Instability of relativistic electron beam with strong magnetic field

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Abstract Stability criteria for a weak relativistic beam-plasma interaction in a strong magnetic field are found. Two beam modes occur, $\omega \approx k_z c$ and $\omega \approx k_z c - \omega_c$. The dispersion equation of electrostatic two-stream is exactly solved with analytical method.

Keywords: instability, relativistic electron beam, two-stream, strong magnetic field, dispersion relation.

Two-stream instability appears in magnetic confinement devices and laser fusion as well as relativistic electron beam fusion. It plays an important role in ion acceleration and ion compression as well as the ignition of target. In order to make full use of the instability and further control it, it is necessary to understand clearly the relationship between the instability and various parameters and the existent region of the instability.

The relativistic two-stream instability has been investigated theoretically in the limit of zero magnetic field[1], but there does not appear to be an analysis in the literature for this mode in a strong magnetic field. In this work we analyze the fluid equations for the beam and background electrons and delineate the stability parameters and growth rates. We find that electromagnetic effects are important over a wide range of parameters. We also discuss electrostatic two-stream instability excited by relativistic cold electron beam. Although the dispersion relation was discussed by many a researcher in the past, only a few of its approximative solutions were obtained[2—6]. In this note we also solve exactly the dispersion relation of electrostatic two-stream by using analytical method and calculate numerically the growth rate.

1 The dispersion relation in strong magnetic field

Both the beam and background plasma are described in the pressureless fluid limits where the beam speed is taken as v_b along the z directed magnetic field B_z while the background electron speed is zero. The self field of the beam is assumed to be much less than B_z and is neglected in this analysis. Similarly, toroidal effects are neglected. We assume ω, $\omega_{pe} \ll \gamma\omega_c$, where $\omega_{pe} \equiv (4\pi n_e e^2/m)^{1/2}$, $\gamma\omega_c \equiv eB_z/mc$, $\gamma = (1 - v_b^2/c^2)^{-1/2}$. n_e is the cold plasma density, m the electronic rest mass, $\gamma\omega_c$ the nonrelativistic cyclotron frequency, v_b the beam velocity. In this limit we can neglect the transverse dynamics of the cold plasma. However, we will include the approximation transverse dynamics of the beam, which is a consistent approximation if $\gamma \gg 1$. The perturbed equations are the form[1)].

1) Berk, H. L., Relativistic beam plasma instability in strong magnetic field, Report in Lawrence Livermor Lab., 1972, preprint 1—24.

$$
\begin{cases}
\left(\frac{\partial}{\partial t}+v_b\frac{\partial}{\partial z}\right)\delta n_b+\nabla\cdot(n_b\delta v_b)=0,\\
\left(\frac{\partial}{\partial t}+v_b\frac{\partial}{\partial z}\right)(\gamma\delta v_b)=\frac{e}{m}\left(E+\frac{v_b}{c}\hat{z}\times\delta B+\frac{\delta v_b}{c}\times B_z\right),\\
\frac{\partial}{\partial t}\delta n_e+\frac{\partial}{\partial z}(n_e\delta v_{ez})=0,\\
\frac{\partial}{\partial t}\delta v_{ez}=\frac{e}{m}E_z,\\
-\frac{1}{c}\frac{\partial}{\partial t}\delta B=\nabla\times E,\\
\nabla\times\delta B=\frac{1}{c}\frac{\partial}{\partial t}E+4\pi\, e[(n_e\delta v_{ez}+\delta n_b v_b)\hat{z}+\delta n_b\delta v_b].
\end{cases}
\tag{1}
$$

