tau_{nx}\cos\alpha_y,$$

$$\tau_{n_0}=\tau_x\cos\alpha_x+\tau_y\cos\alpha_y, \qquad q_{n_0}=q_x\cos\alpha_x+q_y\cos\alpha_y;$$

here, s_0 denotes the tangent at the boundary which forms a right-hand Cartesion coordinate system with the normal n_0;

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The nondimensional problem (14) now reduces to the problem of $\bar{\zeta}$ (18) and the expression for $\overrightarrow{\bar{V}}$ (16), where the eigenfunction F_n is derived from the Sturm-Liouville system (15) on the finite closed interval $[-h, 0]$, as well as the expression for $\bar{w}$ (17).

POSTSCRIPT

Based on a scale analysis for the currents in shallow seas, a nondimensional hydrodynamic problem for baroclinic shallow seas is proposed. A two-dimensional transport model and a three-dimensional model for baroclinic residual currents are developed respectively. The former is a direct generalization from the barotropic model of residual circulation presented by Nihoul and Randy (1975)[10] and Heaps (1978)[8] to the baroclinic one while in the latter the same Sturm-Liouville system as that proposed in the reference [5] is used, which can play a key role in the mathematical treatment for this model. Thus, no the mathematical difficulty will be added to the models proposed for computation.

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