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Characteristic Cohomology of p-Form Gauge Theories

Marc Henneaux[1,2], **Bernard Knaepen**[1,⋆], **Christiane Schomblond**[1]

[1] Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine C.P. 231, B–1050 Bruxelles, Belgium
[2] Centro de Estudios Científicos de Santiago, Casilla 16443, Santiago 9, Chile

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Abstract: The characteristic cohomology $H^k_{char}(d)$ for an arbitrary set of free p-form gauge fields is explicitly worked out in all form degrees $k < n-1$, where n is the spacetime dimension. It is shown that this cohomology is finite-dimensional and completely generated by the forms dual to the field strengths. The gauge invariant characteristic cohomology is also computed. The results are extended to interacting p-form gauge theories with gauge invariant interactions. Implications for the BRST cohomology are mentioned.

1. Introduction

The characteristic cohomology [1] plays a central role in the analysis of any local field theory. The easiest way to define this cohomology, which is contained in the so-called Vinogradov C-spectral sequence [2, 3, 4], is to start with the familiar notion of conserved current. Consider a dynamical theory with field variables ϕ^i ($i = 1, \ldots, M$) and Lagrangian $\mathcal{L}(\phi^i, \partial_\mu \phi^i, \ldots, \partial_{\mu_1 \ldots \mu_k} \phi^i)$. The field equations read

$$\mathcal{L}_i = 0, \tag{1.1}$$

with

$$\mathcal{L}_i = \frac{\delta \mathcal{L}}{\delta \phi^i} = \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)}\Big) + \ldots + (-1)^k \partial_{\mu_1 \ldots \mu_k} \Big(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1 \ldots \mu_k} \phi^i)}\Big). \tag{1.2}$$

A (local) conserved current j^μ is a vector-density which involves the fields and their derivatives up to some finite order and which is conserved modulo the field equations, i.e., which fulfills

$$\partial_\mu j^\mu \approx 0. \tag{1.3}$$

⋆ Aspirant du Fonds National de la Recherche Scientifique (Belgium)

Here and in the sequel, $\approx$ means "equal when the equations of motion hold" or, as one also says equal "on-shell". Thus, (1.3) is equivalent to

$$\partial_\mu j^\mu = \lambda^i \mathcal{L}_i + \lambda^{i\mu}\partial_\mu \mathcal{L}_i + \ldots + \lambda^{i\mu_1\cdots\mu_s}\partial_{\mu_1\ldots\mu_s}\mathcal{L}_i \tag{1.4}$$

for some $\lambda^{i\mu_1\cdots\mu_j}$, $j = 0, \ldots, s$. A conserved current is said to be trivial if it can be written as

$$j^\mu \approx \partial_\nu S^{\mu\nu} \tag{1.5}$$

for some local antisymmetric tensor density $S^{\mu\nu} = -S^{\nu\mu}$. The terminology does not mean that trivial currents are devoid of physical interest, but rather, that they are easy to construct and that they are trivially conserved. Two conserved currents are said to be equivalent if they differ by a trivial one. The characteristic cohomology in degree $n-1$ is defined to be the quotient space of equivalence classes of conserved currents. One assigns the degree $n-1$ because Eqs. (1.3) and (1.5) can be rewritten as $d\omega \approx 0$ and $\omega \approx d\psi$ in terms of the $(n-1)$-form ω and $(n-2)$-form ψ respectively dual to j^μ and $S^{\mu\nu}$.

One defines the characteristic cohomology in degree k ($k < n$) along exactly the same lines, by simply considering other values of the form degree. So, one says that a local k-form ω is a cocycle of the characteristic cohomology in degree k if it is weakly closed,

$$d\omega \approx 0; \quad \text{"cocycle condition,"} \tag{1.6}$$

and that it is a coboundary if it is weakly exact,

$$\omega \approx d\psi, \quad \text{"coboundary condition,"} \tag{1.7}$$

just as it is done for $k = n-1$. For instance, the characteristic cohomology in form degree $n-2$ is defined, in dual notations, as the quotient space of equivalence classes of weakly conserved antisymmetric tensors,

$$\partial_\nu S^{\mu\nu} \approx 0, \; S^{\mu\nu} = S^{[\mu\nu]}, \tag{1.8}$$

where two such tensors are regarded as equivalent iff

$$S^{\mu\nu} - S'^{\mu\nu} \approx \partial_\rho R^{\rho\mu\nu}, \; R^{\rho\mu\nu} = R^{[\rho\mu\nu]}. \tag{1.9}$$

We shall denote the characteristic cohomological groups by $H^k_{char}(d)$.

Higher order conservation laws involving antisymmetric tensors of degree 2 or higher are quite interesting in their own right. In particular, conservation laws of the form (1.8), involving an antisymmetric tensor $S^{\mu\nu}$ have attracted a great deal of interest in the past [5] as well as recently [6, 7] in the context of the mechanism of "charge without charge" of Wheeler [8].

But the characteristic cohomology is also important for another reason: it appears as an auxiliary cohomology in the calculation of the local BRST cohomology [9]. This local BRST cohomology, in turn, is quite useful in the determination of the structure of the counterterms [10, 11] and the anomalies [12] in the quantum theory. It plays also a central role classically, in constraining the form of the consistent deformations of the action [13]. It is by establishing vanishing theorems for the characteristic cohomology that the problem of consistent deformations and of candidate anomalies has been completely solved in the cases of Yang-Mills gauge theories and of gravity [14, 6]. For this reason, it is an important question to determine the characteristic cohomological groups for any given theory.

The purpose of this paper is to carry out this task for a system of free antisymmetric tensor fields $B^a_{\mu_1\dots\mu_{p_a}}$, $a = 1, \dots, N$, with Lagrangian

$$\mathcal{L} = \sum_a \Big(\frac{-1}{2(p_a+1)!} H^a_{\mu_1\dots\mu_{p_a+1}} H^{a\mu_1\dots\mu_{p_a+1}} \Big), \tag{1.10}$$

where the H^a's are the "field strengths" or "curvatures",

$$H^a = \frac{1}{(p_a+1)!} H^a_{\mu_1\dots\mu_{p_a+1}} dx^{\mu_1} \dots dx^{\mu_{p_a+1}} = dB^a, \tag{1.11}$$

$$B^a = \frac{1}{p_a!} B^a_{\mu_1\dots\mu_{p_a}} dx^{\mu_1} \dots dx^{\mu_{p_a}}. \tag{1.12}$$

The equations of motion, obtained by varying the fields $B^a_{\mu_1\dots\mu_{p_a}}$, are given by

$$\partial_\rho H^{a\rho\mu_1\dots\mu_{p_a}} = 0. \tag{1.13}$$

We consider simultaneously antisymmetric tensors of different degrees, but we assume $1 \leq p_a$. We also assume $n > p_a + 1$ for each a so that the fields $B^a_{\mu_1\dots\mu_{p_a}}$ all carry local degrees of freedom. Modifications of the Lagrangian by gauge invariant interactions are treated at the end of the paper.

We give complete results for the characteristic cohomology in degree $< n-1$, that is, we determine all the solutions to the equation $\partial_\mu S^{\mu\nu_1\dots\nu_s} \approx 0$ with $s > 0$. Although we do not solve the characteristic cohomology in degree $n-1$, we comment on the gauge invariance properties of the conserved currents and provide an infinite number of them, generalizing earlier results of the Maxwell case [15, 16, 17][1]. The results of this paper will be used in [18] to compute the BRST cohomology of free, antisymmetric tensor fields. This is a necessary step not only for determining the possible consistent interactions that can be added to the free Lagrangian, but also for analyzing completely the BRST cohomology in the interacting case. Our results have already been used and partly announced in [19] to show the uniqueness of the Freedman-Townsend deformation of the gauge symmetries of a system of antisymmetric tensors of degree 2 in four dimensions.

Antisymmetric tensor fields - or, as one also says, p-form gauge fields - have been much studied in the past [20, 21, 22, 23, 24] and are crucial ingredients of string theory and of various supergravity models [25]. The main feature of theories involving p-form gauge fields is that their gauge symmetries are *reducible*. More precisely, in the present case, the Lagrangian (1.10) is invariant under the gauge transformations

$$B^a \to B'^a = B^a + d\Lambda^a, \tag{1.14}$$

where Λ^a are arbitrary $(p_a - 1)$-forms. Now, if $\Lambda^a = d\epsilon^a$, then, the variation of B^a vanishes identically. Thus, the gauge parameters Λ^a do not all provide independent gauge symmetries: the gauge transformations (1.14) are reducible. In the same way, if

[1] The determination of all the conserved currents is of course also an interesting question, but it is not systematically pursued here for two reasons. First the characteristic cohomology $H^{n-1}_{char}(d)$ is infinite-dimensional for the free theories considered here and does not appear to be completely known even in the Maxwell case in an arbitrary number of dimensions. By contrast, the cohomological groups $H^k_{char}(d)$, $k < n-1$, are all finite-dimensional and can be explicitly computed. Second, the group $H^{n-1}_{char}(d)$ plays no role in the analysis of the consistent interactions of antisymmetric tensor fields of degree > 1, as well as in the analysis of candidate anomalies if the antisymmetric tensor fields all have degree > 2 [18].

ϵ^a is equal to $d\mu^a$, then, it yields a vanishing Λ^a. There is "reducibility of reducibility" unless ϵ^a is a zero form. If ϵ^a is not a zero form, the process keeps going until one reaches 0-forms. For the theory with Lagrangian (1.10), there are thus $p_M - 1$ stages of reducibility of the gauge transformations (Λ^a is a $(p_a - 1)$-form), where p_M is the degree of the form of highest degree occurring in (1.10) [26, 27, 28, 29]. One says that the theory is a reducible gauge theory of reducibility order $p_M - 1$.

General vanishing theorems have been established in [1, 2, 3, 9] showing that the characteristic cohomology of reducible theories of reducibility order $p - 1$ vanishes in form degree strictly smaller than $n - p - 1$. Accordingly, in the case of p-form gauge theories, there can be a priori non-vanishing characteristic cohomology only in form degree $n - p_M - 1$, $n - p_M$, etc., up to form degree $n - 1$ (conserved currents). In the 1-form case, these are the best vanishing theorems one can prove, since a set of free gauge fields A^a_μ has characteristic cohomology both in form degree $n-1$ and $n-2$ [9]. Representatives of the cohomology classes in form degree $n - 2$ are given by the duals to the field strengths, which are indeed closed on-shell due to Maxwell equations.

Our main result is that the general vanishing theorems of [1, 2, 3, 9] can be considerably strengthened when $p > 1$. For instance, if there is a single p-form gauge field and if $n-p-1$ is odd, there is only one non-vanishing group of the characteristic cohomology in degree $< n - 1$. This is $H^{n-p-1}_{char}(d)$, which is one-dimensional. All the other groups $H^k_{char}(d)$ of the characteristic cohomology with $n - p - 1 < k < n - 1$ are zero, even though the general theorems of [1, 2, 3, 9] leave open the possibility that they do not vanish. As we shall show in [18], it is the presence of these additional zeros that give p-form gauge fields and gauge transformations their strong rigidity.

Besides the standard characteristic cohomology, one may consider the invariant characteristic cohomology, in which the local forms ω and ψ occurring in (1.6) and (1.7) are required to be invariant under the gauge transformations (1.14). We also completely determine in this paper the invariant characteristic cohomology in form degree $< n - 1$.

Our method for computing the characteristic cohomology is based on the reformulation performed in [9] of the characteristic cohomology in form degree k in terms of the cohomology $H^n_{n-k}(\delta|d)$ of the Koszul-Tate differential δ modulo the spacetime exterior derivative d. Here, n is the form degree and $n-k$ is the antighost number. This approach is strongly motivated by the BRST construction and appears to be particularly attractive and powerful.