We consider perturbations to be of the form $\hat{f}\exp[-\mathrm{i}(\omega t-k_z z-k_x x)]$. From the perturbed equations of mass and motion and electromagnetic field we obtain the following dispersion relation:

$$
\left(1-\frac{\omega_{pe}^2}{\omega^2}-\frac{\omega_b^2}{\gamma^2\tilde{\omega}^2}\right)(\omega^2-k_z^2c^2-\tilde{\omega}^2\Lambda)-k_x^2c^2(1-\Lambda/\gamma^2)=0, \tag{2}
$$

where

$$
\Lambda=\frac{\omega_B^2}{\tilde{\omega}^2-\omega_c^2}[(k^2c^2-\omega^2+\omega_b^2)/(k^2c^2-\omega^2+\omega_b^2\tilde{\omega}^2/(\tilde{\omega}^2-\tilde{\omega}_c^2))],
$$

$$
\tilde{\omega}=\omega-k_z v_b, \omega_b=(4\pi\, n_b e^2/m\gamma)^{1/2}.
$$

2 Analysis of instability

To analyze the stability of eq. (2), we assume $\omega_b<<\omega_{pe}$, ω_c. Therefore in eq. (2) we need only keep terms linear in ω_b^2. Hence we take $\Lambda=\omega_b^2/(\tilde{\omega}^2-\omega_c^2)$. The interesting regimes for stability then occur near the resonances where (A) $\tilde{\omega}=0$ and (B) $\tilde{\omega}^2=\omega_c^2$. For case (A) we can set Λ=0 and for case (B) we can set $\omega_b^2/\gamma^2\tilde{\omega}^2=0$.

(i) For case (A) $\tilde{\omega}\approx 0$. (2) can be approximated as

$$
k_z^2=\omega_{pe}^2/\xi^2+\omega_b^2/[\gamma^2(\xi-v_b)^2]+k_x^2c^2/(\xi^2-c^2), \tag{3}
$$

where $\xi=\omega/k_z$. If $k_x^2=0$, eq. (3) is unstable. Thus we have

$$
1-\omega_{pe}^2/\omega^2-\omega_b^2/[\gamma^2(\omega-k_2v_b)^2]=0. \tag{4}
$$

This is a purely electrostatic case. It will be discussed in sec. 3 for cold plasma beam.

If we choose $\omega=\omega_{pe}+\delta\omega$ and $\omega_{pe}/k_z=v_b$, we then find from (2), that the unstable root is given by

$$
\delta\omega=2^{-4/3}\frac{\omega_{pe}}{\gamma}\left(\frac{n_b}{n_e}\right)^{1/3}(-1+\mathrm{i}\sqrt{3}). \tag{5}
$$

If k_x^2 is sufficiently large, eq. (3) is stable (see fig. 1). In fig. 1 typical curves are shown. For $0<\xi<v_b$, two possibilities are evident, either the minimum occurs for $k_z^2>0$ (solid curve) or for $k_z^2<0$ (dotted curve).

Marginal stability corresponds to the minimum lying on the ξ axis. To find the stability let us assume $\omega_b<<\omega_{pe}$, and $\gamma>>1$. Therefore $\xi_{min}\approx v_b\approx c$. We can write eq. (3) approximately as

$$k_z^2 = \omega_{pe}^2 / v_b^2 + \omega_b^2 /[\gamma^2 (\xi - v_b)^2] + k_x^2 c /[2(\xi - c)]. \quad (6)$$

Stability occurs for (a) $k_x > \frac{\omega_{pe}}{\gamma c}$, if $\gamma^2 \omega_b^2 < \omega_{pe}^2 / 8$, (b) $k_x > 2 \frac{\omega_{pe}}{\gamma^{3/4} c} \left(\frac{n_b}{n_e} \right)^{1/4}$, if $\gamma^2 \omega_b^2 >> \omega_{pe}^2 / 4$.