Our paper is organized as follows. In the next section, we formulate precisely our main results, which are (i) that the characteristic cohomology $H^k_{char}(d)$ with $k < n - 1$ is generated (in the exterior product) by the exterior forms $\overline{H}^a$ dual to the field strengths H^a; these are forms of degree $n - p_a - 1$; and (ii) that the invariant characteristic cohomology $H^{k,inv}_{char}(d)$ with $k < n - 1$ is generated (again in the exterior product) by the exterior forms H^a and $\overline{H}^a$. We then review, in Sects. 3 and 4, the definition and properties of the Koszul-Tate complex. Section 5 is of a more technical nature and relates the characteristic cohomology to the cohomology of the differential $\delta + d$, where δ is the Koszul-Tate differential. Section 6 analyses the gauge invariance properties of δ-boundaries modulo d. In Sect. 7, we determine the characteristic cohomology for a single p-form gauge field. The results are then extended to an arbitrary system of p-form gauge fields in Sect. 8. The invariant cohomology is analyzed in Sect. 9. Section 10 discusses in detail the cohomological groups $H^*(\delta|d)$, which play a key role in the calculation of the local BRST cohomological groups $H^*(s|d)$. In Sect. 11, we show that the existence of representatives expressible in terms of the $\overline{H}^a$'s does not extend to the characteristic cohomology in form degree $n - 1$, by exhibiting an infinite number of (inequivalent) conserved currents which are not of that form. We show next in Sect. 12 that the results on the free characteristic cohomology in degree $< n-1$ can be generalized straightforwardly if one adds to the free Lagrangian (1.10) gauge invariant interaction terms that involve the fields $B^a_{\mu_1\dots\mu_{p_a}}$ and their derivatives only through the gauge invariant field strength components and their derivatives (which are in general the only consistent interactions that one can add). We conclude in Sect. 13 by summarizing our results and indicating future lines of research.

We assume throughout this paper that spacetime is the n-dimensional Minkowski space, so that the indices in (1.10) are raised with the inverse $\eta^{\mu\nu}$ of the flat Minkowski metric $\eta_{\mu\nu}$. However, because of their geometrical character, our results generalize straightforwardly to curved backgrounds.

2. Results

2.1. Characteristic cohomology. The equations of motion (1.13) can be rewritten as

$$d\overline{H}^a \approx 0 \tag{2.1}$$

in terms of the $(n-p_a-1)$-forms $\overline{H}^a$ dual to the field strengths. It then follows that any polynomial in the $\overline{H}^a$'s is closed on-shell and thus defines a cocycle of the characteristic cohomology.

The remarkable feature is that these polynomials are not only inequivalent in cohomology, but also *completely exhaust the characteristic cohomology in form degree strictly smaller than* $n-1$. Indeed, one has:

Theorem 2.1. *Let $\overline{\mathcal{H}}$ be the algebra generated by the $\overline{H}^a$'s and let $\mathcal{V}$ be the subspace containing the polynomials in the $\overline{H}^a$'s with no term of form degree exceeding $n-2$. The subspace $\mathcal{V}$ is isomorphic to the characteristic cohomology in form degree $< n-1$.*

We stress again that the theorem does not hold in degree $n-1$ because there exist conserved currents not expressible in terms of the $\overline{H}^a$'s.

Since the form degree is limited by the spacetime dimension n, and since $\overline{H}^a$ has form degree $n-p_a-1$, which is strictly positive (as explained in the introduction, we assume $n-p_a-1>0$ for each a), the algebra $\overline{\mathcal{H}}$ is finite-dimensional. In that algebra, the $\overline{H}^a$ with even $n-p_a-1$ commute with all the other generators, while the $\overline{H}^a$ with odd $n-p_a-1$ are anticommuting objects.

2.2. Invariant characteristic cohomology. While the cocycles of Theorem 2.1 are all gauge invariant, there exists coboundaries of the characteristic cohomology that are gauge invariant, i.e., that involve only the field strength components and their derivatives, but which cannot, nevertheless, be written as coboundaries of gauge invariant local forms, even weakly. Examples are given by the field strengths $H^a = dB^a$ themselves. For this reason, the invariant characteristic cohomology and the characteristic cohomology do not coincide. We shall denote by $\mathcal{H}$ the finite-dimensional algebra generated by the (p_a+1)-forms H^a, and by $\mathcal{J}$ the finite-dimensional algebra generated by the field strengths H^a and their duals $\overline{H}^a$. One has

Theorem 2.2. *Let $\mathcal{W}$ be the subspace of $\mathcal{J}$ containing the polynomials in the H^a's and the $\overline{H}^a$'s with no term of form degree exceeding $n-2$. The subspace $\mathcal{W}$ is isomorphic to the invariant characteristic cohomology in form degree $< n-1$.*

Our paper is devoted to proving these theorems.

2.3. Cohomologies in algebra of x-independent forms. The previous theorems hold as they are formulated in the algebra of local forms that are allowed to have an explicit x-dependence. The explicit x-dependence enables one to remove the constant k-forms ($k > 0$) from the cohomology, since these are exact, $c_{i_1 i_2 \ldots i_k} dx^{i_1} dx^{i_2} \ldots dx^{i_k} = d(c_{i_1 i_2 \ldots i_k} x^{i_1} dx^{i_2} \ldots dx^{i_k})$. If one restricts one's attention to the algebra of local forms with no explicit dependence on the spacetime coordinates, then, one must replace in the above theorems the polynomials in the curvatures and their duals with coefficients that are *numbers* by the polynomials in the curvatures and their duals with coefficients that are *constant exterior forms*.

Note that the constant exterior forms can be alternatively gotten rid of without introducing an explicit x-dependence, by imposing Lorentz invariance (there is no Lorentz-invariant constant k-form for $0 < k < n$).

3. Koszul-Tate Complex

The definition of the cocycles of the characteristic cohomology $H^k_{char}(d)$ involves "weak" equations holding only on-shell. It is convenient to replace them by "strong" equations holding everywhere in field space, and not just when the equations of motion are satisfied. The reason is that the coefficients of the equations of motion in the conservation laws are not arbitrary, but are subject to restrictions whose analysis yields useful insight on the conservation laws themselves. From this point of view, Eq. (1.4) involving the coefficients $\lambda^{i\mu_1 \cdots \mu_j}$ is a more interesting starting point than Eq. (1.3). One useful way to replace weak equations by strong equations is to introduce the Koszul-Tate resolution associated with the equations of motion (1.13).

The details of the construction of the Koszul-Tate differential δ can be found in [30]. Because the present theory is reducible, we must introduce the following set of BV-antifields [31]:

$$B^{*a\mu_1 \ldots \mu_{p_a}}, B^{*a\mu_1 \ldots \mu_{p_a - 1}}, \ldots, B^{*a\mu_1}, B^{*a}. \tag{3.1}$$

The Grassmann parity and the *antighost* number of the antifields $B^{*a\mu_1 \cdots \mu_{p_a}}$ associated with the fields $B^a_{\mu_1 \ldots \mu_{p_a}}$ are equal to 1. The Grassmann parity and the *antighost* number of the other antifields is determined according to the following rule. As one moves from one term to the next one to its right in (3.1), the Grassmann parity changes and the antighost number increases by one unit. Therefore the parity and the antighost number of a given antifield $B^{*a\mu_1 \cdots \mu_{p-j}}$ are respectively $j+1$ modulo 2 and $j+1$.

The Koszul-Tate differential acts in the algebra $\mathcal{P}$ of local exterior forms. By definition, a local exterior form ω reads

$$\omega = \sum \omega_{\mu_1 \ldots \mu_J} dx^{\mu_1} \ldots dx^{\mu_J}, \tag{3.2}$$

where the coefficients $\omega_{\mu_1 \ldots \mu_J}$ are smooth functions of the coordinates x^μ, the fields $B^a_{\mu_1 \ldots \mu_{p_a}}$, the antifields (3.1), and their derivatives up to a finite order. Although this is not strictly necessary, we shall actually assume polynomiality in the fields, the antifields and their derivatives, as this is the situation encountered in field theory.

The Koszul-Tate differential is defined by its action on the fields and the antifields as follows:

$$\delta B^a_{\mu_1\dots\mu_{p_a}} = 0, \tag{3.3}$$
$$\delta B^{*a\mu_1\dots\mu_{p_a}} = \partial_\rho H^{a\rho\mu_1\dots\mu_{p_a}}, \tag{3.4}$$
$$\delta B^{*a\mu_1\dots\mu_{p_a-1}} = \partial_\rho B^{*a\rho\mu_1\dots\mu_{p_a-1}}, \tag{3.5}$$
$$\vdots$$
$$\delta B^{*a\mu_1} = \partial_\rho B^{*a\rho\mu_1}, \tag{3.6}$$
$$\delta B^{a*} = \partial_\rho B^{*a\rho}. \tag{3.7}$$

Furthermore we have,

$$\delta x^\mu = 0, \ \ \delta(dx^\mu) = 0. \tag{3.8}$$

The action of δ is extended to an arbitrary element in $\mathcal{P}$ by using the rule

$$\delta\partial_\mu = \partial_\mu\delta, \tag{3.9}$$

and the fact that δ is an odd derivation which we take here to act from the left,

$$\delta(ab) = (\delta a)b + (-)^{\epsilon_a}a(\delta b). \tag{3.10}$$

In 3.10, ϵ_a is the Grassmann parity of the (homogeneous) element a. These rules make δ a differential and one has the following important property [32, 33, 30, 34]:

Theorem 3.1. *$H_i(\delta) = 0$ for $i > 0$, where i is the antighost number, i.e, the cohomology of δ is empty in antighost number strictly greater than zero.*

One can also show that in degree zero, the cohomology of δ is the algebra of "on-shell functions" [32, 33, 30, 34]. Thus, the Koszul-Tate complex provides a resolution of that algebra. For the reader unaware of the BRST developments, one may view this property as the motivation for the definitions (3.3) through (3.7).

One has a similar theorem for the cohomology of the exterior derivative d (for which we also take a left action, $d(ab) = (da)b + (-)^{\epsilon_a}a(db)$).

Theorem 3.2. *The cohomology of d in the algebra of local forms is given by,*

$$H^0(d) \simeq R, \tag{3.11}$$
$$H^k(d) = 0 \text{ for } k \neq 0, k \neq n, \tag{3.12}$$
$$H^n(d) \simeq \text{ space of equivalence classes of local forms}, \tag{3.13}$$

where k is the form degree and n the spacetime dimension. In (3.13), two local forms are said to be equivalent if and only if they have identical Euler-Lagrange derivatives with respect to all the fields and the antifields.

Proof. This theorem is known as the algebraic Poincaré Lemma. For various proofs, see [2, 35, 36, 37]. It should be mentioned that the theorem holds as such because we allow for an explicit x-dependence of the local exterior forms (3.2). If the local forms had no explicit x-dependence, then (3.12) would have to be amended as

$$H^k(d) \simeq \{\text{constant forms}\} \text{ for } k \neq 0, k \neq n, \tag{3.14}$$

where the constant forms are by definition the local exterior forms (3.2) with constant coefficients. We shall denote in the sequel the algebra of constant forms by Λ^* and the subspace of constants forms of degree k by Λ^k. The following formulation of the Poincaré lemma is also useful. □

Theorem 3.3. *Let a be a local, closed k-form ($k < n$) that vanishes when the fields and the antifields are set equal to zero. Then, a is d-exact.*

Proof. The condition that a vanishes when the fields and the antifields are set equal to zero eliminates the constants.

This form of the Poincaré lemma holds in both the algebras of x-dependent and x-independent local exterior forms. □

4. Characteristic Cohomology and Koszul-Tate Complex

Our analysis of the characteristic cohomology relies upon the isomorphism established in [9] between $H^*_{char}(d)$ and the cohomology $H^*_*(\delta|d)$ of δ modulo d. The cohomology $H^k_i(\delta|d)$ in form degree k and antighost number i is obtained by solving in the algebra $\mathcal{P}$ of local exterior forms the equation,

$$\delta a^k_i + d b^{k-1}_{i-1} = 0, \tag{4.1}$$

and by identifying solutions which differ by δ-exact and d-exact terms, i.e,

$$a^k_i \sim a'^k_i = a^k_i + \delta n^k_{i+1} + d m^{k-1}_i. \tag{4.2}$$

One has

Theorem 4.1.

$$H^k_{char}(d) \simeq H^n_{n-k}(\delta|d),\ 0 < k < n, \tag{4.3}$$

$$\frac{H^0_{char}(d)}{R} \simeq H^n_n(\delta|d), \tag{4.4}$$

$$0 \simeq H^n_{n+k}(\delta|d),\ k > 0. \tag{4.5}$$

Proof. Although the proof is standard and can be found in [37, 9], we shall repeat it explicitly here because it involves ingredients which will be needed below. Let α be a class of $H^k_{char}(d)$ ($k < n$) and let a^k_0 be a representative of α, $\alpha = [a^k_0]$. One has

$$\delta a^{k+1}_1 + d a^k_0 = 0 \tag{4.6}$$

for some a^{k+1}_1 since any antifield-independent form that is zero on-shell can be written as the δ of something. By acting with d on this equation, one finds that $d a^{k+1}_1$ is δ-closed and thus, by Theorem 3.1, that it is δ-exact, $\delta a^{k+2}_2 + d a^{k+1}_1 = 0$ for some a^{k+2}_2. One can repeat the procedure until one reaches degree n, the last term a^n_{n-k} fulfilling

$$\delta a^n_{n-k} + d a^{n-1}_{n-1-k} = 0, \tag{4.7}$$

and, of course, $da^n_{n-k} = 0$ (it is a n-form). For future reference we collect all the terms appearing in this tower of equations as

$$a^k = a^n_{n-k} + a^{n-1}_{n-1-k} + \ldots + a^{k+1}_1 + a^k_0. \tag{4.8}$$

Equation (4.7) shows that a^n_{n-k} is a cocycle of the cohomology of δ modulo d, in form-degree n and antighost number $n-k$. Now, given the cohomological class α of $H^k_{char}(d)$, it is easy to see, using again Theorem 3.1, that the corresponding element a^n_{n-k} is well-defined in $H^n_{n-k}(\delta|d)$. Consequently, the above procedure defines an non-ambiguous map m from $H^k_{char}(d)$ to $H^n_{n-k}(\delta|d)$.

This map is surjective. Indeed, let a^n_{n-k} be a cocycle of $H^n_{n-k}(\delta|d)$. By acting with d on Eq. (4.7) and using the second form of the Poincaré lemma (Theorem 3.3), one finds that a^{n-1}_{n-1-k} is also δ-closed modulo d. Repeating the procedure all the way down to antighost number zero, one sees that there exists a cocycle a^k_0 of the characteristic cohomology such that $m([a^k_0]) = [a^n_{n-k}]$.

The map m is not quite injective, however, because of the constants. Assume that a^k_0 is mapped on zero. This means that the corresponding a^n_{n-k} is trivial in $H^n_{n-k}(\delta|d)$, i.e., $a^n_{n-k} = \delta b^n_{n-k+1} + d b^{n-1}_{n-k}$. Using the Poincaré lemma (in the second form) one then finds successively that a^{n-1}_{n-k-1} ... up to a^{k+1}_1 are all trivial. The last term a^k_0 fulfills $da^k_0 + \delta d b^k_1 = 0$ and thus, by the Poincaré lemma (Theorem 3.2), $a^k_0 = \delta b^k_1 + d b^{k-1}_0 + c^k$. In the algebra of x-dependent local forms, the constant k-form c^k is present only if $k = 0$. This establishes (4.3) and (4.4). That $H^n_m(\delta|d)$ vanishes for $m > n$ is proved in [34]. □

The proof of the theorem shows also that (4.3) holds as such because one allows for an explicit x-dependence of the local forms. Otherwise, one must take into account the constant forms c^k which appear in the analysis of injectivity and which are no longer exact even when $k > 0$, so that (4.3) becomes

$$\frac{H^k_{char}(d)}{\Lambda^k} \simeq H^n_{n-k}(\delta|d), \tag{4.9}$$

while (4.4) and (4.5) remain unchanged.

5. Characteristic Cohomology and Cohomology of $\Delta = \delta + d$

It is convenient to rewrite the Koszul-Tate differential in form notations. Denoting the duals with an overline to avoid confusion with the antifield *-notation, and redefining the antifields by appropriate multiplicative constants, one finds that Eqs. (3.4) through (3.7) become simply

$$\begin{aligned} \delta \overline{B}^{*a}_1 &+ d\overline{H}^a = 0, \\ \delta \overline{B}^{*a}_2 &+ d\overline{B}^{*a}_1 = 0, \\ &\vdots \\ \delta \overline{B}^{*a}_{p_a+1} &+ d\overline{B}^{*a}_{p_a} = 0. \end{aligned} \tag{5.1}$$

The form $\overline{B}_j^{*a}$ dual to the antisymmetric tensor density $B^{*a\mu_1\cdots\mu_{p_a+1-j}}$ ($j = 1, \ldots, p_a+1$) has (i) form degree equal to $n - p_a - 1 + j$; and (ii) antighost number equal to j. Since $B^{*a\mu_1\cdots\mu_{p_a+1-j}}$ has Grassmann parity j and since the product of $(n - p_a - 1 + j)$ dx's has Grassmann parity $n - p_a - 1 + j$, each $\overline{B}_j^{*a}$ has same Grassmann parity $n - p_a - 1$ (modulo 2), irrespective of j. This is the same parity as that of the $n - p_a - 1$-form $\overline{H}^a$ dual to the field strengths.

Equation (5.1) can be rewritten as

$$\Delta \tilde{H}^a = 0 \tag{5.2}$$

with

$$\Delta = \delta + d \tag{5.3}$$

and

$$\tilde{H}^a = \overline{H}^a + \sum_{j=1}^{p_a+1} \overline{B}_j^{*a}. \tag{5.4}$$

The parity of the exterior form $\tilde{H}^a$ is equal to $n - p_a - 1$. The regrouping of physical fields with ghost-like variables is quite standard in BRST theory [38]. Expressions similar (but not identical) to (5.4) have appeared in the analysis of the Freedman-Townsend model and of string field theory [39, 40], as well as in the context of topological models [41, 42]. Note that for a one-form, expression (5.4) reduces to Ey. (9.8) of [14]. Quite generally, it should be noted that the dual $\overline{H}^a$ to the field strength H^a is the term of lowest form degree in $\tilde{H}^a$. It is also the term of lowest antighost number, namely, zero. At the other end, the term of highest form degree in $\tilde{H}^a$ is $\overline{B}_{p_a+1}^{*a}$, which has form degree n and antighost number $p_a + 1$. If we call the difference between the form degree and the antighost number the "Δ-degree", all the terms present in the expansion of $\tilde{H}^a$ have the same Δ-degree, namely $n - p_a - 1$.

The differential $\Delta = \delta + d$ enables one to reformulate the characteristic cohomology as the cohomology of Δ. Indeed one has

Theorem 5.1. *The cohomology of Δ is isomorphic to the characteristic cohomology,*

$$H^k(\Delta) \simeq H^k_{char}(d), \; 0 \leq k \leq n \tag{5.5}$$

where k in $H^k(\Delta)$ is the Δ-degree, and in $H^k_{char}(d)$ is the form degree.

Proof. Let a_0^k ($k < n$) be a cocycle of the characteristic cohomology. Construct a^k as in the proof of Theorem 4.1, formula (4.8). The form a^k is easily seen to be a cocycle of Δ, $\Delta a^k = 0$, and furthermore, to be uniquely defined in cohomology given the class of a_0^k. We leave it to the reader to check that the map so defined is both injective and surjective. This proves the theorem for $k < n$. For $k = n$, the isomorphism of $H^n(\Delta)$ and $H^n_{char}(d)$ is even more direct ($da_0^n = 0$ is equivalent to $\Delta a_0^n = 0$ and $a_0^n = db_0^{n-1} + \delta b_1^n$ is equivalent to $a_0^n = \Delta(b_0^{n-1} + b_1^n)$). □

Our discussion has also established the following useful rule: the term of lowest form degree in a Δ-cocycle a is a cocycle of the characteristic cohomology. Its form degree is equal to the Δ-degree k of a. For $a = \tilde{H}^a$, this reproduces the rule discussed above Theorem 5.1. Similarly, the term of highest form degree in a has always form degree equal to n if a is not a Δ-coboundary (up to a constant), and defines an element of $H^n_{n-k}(\delta|d)$.

Because Δ is a derivation, its cocycles form an algebra. Therefore, any polynomial in the $\tilde{H}^a$ is also a Δ-cocycle. Since the form degree is limited by the spacetime dimension n, and since the term $\overline{H}^a$ with minimum form degree in $\tilde{H}^a$ has form degree $n - p_a - 1$, which is strictly positive, the algebra generated by the $\tilde{H}^a$ is finite-dimensional. We shall show below that these Δ-cocycles are not exact and that any cocycle of form degree $< n - 1$ is a polynomial in the $\tilde{H}^a$ modulo trivial terms. According to the isomorphism expressed by Theorem 5.1, this is equivalent to proving Theorem 2.1.

Remarks. (i) The Δ-cocycle associated with a conserved current contains only two terms,

$$a = a_1^n + a_0^{n-1}, \tag{5.6}$$

where a_0^{n-1} is the dual to the conserved current in question. The product of such a Δ-cocycle with a Δ-cocycle of Δ-degree k has Δ-degree $n - 1 + k$ and therefore vanishes unless $k = 0$ or 1.

(ii) It will be useful below to introduce another degree N as follows. One assigns N-degree 0 to the undifferentiated fields and N-degree 1 to all the antifields irrespective of their antighost number. One then extends the N-degree to the differentiated variables according to the rule $N(\partial_\mu \Phi) = N(\Phi) + 1$. Thus, N counts the number of derivatives and of antifields. Explicitly,

$$N = \sum_a N_a \tag{5.7}$$

with

$$N_a = \sum_J \left[(|J| \sum_i \partial_J B_i^a \frac{\partial}{\partial_J B_i^a} + (|J| + 1) \sum_\alpha \partial_J \phi_\alpha^{*a} \frac{\partial}{\partial_J \phi_\alpha^{*a}}\right], \tag{5.8}$$

where (i) the sum over J is a sum over all possible derivatives including the zeroth order one; (ii) $|J|$ is the differential order of the derivative ∂_J (i.e., $|J| = k$ for $\partial_{\mu_1 \dots \mu_k}$); (iii) the sum over i stands for the sum over the independent components of B^a; and (iv) the sum over α is a sum over the independent components of all the antifields appearing in the tower associated with B^a (but there is *no* sum over the p-form species a in (5.8)). The differential δ increases N by one unit. The differentials d and Δ have in addition an inhomogeneous piece not changing the N-degree, namely $dx^\mu(\partial^{explicit}/\partial x^\mu)$, where $\partial^{explicit}/\partial x^\mu$ sees only the explicit x^μ-dependence. The forms $\tilde{H}^a$ have N-degree equal to one.

6. Acyclicity and Gauge Invariance

6.1. Preliminary results. Under the gauge transformations (1.14) of the p-form gauge fields, the field strengths and their derivatives are gauge invariant. These are the only invariant objects that can be formed out of the "potentials" $B^a_{\mu_1 \dots \mu_{p_a}}$ and their derivatives. We shall denote by $\mathcal{I}_{Small}$ the algebra of local exterior forms with coefficients $\omega_{\mu_1 \dots \mu_J}$ that depend only on the field strength components and their derivatives (and possibly x^μ). The algebras $\mathcal{H}$, $\overline{\mathcal{H}}$ and $\mathcal{J}$ respectively generated by the $(p_a + 1)$-forms H^a, $(n - p_a - 1)$-forms $\overline{H}^a$ and $(H^a, \overline{H}^a)$ are subalgebras of $\mathcal{I}_{Small}$. Since the field equations are gauge invariant and since d maps $\mathcal{I}_{Small}$ on $\mathcal{I}_{Small}$, one can consider the cohomological problem (1.6), (1.7) in the algebra $\mathcal{I}_{Small}$. This defines the invariant characteristic cohomology $H^{*,inv}_{char}(d)$.

It is natural to decree that the antifields and their derivatives are also invariant. This can be more fully justified within the BRST context, using the property that the gauge transformations are abelian, but here, it can simply be taken as a useful, consistent postulate. With these conventions, the differentials δ, d and Δ map the algebra $\mathcal{I}$ of invariant polynomials in the field strength components, the antifield components and their derivatives on itself. Clearly, $\mathcal{I}_{Small} \subset \mathcal{I}$. The invariant cohomologies $H^{*,inv}(\Delta)$ and $H_j^{n,inv}(\delta|d)$ are defined by considering only local exterior forms that belong to $\mathcal{I}$.

In order to analyze the invariant characteristic cohomology and to prove the nontriviality of the cocycles listed in Theorem 2.1, we shall need some preliminary results on the invariant cohomologies of the Koszul-Tate differential δ and of d.

The variables generating the algebra $\mathcal{P}$ of local forms are, together with x^μ and dx^μ,

$$B_{a\mu_1\ldots\mu_{p_a}}, \partial_\rho B_{a\mu_1\ldots\mu_{p_a}}, \ldots, B^{*a\mu_1\ldots\mu_{p_a-m}}, \partial_\rho B^{*a\mu_1\ldots\mu_{p_a-m}}, \ldots, B^{*a}, \partial_\rho B^{*a}, \ldots .$$

These variables can be conveniently split into two subsets. The first subset of generators will be collectively denoted by the letter χ. They are given by the field strengths $(H_{a\mu_1\ldots\mu_{p_a+1}})$ and their derivatives, the antifields and their derivatives. The field strengths and their derivatives are not independent, since they are constrained by the identity $dH^a = 0$ and its differential consequences, but this is not a difficulty for the considerations of this section. The χ's are invariant under the gauge transformations and they generate the algebra $\mathcal{I}$ of invariant polynomials. In order to generate the full algebra $\mathcal{P}$ we need to add to the χ's some extra variables that will be collectively denoted Ψ. The Ψ's contain the field components $B^{a\mu_1\ldots\mu_{p_a}}$ and their appropriate derivatives not present in the χ's. The explicit form of the Ψ's is not needed here. All we need to know is that the Ψ's are algebraically independent from the χ's and that, in conjunction with the χ's, they generate $\mathcal{P}$.