(ii) For case (B) $\tilde{\omega} \approx \omega_c, \omega_b << \gamma\omega_c$, eq. (2) can be approximated as

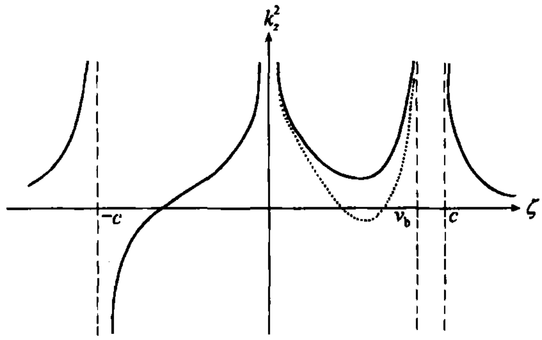

Fig. 1. The curves for k_z^2 vs. ξ.

$$\omega^2 - (\omega_{pe}^2 + k^2 c^2) + k_z^2 c^2 \omega_{pe}^2 / \omega^2 - \frac{\omega_b^2}{\tilde{\omega}^2 - \omega_c^2} \left[\tilde{\omega}^2 \left(1 - \frac{\omega_{pe}^2}{\omega^2} \right) - \frac{k_x^2 c^2}{\gamma^2} \right] = 0. \quad (7)$$

Let us choose ω_0 as the solution of eq. (7) when ω_b^2 is completely neglected. We then obtain

$$\omega_0^2 = \frac{1}{2}(k^2 c^2 + \omega_{pe}^2) \pm \frac{1}{2}[(k^2 c^2 + \omega_{pe}^2)^2 - 4\omega_{pe}^2 k_z^2 c^2]^{1/2}. \quad (8)$$

To simplify the results we use the following approximate relations for ω_0

$$\omega_0^2 = k_z^2 c^2 \omega_{pe}^2 /(\omega_{pe}^2 + k^2 c^2) \text{ and } \omega_0^2 = \omega_{pe}^2 + k^2 c^2. \quad (9)$$

They are pure oscillatory solutions. In the limit $\omega_{pe} << \omega_c$, it is clear that $\omega_0 << \omega_c$, $k_z v_b \approx \omega_c$. Since $kc > k_z v_b \approx \omega_c >> \omega_{pe}$, we have $\omega_0 = k_z \omega_{pe} / k$ which is an electrostatic oscillation.

Instability can be found for $k^2 > \omega_c^2 / v_b^2$. When $k v_b / \omega_c = \sqrt{3}$, the maximum growth rate ω_I is

$$\omega_I = 3^{-1/4} \left(\frac{1}{6} \right)^{1/2} \omega_b \left(\frac{\omega_{pe}}{\omega_c} \right)^{1/2}. \quad (10)$$

Now let us investigate the other limit $\omega_c < \omega_{pe} < \gamma\omega_c$. When $[1+(k^2c^2/\omega_{pe})]=4$, the maximum growth rate ω_I is then

$$\omega_I = \frac{3^{1/2}}{4} \omega_b. \quad (11)$$

If we combine eq. (10) and eq. (11), we find

$$\frac{\omega_I}{\omega_b} = \min \left[3^{-1/4} 6^{-1/2} \left(\frac{\omega_{pe}}{\omega_c} \right)^{1/2}, \frac{3^{1/2}}{4} \right]. \quad (12)$$

3 Electrostatic two-stream instability on cold relativistic electron beam

Let us set $u = v_b, \omega_{Be}^2 = \gamma\omega_b^2 = \frac{4\pi n_b e^2}{m}, \gamma_1 = \frac{1}{\gamma} = (1 - u^2 / c^2)^{1/2}, k_z = k$. Thus, from eq. (4) we have

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\gamma_1^3 \omega_{Be}^2}{(\omega - ku)^2} = 0. \quad (13)$$

Eq. (13) can be written as an algebraic equation of fourth order for ω

$$\omega^4 + a_1 \omega^3 + a_2 \omega^2 + a_3 \omega + a_4 = 0, \quad (14)$$

in which

$$a_1 = -2ku, a_2 = k^2u^2 - \omega_{pe}^2 - \gamma_1^3\omega_{Be}, a_3 = 2ku\omega_{pe}^2, a_4 = -k^2u^2\omega_{pe}^2. \tag{15}$$