Theorem 6.1. *Let a be a polynomial in the χ: $a = a(\chi)$. If $a = \delta b$, then we can choose b such that $b = b(\chi)$. In particular,*

$$H_j^{inv}(\delta) \simeq 0 \text{ for } j > 0. \tag{6.1}$$

Proof. We can decompose b into two parts: $b = \bar{b} + \bar{\bar{b}}$, with $\bar{b} = \bar{b}(\chi) = b(\Psi = 0)$ and $\bar{\bar{b}} = \sum_m R_m(\chi)S_m(\Psi)$, where $S_m(\Psi)$ contains at least one Ψ. Because $\delta\Psi = 0$, we have, $\delta(\bar{b} + \bar{\bar{b}}) = \delta\bar{b}(\chi) + \sum_m \delta R_m(\chi)S_m(\Psi)$. Furthermore if $M = M(\chi)$, then $\delta M(\chi) = (\delta M)(\chi)$. We thus get,

$$a(\chi) = (\delta\bar{b})(\chi) + \sum_m (\delta R_m)(\chi)S_m(\Psi).$$

The above equation has to be satisfied for all the values of the Ψ's and in particular for $\Psi = 0$. This means that $a(\chi) = (\delta\bar{b})(\chi) = \delta\bar{b}(\chi)$. □

Theorem 6.2. *Let $\mathcal{H}^k$ be the subspace of form degree k of the finite dimensional algebra $\mathcal{H}$ of polynomials in the curvature $(p_a + 1)$-forms H^a, $\mathcal{H} = \oplus_k \mathcal{H}^k$. One has*

$$H_j^{k,inv}(d) = 0, \; k < n, \; j > 0 \tag{6.2}$$

and

$$H_0^{k,inv}(d) = \mathcal{H}^k, \; k < n. \tag{6.3}$$

Thus, in particular, if $a = a(\chi)$ with $da = 0$, antighost $a > 0$ and deg $a < n$, then $a = db$ with $b = b(\chi)$. And if a has antighost number zero, then $a = P(H^a) + db$, where $P(H^a)$ is a polynomial in the curvature forms and where $b \in \mathcal{I}_{Small}$.

Proof. The theorem has been proved in [36, 43] for 1-forms. It can be extended straightforwardly to the case of p-forms of odd degree. The even degree case is slightly different because the curvatures $(p + 1)$-forms H^a are then anticommuting. It is fully treated in Appendix A. If the local forms are not taken to be explicitly x-dependent, Eq. (6.3) must be replaced by

$$H_0^{k,inv}(d) = (\Lambda \otimes \mathcal{H})^k. \tag{6.4}$$

□

6.2. Gauge invariant δ-boundaries modulo d. We assume in this section that the antisymmetric tensors $B^a_{\mu_1\mu_2\ldots\mu_p}$ have all the same degree p. This covers, in particular, the case of a single p-form.

Theorem 6.3. *(Valid when the $B^a_{\mu_1\mu_2\ldots\mu_p}$'s have all the same form degree p). Let $a^n_q = a^n_q(\chi) \in \mathcal{I}$ be an invariant local n-form of antighost number $q > 0$. If a^n_q is δ-exact modulo d, $a^n_q = \delta\mu^n_{q+1} + d\mu^{n-1}_q$, then one can assume that μ^n_{q+1} and μ^{n-1}_q only depend on the χ's, i.e., are invariant (μ^n_{q+1} and $\mu^{n-1}_q \in \mathcal{I}$).*

Proof. The proof goes along exactly the same lines as the proof of a similar statement made in [14] (Theorem 6.1) for 1-form gauge fields. Accordingly, it will not be repeated here[2]. □

Remark. The theorem does not hold if the forms have various form degrees (see Theorem 10.1 below).

7. Characteristic Cohomology for a Single p-Form Gauge Field

Our strategy for computing the characteristic cohomology is as follows. First, we compute $H^n_*(\delta|d)$ (cocycle condition, coboundary condition) for a single p-form. We then use the isomorphism theorems to infer $H^*_{char}(d)$. Finally, we solve the case of a system involving an arbitrary (but finite) number of p-forms of various form degrees.

7.1. General theorems. Before we compute $H^n_*(\delta|d)$ for a single abelian p-form gauge field $B_{\mu_1\ldots\mu_p}$, we will recall some general results which will be needed in the sequel. These theorems hold for an arbitrary linear theory of reducibility order $p - 1$.

Theorem 7.1. *For a linear gauge theory of reducibility order p-1, one has,*

$$H^n_j(\delta|d) = 0, \quad j > p + 1. \tag{7.1}$$

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[2] We shall just mention a minor point that has been overlooked in the proof of Theorem 6.1 of [14], namely, that when $p = 1$ in Eq. (6.4) of [14], the form Z' need not vanish (in the notations of [14]). However, this does not invalidate the fact that one can replace Z', X'_μ, etc. by invariant polynomials as the recurrence used in the proof of [14] and the absence of invariant cohomology for d in form degree one indicate. This is just what is needed for establishing the theorem.

Proof. See [9], Theorem 9.1. See also [1, 2, 3]. □

Theorem 7.1 is particularly useful because it limits the number of potentially non-vanishing cohomologies. The calculation of the characteristic cohomology is further simplified by the following theorem:

Theorem 7.2. *Any solution of $\delta a + \partial_\rho b^\rho = 0$ that is at least bilinear in the antifields is necessarily trivial.*

Proof. See [9], Theorem 11.2. □

Both theorems hold whether the local forms are assumed to have an explicit x-dependence or not.

7.2. Cocycles of $H^n_{p+1}(\delta|d)$. We have just seen that the first potentially non-vanishing cohomological group is $H^n_{p+1}(\delta|d)$. We show in this section that this group is one-dimensional and provide explicit representatives. We systematically use the dual notations involving divergences of antisymmetric tensor densities.

Theorem 7.3. *$H^n_{p+1}(\delta|d)$ is one-dimensional. One can take as representatives of the cohomological classes $a = kB^*$, where B^* is the last antifield, of antighost number $p+1$ and where k is a number.*

Proof. Any polynomial of antighost number $p+1$ can be written $a = fB^* + f^\rho \partial_\rho B^* + \ldots + \mu$, where f does not involve the antifields and where μ is at least bilinear in the antifields. By adding a divergence to a, one can remove the derivatives of B^*, i.e., one can assume $f^\rho = f^{\rho\sigma} = \ldots = 0$. The cocycle condition $\delta a + \partial_\rho b^\rho = 0$ implies then $-\partial_\rho f B^{*\rho} + \delta\mu + \partial_\rho(b^\rho + fB^{*\rho}) = 0$. By taking the Euler-Lagrange derivative of this equation with respect to $B^{*\rho}$, one gets

$$-\partial_\rho f + \delta((-1)^p \frac{\delta^L \mu}{\delta B^{*\rho}}) = 0. \tag{7.2}$$

This shows that f is a cocycle of the characteristic cohomology in degree zero since δ(anything of antighost number one) ≈ 0. Furthermore, if f is trivial in $H^0_{char}(d)$, then a can be redefined so as to be at least bilinear in the antifields and thus is also trivial in the cohomology of δ modulo d. Now, the isomorphism of $H^0_{char}(d)/R$ with $H^n_n(\delta|d)$ implies $f = k + \delta g$ with k a constant ($H^n_n(\delta|d) = 0$ because $n > p+1$). As we pointed out, the second term can be removed by adding a trivial term, so we may assume $f = k$. Writing $a = kB^* + \mu$, we see that μ has to be a solution of $\delta\mu + \partial_\rho b'^\rho = 0$ by itself and is therefore trivial by Theorem 7.2. So $H^n_{p+1}(\delta|d)$ can indeed be represented by $a = kB^*$. In form notations, this is just the n-form $k\overline{B}^*_{p+1}$. Note that the calculations are true both in the x-dependent and x-independent cases.

To complete the proof of the theorem, it remains to show that the cocycles $a = kB^*$, which belong to the invariant algebra $\mathcal{I}$ and which contain the undifferentiated antifields, are non-trivial. If they were trivial, one would have according to Theorem 6.3, that $\overline{B}^*_{p+1} = \delta u + dv$ for some u, v also in $\mathcal{I}$. But this is impossible, because both δ and d bring in one derivative of the invariant generators χ while $\overline{B}^*_{p+1}$ does not contain derivatives of χ. [This derivative counting argument is direct if u and v do not involve explicitly the spacetime coordinates x^μ. If they do, one must expand u, v and the equation $\overline{B}^*_{p+1} = \delta u + dv$ according to the number of derivatives of the fields in order to reach the conclusion. Explicitly, one sets $u = u_0 + \ldots + u_k$, $v = v_0 + \ldots + v_k$, where k counts the number of derivatives of the $H_{\mu_1 \ldots \mu_{p+1}}$ and of the antifields. The condition $\overline{B}^*_{p+1} = \delta u + dv$ implies in degree $k+1$ in the derivatives that $\delta u_k + d' v_k = 0$, where d' does not differentiate with respect to the explicit dependence on x^μ. This relation implies in turn that u_k is δ-trivial modulo d' since there is no cohomology in antighost number $p+2$. Thus, one can remove u_k by adding trivial terms. Repeating the argument for u_{k-1}, and then for u_{k-2}, etc., leads to the desired conclusion.] □

7.3. Cocycles of $H^n_i(\delta|d)$ with $i \leq p$. We now solve the cocycle condition for the remaining degrees. First we prove

Theorem 7.4. *Let K be the greatest integer such that $n - K(n-p-1) > 1$. The cohomological groups $H^n_j(\delta|d)$ ($j > 1$) vanish unless $j = n - k(n-p-1)$, $k = 1, 2, \ldots, K$. Furthermore, for those values of j, $H^n_j(\delta|d)$ is at most one-dimensional.*

Proof. We already know that $H^n_j(\delta|d)$ is zero for $j > p+1$ and that $H^n_{p+1}(\delta|d)$ is one-dimensional. Assume thus that the theorem has been proved for all j's strictly greater than $J < p+1$ and let us extend it to J. In a manner analogous to what we did in the proof of Theorem 7.3, we can assume that the cocycles of $H^n_J(\delta|d)$ take the form

$$f_{\mu_1 \ldots \mu_{p+1-J}} B^{*\mu_1 \ldots \mu_{p+1-J}} + \mu, \tag{7.3}$$

where $f_{\mu_1 \ldots \mu_{p+1-J}}$ does not involve the antifields and defines an element of $H^{p+1-J}_{char}(d)$. Furthermore, if $f_{\mu_1 \ldots \mu_{p+1-J}}$ is trivial, then the cocycle (7.3) is also trivial. Now, using the isomorphism $H^{p+1-J}_{char}(d) \simeq H^n_{n-p-1+J}(\delta|d)$ ($p+1-J > 0$), we see that f is trivial unless $j' = n-p-1+J$, which is strictly greater than J and is of the form $j' = n-k(n-p-1)$. In this case, $H^n_{j'}$ is at most one-dimensional. Since $J = j' - (n-p-1) = n-(k+1)(n-p-1)$ is of the required form, the property extends to J. This proves the theorem. □

Because we explicitly used the isomorphism $H^{p+1-J}_{char}(d) \simeq H^n_{n-p-1+J}(\delta|d)$, which holds only if the local forms are allowed to involve explicitly the coordinates x^μ, the theorem must be amended for x-independent local forms. This will be done in Sect. 7.5.

Theorem 7.4 goes beyond the vanishing theorems of [1, 2, 3, 9] since it sets further cohomological groups equal to zero, in antighost number smaller than $p+1$. This is done by viewing the cohomological group $H^n_i(\delta|d)$ as a subset of $H^n_{n-p-1+i}(\delta|d)$ at a *higher* value of the antighost number, through the form (7.3) of the cocycle and the isomorphism between $H^{p+1-i}_{char}(d)$ and $H^n_{n-p-1+i}(\delta|d)$. In that manner, the known zeros at values of the ghost number greater than $p+1$ are "propagated" down to values of the ghost number smaller than $p+1$.

To proceed with the analysis, we have to consider two cases:

(i) Case I: $n-p-1$ is even.
(ii) Case II: $n-p-1$ is odd.