According to the formulae of solving fourth order equation[7], we can obtain

$$\omega_j = x_j + \frac{1}{2}ku \qquad (j = 1,\ 2,\ 3,\ 4), \tag{16}$$

$$\begin{cases} x_1 = \frac{1}{2}\left\{-\left[(A+B) - \frac{2}{3}p\right]^{\frac{1}{2}} + 2\rho\cos\frac{\varphi}{2}\right\}, \\ x_2 = \frac{1}{2}\left\{\left[(A+B) - \frac{2}{3}p\right]^{\frac{1}{2}} - 2\rho\cos\frac{\varphi}{2}\right\}, \\ x_3 = \frac{1}{2}\left\{\left[(A+B) - \frac{2}{3}p\right]^{\frac{1}{2}} + \mathrm{i}2\rho\sin\frac{\varphi}{2}\right\}, \\ x_4 = \frac{1}{2}\left\{\left[(A+B) - \frac{2}{3}p\right]^{\frac{1}{2}} - \mathrm{i}2\rho\sin\frac{\varphi}{2}\right\}, \end{cases} \tag{17}$$

where

$$\begin{cases} A = -\frac{1}{3}\{[\omega_{pe}^2 + \gamma_1^3\omega_{Be}^2 - k^2u^2)^3 + 54k^2u^2\omega_{pe}^2\gamma_1^3\omega_{Be}^2] \\ \qquad -6\sqrt{3}ku\omega_{pe}\gamma_1^{3/2}\omega_{Be}[(\omega_{pe}^2 + \gamma_1^3\omega_{Be}^2 - k^2u^2)^3 + 27k^2u^2\omega_{pe}^2\gamma_1^3\omega_{Be}^2]^{1/2}\}^{1/3}, \\ B = -\frac{1}{3}\{[\omega_{pe}^2 + \gamma_1^3\omega_{Be}^2 - k^2u^2)^3 + 54k^2u^2\omega_{pe}^2\gamma_1^3\omega_{Be}^2] \\ \qquad +6\sqrt{3}ku\omega_{pe}\gamma_1^{3/2}\omega_{Be}[(\omega_{pe}^2 + \gamma_1^3\omega_{Be}^2 - k^2u^2)^3 + 27k^2u^2\omega_{pe}^2\gamma_1^3\omega_{Be}^2]^{1/2}\}^{1/3}, \\ \rho = \left[\left(\frac{1}{2}A + \frac{1}{2}B + \frac{2}{3}p\right)^2 + \frac{3}{4}(A-B)^2\right]^{1/4}, \quad p = -\frac{1}{2}k^2u^2 - \omega_{pe}^2 - \gamma_1^3\omega_{Be}^2, \\ \cos\varphi = -\left[\frac{1}{2}(A+B) + \frac{2}{3}p\right]\Big/\rho^2, \quad \sin\varphi = \sqrt{\frac{3}{2}}(A-B)\Big/\rho^2. \end{cases} \tag{18}$$

From the condition (15) satisfied by cubic equation with complex conjugate roots, it can be found that boundary of instability and damping is valid for

$$(\omega_{pe}^2 + \gamma_1^3\omega_{Be}^2 - k^2u^2)^3 + 27k^2u^2\omega_{pe}^2\gamma_1^3\omega_{Be}^2 = 0. \tag{19}$$

This is a cubic equation for k^2u^2. From Cadar formula we have found

$$\left(\frac{k^2u^2}{\omega_{pe}^2}\right)_c = 1 + \frac{\gamma_1^3\omega_{Be}^2}{\omega_{pe}^2} + 3\left[\frac{1}{2}\left(1 + \frac{\gamma^3\omega_{Be}^2}{\omega_{pe}^2}\right)\frac{\gamma^3\omega_{Be}^2}{\omega_{pe}^2}\right]^{1/3}\left\{\left[1 + \left(1 - \frac{4\gamma_1^3\omega_{Be}^2/\omega_{pe}^2}{(1+\gamma_1^3\omega_{Be}^2/\omega_{pe}^2)^2}\right)^{1/2}\right]^{1/3} + \left[1 - \left(1 - \frac{4\gamma_1^2\omega_{Be}^2/\omega_{pe}^2}{(1+\gamma_1^3\omega_{Be}^2/\omega_{pe}^2)^2}\right)^{1/2}\right]^{1/3}\right\}. \tag{20}$$

When $\gamma_1^3\omega_{Be}^2/\omega_{pe}^2 \ll 1$, we have

$$k_c \simeq \frac{\omega_{pe}}{u}\left[1+\frac{3}{2}\gamma_1\left(\frac{\omega_{Be}}{\omega_{pe}}\right)^{2/3}\right]. \tag{21}$$

Obviously, $\dfrac{ku}{\omega_{pe}} \le \left(\dfrac{ku}{\omega_{pe}}\right)_c$.