We start with the simplest case, namely, Case I. In that case, $\tilde{H}$ is a commuting object and we can consider its various powers $(\tilde{H})^k$, $k = 1, 2, \ldots, K$ with K as in Theorem 7.4. These powers have Δ-degree $k(n-p-1)$. By Theorem 5.1, the term of form degree n in $(\tilde{H})^k$ defines a cocycle of $H^n_{n-k(n-p-1)}(\delta|d)$, which is non-trivial as the same invariance argument used in the previous subsection indicates. Thus, $H^n_{n-k(n-p-1)}(\delta|d)$, which we know is at most one-dimensional, is actually exactly one-dimensional and one may take as representative the term of form degree n in $(\tilde{H})^k$. This settles the case when $n-p-1$ is even.

In the case when $n-p-1$ is odd, $\tilde{H}$ is an anticommuting object and its powers $(\tilde{H})^k$, $k > 0$ all vanish unless $k = 1$. We want to show that $H^n_{n-k(n-p-1)}(\delta|d)$ similarly vanishes unless $k = 1$. To that end, it is enough to prove that $H^n_{n-2(n-p-1)}(\delta|d) = H^n_{2p+2-n}(\delta|d) = 0$ as the proof of Theorem 7.4 indicates (we assume, as before, that $2p + 2 - n > 1$ since we only investigate here the cohomological groups $H^n_i(\delta|d)$ with $i > 1$). Now, as we have seen, the most general cocycle in $H^n_{2p+2-n}(\delta|d)$ may be assumed to take the form $a = f_{\mu_{p+2}\dots\mu_n} B^{*\mu_{p+2}\dots\mu_n} + \mu$, where μ is at least quadratic in the antifields and where $f_{\mu_{p+2}\dots\mu_n}$ does not involve the antifields and defines an element of $H^{n-p-1}_{char}(d)$. But $H^{n-p-1}_{char}(d) \simeq H^n_{p+1}(\delta|d)$ is one-dimensional and one may take as representative of $H^{n-p-1}_{char}(d)$ the dual $k\epsilon_{\mu_1\dots\mu_n} H^{\mu_1\dots\mu_{p+1}}$ of the field strength. This means that a is necessarily of the form,

$$a = k\epsilon_{\mu_1\dots\mu_n} H^{\mu_1\dots\mu_{p+1}} B^{*\mu_{p+2}\dots\mu_n} + \mu, \tag{7.4}$$

and the question to be answered is: for which value of k can one adjust μ in (7.4) so that

$$k\epsilon_{\mu_1\dots\mu_n} H^{\mu_1\dots\mu_{p+1}} \delta B^{*\mu_{p+2}\dots\mu_n} + \delta\mu + \partial_\rho b^\rho = 0? \tag{7.5}$$

In (7.5), μ does not contain $B^{*\mu_{p+2}\dots\mu_n}$ and is at least quadratic in the antifields. Without loss of generality, we can assume that it is exactly quadratic in the antifields and that it does not contain derivatives, since δ and ∂ are both linear and bring in one derivative. [That μ can be assumed to be quadratic is obvious. That it can also be assumed not to contain the derivatives of the antifields is a little bit less obvious since we allow for explicit x-dependence, but can be easily checked by expanding μ and b^ρ according to the number of derivatives of the variables and using the triviality of the cohomology of δ modulo d in the space of local forms that are at least quadratic in the fields and the antifields.] Thus, we can write

$$\mu = \sigma_{\mu_1\dots\mu_n} B^{*\mu_1\dots\mu_p}_{(1)} B^{*\mu_{p+1}\dots\mu_n}_{(2p+1-n)} + \mu',$$

where μ' involves neither $B^{*\mu_{p+2}\dots\mu_n}_{(2p+2-n)}$ nor $B^{*\mu_{p+1}\dots\mu_n}_{(2p+1-n)}$. We have explicitly indicated the antighost number in parentheses in order to keep track of it. Inserting this form of μ in (7.5) one finds that $\sigma_{\mu_1\dots\mu_n}$ is equal to $k\epsilon_{\mu_1\dots\mu_n}$ if $2p + 1 - n > 1$ (if $2p + 1 - n = 1$, see below). One can then successively eliminate $B^{*\mu_p\dots\mu_n}_{(2p-n)}$, $B^{*\mu_{p-1}\dots\mu_n}_{(2p-n-1)}$, etc. from μ, so that the question ultimately boils down to: is

$$k\epsilon_{\mu_1\dots\mu_{2j}} B^{*\mu_1\dots\mu_j}_{(p+1-j)} \delta B^{*\mu_{j+1}\dots\mu_{2j}}_{(p+1-j)}$$

(n even $= 2j$) or

$$k\epsilon_{\mu_1\dots\mu_{2j+1}} B^{*\mu_1\dots\mu_{j+1}}_{(p-j)} \delta B^{*\mu_{j+2}\dots\mu_{2j+1}}_{(p+1-j)}$$

(n odd $= 2j + 1$) δ-exact modulo d, i.e., of the form $\delta\nu + \partial_\rho c^\rho$, where ν does not involve the antifields B^*_s for $s > p+1-j$ (n even) or $s > p - j$ (n odd)? That the answer to this question is negative unless $k = 0$ and a accordingly trivial, which is the desired result, is easily seen by trying to construct explicitly ν. We treat for definiteness the case n even ($n = 2j$). One has

$$\nu = \lambda_{\mu_1\dots\mu_{2j}} B^{*\mu_1\dots\mu_j}_{(p+1-j)} B^{*\mu_{j+1}\dots\mu_{2j}}_{(p+1-j)},$$

where $\lambda_{\mu_1\ldots\mu_{2j}}$ is antisymmetric (respectively symmetric) for the exchange of $(\mu_1\ldots\mu_j)$ with $(\mu_{j+1}\ldots\mu_{2j})$ if j is even (respectively odd) (the j-form $\overline{B}^*_{(p+1-j)}$ is odd by assumption and this can happen only if the components $B^{*\mu_1\ldots\mu_j}_{(p+1-j)}$ are odd for j even, or even for j odd). From the equation

$$k\epsilon_{\mu_1\ldots\mu_{2j}}B^{*\mu_1\ldots\mu_j}_{(p+1-j)}\delta B^{*\mu_{j+1}\ldots\mu_{2j}}_{(p+1-j)} = \delta\nu + \partial_\rho c^\rho, \tag{7.6}$$

one gets

$$k\epsilon_{\mu_1\ldots\mu_{2j}}B^{*\mu_1\ldots\mu_j}_{(p+1-j)}\partial_\rho B^{*\rho\mu_{j+1}\ldots\mu_{2j}}_{(p-j)} = 2\lambda_{\mu_1\ldots\mu_{2j}}B^{*\mu_1\ldots\mu_j}_{(p+1-j)}\partial_\rho B^{*\rho\mu_{j+1}\ldots\mu_{2j}}_{(p-j)} + \partial_\rho c^\rho. \tag{7.7}$$

Taking the Euler-Lagrange derivative of this equation with respect to $B^{*\mu_1\ldots\mu_j}_{(p+1-j)}$ yields next

$$k\big(\epsilon_{\mu_1\ldots\mu_{2j}} - 2\lambda_{\mu_1\ldots\mu_{2j}}\big)\partial_\rho B^{*\rho\mu_{j+1}\ldots\mu_{2j}}_{(p-j)} = 0,$$

which implies $k\epsilon_{\mu_1\ldots\mu_{2j}} = 2\lambda_{\mu_1\ldots\mu_{2j}}$. This contradicts the symmetry properties of $\lambda_{\mu_1\ldots\mu_{2j}}$, unless $k = 0$, as we wanted to prove.

7.4. Characteristic Cohomology. By means of the isomorphism theorem of section 4, our results on $H^n_*(\delta|d)$ can be translated in terms of the characteristic cohomology as follows:

(i) If $n-p-1$ is odd, the only non-vanishing group of the characteristic cohomology in form degree $< n-1$ is $H^{n-p-1}_{char}(d)$, which is one-dimensional. All the other groups vanish. One may take as representatives for $H^{n-p-1}_{char}(d)$ the cocycles $k\overline{H}$. Similarly, the only non-vanishing group $H^j(\Delta)$ with $j < n-1$ is $H^{n-p-1}(\Delta)$ with representatives $k\tilde{H}$ and the only non-vanishing group $H^n_i(\delta|d)$ with $i > 1$ is $H^n_{p+1}(\delta|d)$ with representatives $k\overline{B}^*_{p+1}$.

(ii) If $n-p-1$ is even, there is further cohomology. The degrees in which there is non-trivial cohomology are multiples of $n-p-1$ (considering again values of the form degree strictly smaller than $n-1$). Thus, there is characteristic cohomology only in degrees $n-p-1$, $2(n-p-1)$, $3(n-p-1)$, etc. The corresponding groups are one-dimensional and one may take as representatives $k\overline{H}$, $k(\overline{H})^2$, $k(\overline{H})^3$, etc. There is also non-vanishing Δ-cohomology for the same values of the Δ-degree, with representative cocycles given by $k\tilde{H}$, $k(\tilde{H})^2$, $k(\tilde{H})^3$, etc. By expanding these cocycles according to the form degree and keeping the terms of form degree n, one gets representatives for the only non-vanishing groups $H^n_i(\delta|d)$ (with $i > 1$), which are respectively $H^n_{p+1}(\delta|d)$, $H^n_{p+1-(n-p-1)}$, $H^n_{p+1-2(n-p-1)}$, etc.

An immediate consequence of our analysis is the following useful theorem:

Theorem 7.5. *If the polynomial $P^k(H)$ of form degree $k < n$ in the curvature $(p+1)$-form H is δ-exact modulo d in the invariant algebra $\mathcal{I}$, then $P^k(H) = 0$.*

Proof. The theorem is straightforward in the algebra of x-independent local forms, as a direct derivative counting argument shows. To prove it when an explicit x-dependence is allowed, one proceeds as follows. If $P^k(H) = \delta a_1^k + da_0^{k-1}$ where a_1^k and $a_0^{k-1} \in \mathcal{I}$, then $da_1^k + \delta a_2^{k+1} = 0$ for some invariant a_2^{k+1}. Using the results on the cohomology of δ modulo d that we have just established, this implies that a_1^k differs from the component of form degree k and antighost number 1 of a polynomial $Q(\tilde{H})$ by a term of the form $\delta\rho + d\sigma$, where ρ and σ are both invariant. But then, δa_1^k has the form $d\big([Q(\tilde{H})]_0^{k-1} + \delta\sigma\big)$, which implies $P^k(H) = d\big(-[Q(\tilde{H})]_0^{k-1} - \delta\sigma + a_0^{k-1}\big)$, i.e., $P^k(H) = d$(invariant). According to the theorem on the invariant cohomology of d, this can occur only if $P^k(H) = 0$. □

7.5. Characteristic cohomology in the algebra of x-independent local forms. Let us denote $(\tilde{H})^m$ by P_m $(m = 0, \dots, K)$. We have just shown (i) that the most general cocycles of the Δ-cohomology are given, up to trivial terms, by the linear combinations $\lambda_m P_m$ with λ_m real or complex numbers; and (ii) that if $\lambda_m P_m$ is Δ-exact, then the λ_m are all zero. In establishing these results, we allowed for an explicit x-dependence of the local forms (see comments after the proof of Theorem 7.4). How are our results affected if we work exclusively with local forms with no explicit x-dependence?

In the above analysis, it is in calculating the cocycles that arise in antighost number $< p + 1$ that we used the x-dependence of the local forms, through the isomorphism $H^{p+1-J}_{char}(d) \simeq H^n_{n-p-1+J}(\delta|d)$. If the local exterior forms are not allowed to depend explicitly on x, one must take the constant k-forms $(k > 0)$ into account. The derivation goes otherwise unchanged and one finds that the cohomology of Δ in the space of x-independent local forms is given by the polynomials in the P_m with coefficients λ_m that are constant forms, $\lambda_m = \lambda_m(dx)$. In addition, if $\lambda_m P_m$ is Δ-exact, then, $\lambda_m P_m = 0$ for each m. One cannot infer from this equation that λ_m vanishes, because it is an exterior form. One can simply assert that the components of λ_m of form degree $n - m(n-p-1)$ or lower are zero (when multiplied by P_m, the other components of λ_m yields forms of degree $> n$ that identically vanish, no matter what these other components are).

It will be also useful in the sequel to know the cohomology of Δ', where Δ' is the part of Δ that acts only on the fields and antifields, and not on the explicit x-dependence. One has $\Delta = \Delta' + d_x$, where $d_x \equiv \partial^{explicit}/\partial x^\mu$ sees only the explicit x-dependence. By the above result, the cohomology of Δ' is clearly given by the polynomials in the P_m with coefficients λ_m that are now arbitrary spacetime forms, $\lambda_m = \lambda_m(x, dx)$.

8. Characteristic Cohomology in the General Case

To compute the cohomology $H^n_i(\delta|d)$ for an arbitrary set of p-forms, one proceeds along the lines of the Kunneth theorem. Let us illustrate explicitly the procedure for two fields $B^1_{\mu_1 \dots \mu_{p_1}}$ and $B^2_{\mu_1 \dots \mu_{p_2}}$. One may split the differential Δ as a sum of terms with definite N_a-degrees,

$$\Delta = \Delta_1 + \Delta_2 + d_x \tag{8.1}$$

(see (5.8)). In (8.1), d_x leaves both N_1 and N_2 unchanged. By contrast, Δ_1 increases N_1 by one unit without changing N_2, while Δ_2 increases N_2 by one unit without changing N_1. The differential Δ_1 acts only on the fields B^1 and its associated antifields ("fields and antifields of the first set"), whereas the differential Δ_2 acts only on the fields B^2 and its associated antifields ("fields and antifields of the second set"). Note that $\Delta_1 + \Delta_2 = \Delta'$.

Let a be a cocycle of Δ with Δ-degree $< n-1$. Expand a according to the N_1-degree,

$$a = a_0 + a_1 + a_2 + \dots + a_m, \quad N_1(a_j) = j. \tag{8.2}$$

The equation $\Delta a = 0$ implies $\Delta_1 a_m = 0$ for the term a_m of highest N_1-degree. Our analysis of the Δ'-cohomology for a single p-form yields then $a_m = c_m(\tilde{H}^1)^k + \Delta_1$(something), where c_m involves only the fields and antifields of the second set, as well as dx^μ and possibly x^μ. There can be no conserved current in a_m since we assume the Δ-degree of a – and thus of each a_j – to be strictly smaller than $n-1$. Now, the exact term in a_m can be absorbed by adding to a_m a Δ-exact term, through a redefinition of a_{m-1}. Once this is done, one finds that the next equation for a_m and a_{m-1} following from $\Delta a = 0$ reads

$$[(\Delta_2 + d_x)c_m](\tilde{H}^1)^k + \Delta_1 a_{m-1} = 0. \tag{8.3}$$

But we have seen that $\lambda_m(\tilde{H}^1)^k$ cannot be exact unless it is zero, and thus this last equation implies both

$$[(\Delta_2 + d_x)c_m](\tilde{H}^1)^k = 0 \tag{8.4}$$

and

$$\Delta_1 a_{m-1} = 0. \tag{8.5}$$

Since $(\tilde{H}^1)^k$ has independent form components in degrees $k(n-p-1)$, $k(n-p-1)+1$ up to degree n, we infer from (8.4) that the form components of $(\Delta_2 + d_x)c_m$ of degrees 0 up to degree $n - k(n-p-1)$ are zero. If we expand c_m itself according to the form degree, $c_m = \sum c^i_m$, this gives the equations

$$\delta c^i_m + dc^{i-1}_m = 0, \; i = 1, \ldots, n - k(n-p-1), \tag{8.6}$$

and

$$\delta c^0_m = 0. \tag{8.7}$$

Our analysis of the relationship between the Δ-cohomology and the cohomology of δ modulo d indicates then that one can redefine the terms of form degree $> n-k(n-p-1)$ of c_m in such a way that $\Delta c_m = 0$. This does not affect the product $c_m(\tilde{H}^1)^k$. We shall assume that the (irrelevant) higher order terms in c_m have been chosen in that manner. With that choice, c_m is given, up to trivial terms that can be reabsorbed, by $\lambda_m(\tilde{H}^2)^l$, with λ_m a number, so that $a_m = \lambda_m(\tilde{H}^2)^l(\tilde{H}^1)^k$ is a Δ-cocycle by itself. One next repeats successively the analysis for a_{m-1}, a_{m-2} to reach the desired conclusion that a may indeed be assumed to be a polynomial in the $\tilde{H}^a$'s, as claimed above.

The non-triviality of the polynomials in the $\tilde{H}^a$'s is also easy to prove. If $P(\tilde{H}) = \Delta\rho$, with $\rho = \rho_0 + \rho_1 + \ldots + \rho_m$, $N_1(\rho_k) = k$, then one gets at N_1-degree $m+1$ the condition $(P(\tilde{H}))_{m+1} = \Delta_1\rho_m$, which implies $(P(\tilde{H}))_{m+1} = 0$ and $\Delta_1\rho_m = 0$, since no polynomial in $\tilde{H}^1$ is Δ_1-trivial, except zero. It follows that $\rho_m = u(\tilde{H}^1)^m$ up to trivial terms that play no role, where u is a function of the variables of the second set as well as of x^μ and dx^μ. The equation of order m implies then $(P(\tilde{H}))_m = \big((\Delta_2 + d_x)u\big)(\tilde{H}^1)^m + \Delta_1\rho_{m-1}$. The non-triviality of the polynomials in $\tilde{H}^1$ in Δ_1-cohomology yields next $\Delta_1\rho_{m-1} = 0$ and $(P(\tilde{H}))_m = \big((\Delta_2 + d_x)u\big)(\tilde{H}^1)^m$. Since the coefficient of $(\tilde{H}^1)^m$ in $(P(\tilde{H}))_m$ is a polynomial in $\tilde{H}^2$, which cannot be $(\Delta_2 + d_x)$-exact, one gets in fact $(P(\tilde{H}))_m = 0$ and $(\Delta_2 + d_x)u = 0$. It follows that ρ_m fulfills $\Delta\rho_m = 0$ and can be dropped. The analysis goes on in the same way at the lower values of the Δ_1-degree, until one reaches the desired conclusion that the exact polynomial $P(\tilde{H})$ indeed vanishes.

In view of the isomorphism between the characteristic cohomology and $H^*(\Delta)$, this completes the proof of Theorem 2.1 in the case of two p-forms. The case of more p-forms is treated similarly and left to the reader.

9. Invariant Characteristic Cohomology

9.1. Isomorphism theorems for the invariant cohomologies. To compute the invariant characteristic cohomology, we proceed as follows. First, we establish isomorphism theorems between $H^{k,inv}_{char}(d)$, $H^{n,inv}_{n-k}(\delta|d)$ and $H^{k,inv}(\Delta)$. Then, we compute $H^{k,inv}(\Delta)$ for a single p-form. Finally, we extend the calculation to an arbitrary systems of p-forms.

Theorem 9.1.

$$\frac{H^{k,inv}_{char}(d)}{\mathcal{H}^k} \simeq H^{n,inv}_{n-k}(\delta|d),\ 0 \leq k < n, \tag{9.1}$$

$$0 \simeq H^{n,inv}_{n+k}(\delta|d),\ k > 0. \tag{9.2}$$

Theorem 9.2. *The invariant cohomology of Δ is isomorphic to the invariant characteristic cohomology,*

$$H^{k,inv}(\Delta) \simeq H^{k,inv}_{char}(d),\ 0 \leq k \leq n. \tag{9.3}$$

Proof. First we prove (9.1). To that end we observe that the map m introduced in the demonstration of Theorem 4.1 maps $H^{k,inv}_{char}(d)$ on $H^{n,inv}_{n-k}(\delta|d)$. Indeed, in the expansion (4.8) for a, all the terms can be assumed to be invariant on account of Theorem 6.1. The surjectivity of m is also direct, provided that the polynomials in the curvature $P(H)$ are not trivial in $H^*(\delta|d)$, which is certainly the case if there is a single p-form (Theorem 7.5). We shall thus use Theorem 9.1 first only in the case of a single p-form. We shall then prove that Theorem 7.5 extends to an arbitrary system of forms of various form degrees, so that the proof of Theorem 9.1 will be completed.

To compute the kernel of m, consider an element $a^k_0 \in \mathcal{I}$ such that the corresponding a^n_{n-k} is trivial in $H^{n,inv}_{n-k}(\delta|d)$. Then, again as in the proof of Theorem 4.1, one finds that all the terms in the expansion (4.8) are trivial, except perhaps a^k_0, which fulfills $da^k_0 + \delta db^k_1 = 0$, where $b^k_1 \in \mathcal{I}$ is the k-form appearing in the equation expressing the triviality of a^{k+1}_1, $a^{k+1}_1 = db^k_1 + \delta b^{k+1}_2$. This implies $d(a^k_0 - \delta b^k_1) = 0$, and thus, by Theorem 6.2, $a^k_0 = P + db^{k-1}_0 + \delta b^k_1$ with $P \in \mathcal{H}^k$ and $b^{k-1}_0 \in \mathcal{I}$. This proves (9.1), since P is not trivial in $H^*(\delta|d)$ (Theorem 7.5). [Again, we are entitled to use this fact only for a single p-form until we have proved the non-triviality of P in the general case.]

The proof of (9.2) is a direct consequence of Theorem 6.1 and parallels step by step the proof of a similar statement demonstrated for 1-forms in [14] (Lemma 6.1). It will not be repeated here. Finally, the proof of Theorem 9.2 amounts to observing that the map m' that sends $[a^k_0]$ on $[a]$ (Eq. (4.8)) is indeed well defined in cohomology, and is injective as well as surjective (independently of whether $P(H)$ is trivial in the invariant cohomology of δ modulo d).

Note that if the forms do not depend explicitly on x, one must replace (9.1) by

$$\frac{H^{k,inv}_{char}(d)}{(\Lambda \otimes \mathcal{H})^k} \simeq H^{n,inv}_{n-k}(\delta|d). \qquad \square \tag{9.4}$$

9.2. Case of a single p-form gauge field. Theorem 6.3 enables one to compute also the invariant characteristic cohomology for a single p-form gauge field. Indeed, this theorem implies that $H^{n,inv}_{n-k}(\delta|d)$ and $H^n_{n-k}(\delta|d)$ actually coincide since the cocycles of

$H^n_{n-k}(\delta|d)$ are invariant and the coboundary conditions are equivalent. The isomorphism of Theorem 9.1 shows then that the invariant characteristic cohomology for a single p-form gauge field in form degree $< n - 1$ is isomorphic to the subspace of form degree $< n - 1$ of the direct sum $\mathcal{H} \oplus \overline{\mathcal{H}}$. Since the product $H \wedge \overline{H}$ has form degree n, which exceeds $n - 1$, this is the same as the subspace $\mathcal{W}$ of Theorem 2.2. The invariant characteristic cohomology in form degree $k < n-1$ is thus given by $(\mathcal{H} \otimes \overline{\mathcal{H}})^k$, i.e., by the invariant polynomials in the curvature H and its dual $\overline{H}$ with form degree $< n - 1$. Similarly, by the isomorphism of Theorem 9.2, the invariant cohomology $H^{k,inv}(\Delta)$ of Δ is given by the polynomials in $\tilde{H}$ and H with Δ-degree smaller than $n - 1$.

9.3. Invariant cohomology of Δ in the general case. The invariant Δ-cohomology for an arbitrary system of p-form gauge fields follows again from a straightforward application of the Kunneth formula and is thus given by the polynomials in the $\tilde{H}^a$'s and H^a's with Δ-degree smaller than $n - 1$. The explicit proof of this statement works as in the non-invariant case (for that matter, it is actually more convenient to use as degrees not N_1 and N_2, but rather, degrees counting the number of derivatives of the invariant variables χ's. These degrees have the advantage that the cohomology is entirely in degree zero). In particular, none of the polynomials in the $\tilde{H}^a$'s and H^a's is trivial.

The isomorphism of Theorem 9.2 implies next that the invariant characteristic cohomology $H^{k,inv}_{char}(d)$ $(k < n - 1)$ is given by the polynomials in the curvatures H^a and their duals $\overline{H}^a$, restricted to form degree smaller than $n - 1$. Among these, those that involve the curvatures H^a are weakly exact, but not invariantly so. The property of Theorem 7.5 thus extends as announced to an arbitrary system of dynamical gauge forms of various form degrees.

Because the forms have now different form degrees, one may have elements in $H^{k,inv}_{char}(d)$ $(k < n - 1)$ that involve both the curvatures and their duals. For instance, if B^1 is a 2-form and B^2 is a 4-form, the cocycle $H^1 \wedge \overline{H}^2$ is a $(n - 2)$-form. It is trivial in $H^k_{char}(d)$, but not in $H^{k,inv}_{char}(d)$.

10. Invariant Cohomology of δ mod d

The easiest way to work out explicitly $H^{n,inv}_{n-k}(\delta|d)$ in the general case is to use the above isomorphism theorems, which we are now entitled to do. Thus, one starts from $H^{k,inv}(\Delta)$ and one works out the component of form degree n in the associated cocycles.

Because one has elements in $H^{k,inv}(\Delta)$ that involve simultaneously both the curvature and its Δ-invariant dual $\tilde{H}$, the property that $H^{n,inv}_{n-k}(\delta|d)$ and $H^n_{n-k}(\delta|d)$ coincide may no longer hold. In the previous example, one would find that $H^{(1)}_{\lambda\mu\nu}B^{*(2)\lambda\mu\nu}$, which has antighost number two, is a δ-cocycle modulo d, but it cannot be written invariantly so. An important case where the isomorphism $H^{n,inv}_{n-k}(\delta|d) \simeq H^n_{n-k}(\delta|d)$ $(k > 1)$ does hold, however, is when the forms have all the same degrees.

To write down the generalization of Theorem 6.3 in the case of p-forms of different degrees, let $P(H^a, \tilde{H}^a)$ be a polynomial in the curvatures $(p_a + 1)$-forms H^a and their Δ-invariant duals $\tilde{H}^a$. One has $\Delta P = 0$. We shall be interested in polynomials of Δ-degree $< n$ that are of degree > 0 in both H^a and $\tilde{H}^a$. The condition that P be of degree > 0 in H^a implies that it is trivial (but not invariantly so), while the condition that it be of degree > 0 in $\tilde{H}^a$ guarantees that when expanded according to the antighost number, P has non-vanishing components of antighost number > 0,

$$P = \sum_{j=k}^{n} [P]^j_{j-k}. \tag{10.1}$$

From $\Delta P = 0$, one has $\delta[P]^n_{n-k} + d[P]^{n-1}_{n-k-1} = 0$.

There is no polynomial in H^a and $\tilde{H}^a$ with the required properties if all the antisymmetric tensors $B^a_{\mu_1\dots\mu_{p_a}}$ have the same form degree ($p_a = p$ for all a's) since the product $H^a\tilde{H}^b$ has necessarily Δ-degree n. When there are tensors of different form degrees, one can construct, however, polynomials P with the given features.

The analysis of the previous subsection implies straightforwardly.

Theorem 10.1. *Let $a^n_q = a^n_q(\chi) \in \mathcal{I}$ be an invariant local n-form of antighost number $q > 0$. If a^n_q is δ-exact modulo d, $a^n_q = \delta\mu^n_{q+1} + d\mu^{n-1}_q$, then one has*

$$a^n_q = [P]^n_q + \delta\mu'^n_{q+1} + d\mu'^{n-1}_q \tag{10.2}$$

for some polynomial $P(H^a, \tilde{H}^a)$ of degree at least one in H^a and at least one in $\tilde{H}^a$, and where μ'^n_{q+1} and μ'^{n-1}_q can be assumed to depend only on the χ's, i.e., to be invariant. In particular, if all the p-form gauge fields have the same form degree, $[P]^n_q$ is absent and one has

$$a^n_q = \delta\mu'^n_{q+1} + d\mu'^{n-1}_q, \tag{10.3}$$

where one can assume that μ'^n_{q+1} and μ'^{n-1}_q are invariant (μ'^n_{q+1} and $\mu'^{n-1}_q \in \mathcal{I}$).

11. Remarks on Conserved Currents

That the characteristic cohomology is finite-dimensional and entirely generated by the duals $\overline{H}^a$'s to the field strengths holds only in form degree $k < n-1$. This property is not true in form degree equal to $n-1$, where there are conserved currents that cannot be expressed in terms of the forms $\overline{H}^a$, even up to trivial terms.

An infinite number of conserved currents that cannot be expressible in terms of the forms $\overline{H}^a$ are given by

$$T_{\mu\nu\alpha_1\dots\alpha_s\beta_1\dots\beta_r} = -\frac{1}{2}\Big(\frac{1}{p!}H_{\mu\rho_1\dots\rho_p,\alpha_1\dots\alpha_s}H_\nu{}^{\rho_1\dots\rho_p}{}_{,\beta_1\dots\beta_r}$$
$$-\frac{1}{(n-p-2)!}H^*_{\mu\rho_2\dots\rho_{n-p-1},\alpha_1\dots\alpha_s}H^{*\rho_2\dots\rho_{n-p-1}}_\nu{}_{,\beta_1\dots\beta_r}\Big). \tag{11.1}$$

These quantities are easily checked to be conserved

$$T^\mu{}_{\nu\alpha_1\dots\alpha_s\beta_1\dots\beta_r,\mu} \equiv 0 \tag{11.2}$$

and generalize the conserved currents given in [15, 16, 17] for free electromagnetism. They are symmetric for the exchange of μ and ν and are duality invariant in the critical dimension $n = 2p+2$, where the field strength and its dual have the same form degree $p+1$. In this critical dimension, there are further conserved currents which generalize the "zilches",

$$Z^{\mu\nu\alpha_1\dots\alpha_r\beta_1\dots\beta_s} = H^{\mu\sigma_1\dots\sigma_p,\alpha_1\dots\alpha_r}H^{*\nu}{}_{\sigma_1\dots\sigma_p}{}^{,\beta_1\dots\beta_s}$$
$$-H^{*\mu\sigma_1\dots\sigma_p,\alpha_1\dots\alpha_r}H^\nu{}_{\sigma_1\dots\sigma_p}{}^{,\beta_1\dots\beta_s}. \tag{11.3}$$

Let us prove that the conserved currents (11.1) which contain an even total number of derivatives are not trivial in the space of x-independent local forms. To avoid cumbersome notations we will only look at the currents with no β indices. One may reexpress (11.1) in terms of the field strengths as

$$T^{\mu\nu\alpha_1\ldots\alpha_m} = -\frac{1}{2p!}(H^{\mu\sigma_1\ldots\sigma_p,\alpha_1\ldots\alpha_m}H^\nu_{\ \sigma_1\ldots\sigma_p} + H^{\mu\sigma_1\ldots\sigma_p}H^\nu_{\ \sigma_1\ldots\sigma_p}{}^{,\alpha_1\ldots\alpha_m}) + \eta^{\mu\nu}\frac{1}{2(p+1)!}H_{\sigma_1\ldots\sigma_{p+1}}H^{\sigma_1\ldots\sigma_{p+1},\alpha_1\ldots\alpha_m}. \tag{11.4}$$

If one takes the divergence of this expression one gets,

$$T^{\mu\nu\alpha_1\ldots\alpha_m}{}_{,\mu} = \delta K^{\nu\alpha_1\ldots\alpha_m}, \tag{11.5}$$

where $K^{\nu\alpha_1\ldots\alpha_m}$ differs from $kH^\nu_{\ \sigma_1\ldots\sigma_p}{}^{,\alpha_1\ldots\alpha_m}B^{*\sigma_1\ldots\sigma_p}$ by a divergence. It is easy to see that $T^{\mu\nu\alpha_1\ldots\alpha_m}$ is trivial if and only if $H^\nu_{\ \sigma_1\ldots\sigma_p}{}^{,\alpha_1\ldots\alpha_m}\,B^{*\sigma_1\ldots\sigma_p}$ is trivial. So the question is: can we write,

$$H^\nu_{\ \sigma_1\ldots\sigma_p}{}^{,\alpha_1\ldots\alpha_m}B^{*\sigma_1\ldots\sigma_p} = \delta M^{\nu\alpha_1\ldots\alpha_m} + \partial_\rho N^{\rho\nu\alpha_1\ldots\alpha_m} \tag{11.6}$$

for some $M^{\nu\alpha_1\ldots\alpha_m}$ and $N^{\rho\nu\alpha_1\ldots\alpha_m}$? Without loss of generality, one can assume that M and N have the Lorentz transformation properties indicated by their indices (the parts of M and N transforming in other representations would cancel by themselves). Moreover, by Theorem 6.3, one can also assume that M and N are gauge invariant, i.e., belong to $\mathcal{I}$. If one takes into account all the symmetries of the left-hand side and use the identity $dH = 0$, the problem reduces to the determination of the constant c in,

$$H_{\nu\sigma_1\ldots\sigma_p,\alpha_1\ldots\alpha_m}B^{*\sigma_1\ldots\sigma_p} = \delta(cH_{\nu\sigma_1\ldots\sigma_{p-1}(\alpha_1,\alpha_2\ldots\alpha_m)}B^{*\sigma_1\ldots\sigma_{p-1}}) + \partial_\rho N^\rho_{\ \nu\alpha_1\ldots\alpha_m} + \text{terms that vanish on-shell.} \tag{11.7}$$

If one takes the Euler-Lagrange derivative of this equation with respect to $B^{*\sigma_1\ldots\sigma_p}$ one gets,

$$H_{\nu\sigma_1\ldots\sigma_p,\alpha_1\ldots\alpha_m} \approx (-)^{p+1}cH_{\nu[\sigma_1\ldots\sigma_{p-1}|(\alpha_1,\alpha_2\ldots\alpha_m)|\sigma_p]}, \tag{11.8}$$

where the right-hand side is symmetrized in $\alpha_1\ldots\alpha_m$ and antisymmetrized in $\sigma_1\ldots\sigma_p$. The symmetry properties of the two sides of this equation are not compatible unless $c = 0$. This proves that $T^{\mu\nu\alpha_1\ldots\alpha_m}$ (with m even) is not trivial in the algebra of x-independent local forms. It then follows, by a mere counting of derivative argument, that the $T^{\mu\nu\alpha_1\ldots\alpha_m}$ define independent cohomological classes and cannot be expressed as polynomials in the undifferentiated dual to the field strengths $\overline{H}$ with coefficients that are constant forms.

The fact that the conserved currents are not always expressible in terms of the forms $\overline{H}^a$ makes the validity of this property for higher order conservation laws more striking. In that respect, it should be indicated that the computation of the characteristic cohomology in the algebra generated by the $\overline{H}^a$ is clearly a trivial question. The non-trivial issue is to demonstrate that this computation does not miss other cohomological classes in degree $k < n - 1$.

Finally, we point out that the conserved currents can all be redefined so as to be strictly gauge-invariant, apart from a few of them whose complete list can be systematically determined for each given system of p-forms. This point will be fully established in [18], and extends to higher degree antisymmetric tensors a property established in [44] for one-forms (see also [45] in this context).

12. Introduction of Gauge Invariant Interactions

The analysis of the characteristic cohomology proceeds in the same fashion if one adds to the Lagrangian (1.10) interactions that involve higher dimensionality gauge invariant terms. As we shall show in [18], these are in general the only consistent interactions. These interactions may increase the derivative order of the field equations. The resulting theories should be regarded as effective theories and can be handled through a systematic perturbation expansion [46].

The new equations of motion read

$$\partial_\mu \mathcal{L}^{a\mu\mu_1\mu_2\cdots\mu_{p_a}} = 0, \tag{12.1}$$

where $\mathcal{L}^{a\mu\mu_1\mu_2\cdots\mu_{p_a}}$ are the Euler-Lagrange derivatives of the Lagrangian with respect to the field strengths (by gauge invariance, $\mathcal{L}$ involves only the field strength components and their derivatives). These equations can be rewritten as

$$d\overline{\mathcal{L}}^a \approx 0, \tag{12.2}$$

where $\overline{\mathcal{L}}^a$ is the $(n - p_a - 1)$-form dual to the Euler-Lagrange derivatives.

The Euler-Lagrange equations obey the same Noether identities as in the free case, so that the Koszul-Tate differential takes the same form, with $\overline{H}^a$ replaced everywhere by $\overline{\mathcal{L}}^a$. It then follows that

$$\tilde{\mathcal{L}}^a = \overline{\mathcal{L}}^a + \sum_{j=1}^{p+1} \overline{B}_j^{*a} \tag{12.3}$$

fulfills

$$\Delta\tilde{\mathcal{L}}^a = 0. \tag{12.4}$$

This implies, in turn, that any polynomial in the $\tilde{\mathcal{L}}^a$ is Δ-closed. It is also clear that any polynomial in the $\overline{\mathcal{L}}^a$ is weakly d-closed. By making the regularity assumptions on the higher order terms in the Lagrangian explained in [9], one easily verifies that these are the only cocycles in form degree $< n-1$, and that they are non-trivial. The characteristic cohomology of the free theory possesses therefore some amount of "robustness" since it survives deformations. By contrast, the infinite number of non-trivial conserved currents is not expected to survive interactions (even gauge-invariant ones).

[In certain dimensions, one may add Chern-Simons terms to the Lagrangian. These interactions are not strictly gauge invariant, but only gauge-invariant up to a surface term. The equations of motion still take the form d(something) ≈ 0, but now, that "something" is not gauge invariant. Accordingly, with such interactions, some of the cocycles of the characteristic cohomology are no longer gauge invariant. These cocycles are removed from the invariant cohomology, but the discussion proceeds otherwise almost unchanged and is left to the reader.]

13. Summary of Results and Conclusions

In this paper, we have completely worked out the characteristic cohomology $H^k_{char}(d)$ in form degree $k < n - 1$ for an arbitrary collection of free, antisymmetric tensor theories. We have shown in particular that the cohomological groups $H^k_{char}(d)$ are finite-dimensional and take a simple form, in sharp contrast with $H^{n-1}_{char}(d)$, which is infinite-dimensional and appears to be quite complex. Thus, even though one is dealing with free theories, which have an infinite number of conserved local currents, the existence of higher degree local conservation laws is quite constrained. For instance, in ten dimensions, there is one and only one (non-trivial) higher degree conservation law for a single 2-, 3-, 4-, 6-, or 8-form gauge field, in respective form degrees 7, 6, 5, 3 and 1. It is $d\overline{H} \approx 0$. For a 5-form, there are two higher degree conservation laws, namely $d\overline{H} \approx 0$ and $d(\overline{H})^2 \approx 0$, in form degrees 4 and 8. For a 7-form, there are four higher degree conservation laws, namely $d\overline{H} \approx 0$, $d(\overline{H})^2 \approx 0$, $d(\overline{H})^3 \approx 0$ and $d(\overline{H})^4 \approx 0$, in form degrees 2, 4, 6 and 8.

Our results provide at the same time the complete list of the isomorphic groups $H^k(\varDelta)$, as well as of $H^n_{n-k}(\delta|d)$. We have also worked out the invariant characteristic cohomology, which is central in the investigation of the BRST cohomology since it controls the antifield dependence of BRST cohomological classes [14].

An interesting feature of the characteristic cohomology in form degree $< n-1$ is its "robustness" to the introduction of gauge invariant interactions, in contrast to the conserved currents.

As we pointed out in the introduction, the characteristic cohomology is interesting for its own sake since it provides higher degree local conservation laws. But it is also useful in the analysis of the BRST cohomology. The consequences of our study will be fully investigated in a forthcoming paper [18], where consistent interactions and anomalies will be studied (see [47] for the 2-form case in this context). In particular, it will be pointed out how rigid the gauge symmetries are. We will also apply our results to compute the BRST cohomology of the coupled Yang-Mills-two-form system, where the field strength of the 2-form is modified by the addition of the Chern-Simons 3-form of the Yang-Mills field [48]. This computation will use both the present results and the analysis of [50, 36, 49, 14].

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A. Proof of Theorem 6.2

To prove Theorem 6.2, it is convenient to follow the lines of the BRST formalism. In that approach, gauge invariance is controlled by the so-called longitudinal exterior derivative operator γ, which acts on the fields and further variables called ghosts. The construction of γ can be found in [31, 30]. For simplicity, we consider throughout this appendix the case of a single p-form; the general case is covered by means of the Kunneth formula. The important point here is the reducibility of the gauge transformations. Because of this, we need to introduce p ghost fields:

$$C_{\mu_1\dots\mu_{p-1}}, \dots, C_{\mu_1\dots\mu_{p-j}}, \dots, C. \tag{A.1}$$

These ghosts carry a degree called the pure ghost number. The pure ghost number of $C_{\mu_1\dots\mu_{p-1}}$ is equal to 1 and increases by one unit up to p as one moves from the left to the right of (A.1). The action of γ on the fields and the ghosts is given by,

$$\gamma B = dC_1, \tag{A.2}$$

$$\gamma C_1 = dC_2, \tag{A.3}$$

$$\vdots$$

$$\gamma C_{p-1} = dC_p, \quad (A.4)$$

$$\gamma C_p = 0, \quad (A.5)$$

$$\gamma(antifield) = 0. \quad (A.6)$$

In the above equations, C_j is the $p-j$ form whose components are $C_{\mu_1\ldots\mu_{p-j}}$. For p even, C_p is a commuting object.

One extends γ such that it is a differential that acts from the left and anticommutes with d.

The motivation behind the above definition is essentially contained in the following theorem:

Theorem A.1. *The cohomology of γ is given by,*

$$H(\gamma) = \mathcal{I} \otimes \mathcal{C}_p, \quad (A.7)$$

where $\mathcal{C}_p$ is the algebra generated by the last, undifferentiated *ghost C_p. In particular, in antighost and pure ghost numbers equal to zero, one can take as representatives of the cohomological class the gauge invariant functions, i.e, the functions which depend solely on the field strengths and their derivatives. [It is in that sense that the differential γ incorporates gauge invariance.]*

The proof of this theorem follows the lines given in [43], by redefining the generators of the algebra so that γ takes the standard form $\gamma x_i = y_i$, $\gamma y_i = 0$, $\gamma z_\alpha = 0$ in terms of the new generators x_i, y_i, z_α. The paired variables x_i, y_i disappear from the cohomology, which is entirely generated by the unpaired variables z_α. In the present case, one easily convinces oneself that the generators of $\mathcal{I} \otimes \mathcal{C}_p$ are precisely of the z_α-type, while the other generators come in pairs. The derivatives of the last ghost C_p are paired with the symmetrized derivatives of the next-to-last ghost $C_{p-1\mu}$, the other derivatives of the next-to-last ghost $C_{p-1\mu}$ which may be expressed as derivatives of the "curvatures" $\partial_\mu C_{p-1\nu} - \partial_\nu C_{p-1\mu}$, are paired with the derivatives of the previous ghost $\partial_{\alpha_1\ldots\alpha_k} C_{(p-2),\mu\nu}$ involving a symmetrization, say on α_1 and μ, etc. The details present no difficulty and are left to the reader.

According to the theorem, any solution of the equation $\gamma a = 0$ can be written,

$$a = \sum_l \alpha_l(\chi) C^l + \gamma b. \quad (A.8)$$

Furthermore, if a is γ-exact, then one has $\alpha_l \approx 0$ since the various powers of C are linearly independent.

The previous theorem holds independently of whether p is even or odd. We now assume that p is even, so that the curvature $(p+1)$-form H is anticommuting and the last ghost C_p is commuting, and prove Theorem 6.2 in that case (the case when p is odd parallels the 1-form case and so need not be treated here).

Assume that $da_0^k = 0$ with a_0^k a polynomial in the field strengths and their derivatives. By the Poincaré lemma we have $a_0^k = da_0^{k-1}$, but there is no guarantee that a_0^{k-1} is also in $\mathcal{I}_{small}$. Acting with γ on this equation we get, again using the Poincaré lemma, $\gamma a_0^{k-1} + da_1^{k-2} = 0$. One can thus construct a tower of equations which take the form,

$$
\begin{aligned}
a_0^k &= d a_0^{k-1}, && (A.9)\\
\gamma a_0^{k-1} + d a_1^{k-2} &= 0, && (A.10)\\
&\vdots \\
\gamma a_q^{k-1-q} + d a_{q+1}^{k-2-q} &= 0, && (A.11)\\
\gamma a_{q+1}^{k-2-q} &= 0. && (A.12)
\end{aligned}
$$

Let $r = k - 2 - q$ and $q + 1 = m$. If $m = pl$ then the last equation of the tower implies,

$$a_m^r = C_p^l P + \gamma a_{m-1}^r, \tag{A.13}$$

with $P \in \mathcal{I}_{small}$.

If $m \neq pl$ then we simply have $a_m^r = \gamma a_{m-1}^r$. In that case, an allowed redefinition of the tower allows one to suppose that the tower stops earlier with $\gamma a_{m'}^r = 0$ and $m' = pl$. An allowed redefinition of the tower simply adds to a_0^k a term of the form db_0^{k-1}, where b_0^{k-1} is gauge invariant.

So from now, we shall assume that indeed $m = pl$. If we substitute (A.13) in (A.11) we get,

$$\gamma(a_{m-1}^{r+1} + lC_{p-1}C_p^{l-1}) + C_p^l dP = 0 \tag{A.14}$$

(the trivial term γa_{m-1}^r is absorbed in an allowed redefinition of the tower). Since the action of d is well defined in $\mathcal{I}_{small}$ this implies $dP = 0$. The form degree of P is strictly smaller than the form degree of a_0^k, so let us make the recurrence hypothesis that the theorem holds for P. Because we treat the case p even, H is odd and $P = c'H + c + dQ$, where c and c' are constants and $Q \in \mathcal{I}_{small}$. We thus have,

$$a_m^r = cC_p^l + c'C_p^l H + dQC_p^l. \tag{A.15}$$

The last two terms in (A.15) are trivial. For the first one we have,

$$dBC_p^l = d(QC_p^l) - \gamma(QlC_{p-1}C_p^l). \tag{A.16}$$

Then we note that,

$$
\begin{aligned}
HC_p^l &= \frac{1}{l+1}\Big(d\Big(\sum_{\substack{i_1+\ldots+i_{l+1}=pl\\ 0\le i_1\le p,\ldots,0\le i_{l+1}\le p}} C_{i_1}C_{i_2}\ldots C_{i_{l+1}}\Big) && (A.17)\\
&\quad -\gamma\Big(\sum_{\substack{i_1+\ldots+i_{l+1}=pl-1\\ 0\le i_1\le p,\ldots,0\le i_{l+1}\le p}} C_{i_1}C_{i_2}\ldots C_{i_{l+1}}\Big)\Big). && (A.18)
\end{aligned}
$$

To prove the above identity one sets $C_0 = B$ and $C_{p+1} \equiv 0$ since the ghost of highest pureghost is C_p. One also uses the fact that $\gamma C_i = dC_{i+1}$.

This shows that we only need to consider the bottom $a_m^r = cC_p^l$. Let us now prove that if $l > 1$ then $c = 0$. We can write,

$$a_0^{pl} = c \sum_{\substack{i_1+\ldots+i_l=pl\\ 0\le i_1\le p,\ldots,0\le i_{l+1}\le p}} C_{i_1}C_{i_2}\ldots C_{i_l}. \tag{A.19}$$

It is easy to show that for $p(l-1)+1 \le k \le pl$ we have,

$$d\Big(\sum_{\substack{i_1+\ldots+i_l=k \\ 0\leq i_1\leq p,\ldots,0\leq i_l\leq p}} C_{i_1}C_{i_2}\ldots C_{i_l}\Big) = \gamma\Big(\sum_{\substack{i_1+\ldots+i_l=k-1 \\ 0\leq i_1\leq p,\ldots,0\leq i_l\leq p}} C_{i_1}C_{i_2}\ldots C_{i_l}\Big). \quad \text{(A.20)}$$

This implies that,

$$a_p^{p(l-1)} = (-)^p c \sum_{\substack{i_1+\ldots+i_l=p(l-1) \\ 0\leq i_1\leq p,\ldots,0\leq i_{l+1}\leq p}} C_{i_1}C_{i_2}\ldots C_{i_l} + C_p^{l-1}Q, \quad \text{(A.21)}$$

where $Q \in \mathcal{I}_{small}$. If we substitute this in $\gamma a_{p+1}^{p(l-1)-1} + da_p^{p(l-1)} = 0$ we get,

$$(-1)^p clHC_p^{l-1} + dQC_p^{l-1} + \gamma\Big((-)^p c \sum_{\substack{i_1+\ldots+i_l=p(l-1)-1 \\ 0\leq i_1\leq p,\ldots,0\leq i_{l+1}\leq p}} C_{i_1}C_{i_2}\ldots C_{i_l}$$
$$+Q(l-1)C_{p-1}C_p^{l-2} + a_{m+p+1}^{p(l-1)-1}\Big) = 0. \quad \text{(A.22)}$$

The above equation implies,

$$dQ + (-1)^p clH = 0. \quad \text{(A.23)}$$

Because the form degree of H is strictly smaller than the form degree of a_0^k, the above recurrence hypothesis tells us that this equation is impossible unless $c = 0$.

To show that $c = 0$ we had to lift the bottom a_0^{pl} $p+1$ times. This is only possible when the tower has $p+1$ steps, which is the case when $l > 1$. If $l = 1$ then the bottom is $a_0^l = cC_p$. This bottom can be lifted to the top of the tower and yields $a_0^{p+1} = cH + dN$, $N \in \mathcal{I}_{small}$.

To validate the recurrence hypothesis we observe that if a_0^k is of form degree 0 then necessarily $a_0^0 = k$.

This ends the proof of the theorem and shows that for p even we have $a = c + c'H + dN$, $N \in \mathcal{I}_{small}$, as desired.

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