In the case of collision against each other for $n_{Be}= n_e$, we have

$$\left(\frac{ku}{\omega_{pe}}\right)_c \simeq (1+\gamma_1)^3.$$

For convenience's sake, write

$$u_1 = \frac{ku}{\omega_{pe}}, \quad u_2 = \frac{\omega_{Be}^2}{\omega_{pe}^2} = \frac{n_b}{n_e}. \tag{22}$$

Thus

$$\begin{cases} a = (1/\gamma_1^3 u_2 - u_1^2)^3 + 54u_1^2\gamma_1^3 u_2, \\ b = 6\sqrt{3}u_1\gamma_1^{3/2}u_2^{1/2}[(1+\gamma_1^3 u_2 - u_1^2)^3 + 27u_1^2\gamma_1^3 u_2]^{1/2}, \\ c = 2\left(1+\frac{1}{2}u_1^2 + \gamma_1^3 u_2\right), \quad \rho = 6^{-1/2}\omega_{pe}\rho_0, \\ \rho_0 = \{[(a+b)^{1/3} + (a-b)^{1/3} + 2c]^2 + 3[(a+b)^{1/3} - (a-b)^{1/3}]^2\}^{1/4}, \\ \varphi = \mathrm{tg}^{-1}\dfrac{\sqrt{3}[(a+b)^{1/3} - (a-b)^{1/3}]}{(a+b)^{1/3} + (a-b)^{1/3} + 2c}. \end{cases} \tag{23}$$

Hence we have found from (16)—(18), (22), (23) that

$$\omega_I/\omega_{pe} = I_m\omega/\omega_{pe} = \rho\sin\frac{\varphi}{2}/\omega_{pe}$$
$$= \left\{\rho_0^2 - [(a+b)^{\frac{1}{3}} + (a-b)^{\frac{1}{3}} + 2c]\right\}^{\frac{1}{3}} / 2\sqrt{3}, \tag{24}$$

$$\omega_\gamma/\omega_{pe} = R_e\omega/\omega_{pe}$$
$$= \left\{u_1 + 3^{-\frac{1}{2}}[c - ((a+b)^{\frac{1}{3}} + (a-b)^{\frac{1}{3}})]^{\frac{1}{2}}\right\}/2. \tag{25}$$

Since electron energy $E=mc^2/\gamma_1$, $\gamma_1=mc^2/E=0.511/E$ [MeV].

The values of ω_I/ω_{pe} and $\omega_\gamma/\omega_{pe}$ can then be calculated from (16), (17), (23)—(25) as E, u_2 are given (see figs. (2—4)).

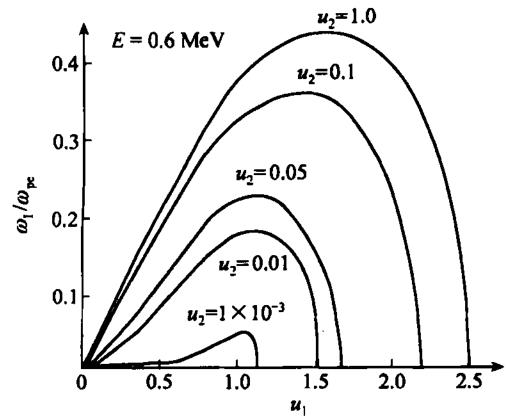

Fig. 2. The curves for ω_I/ω_{pe} vs. u_1 (as E=0.6 MeV, u_2 varies).

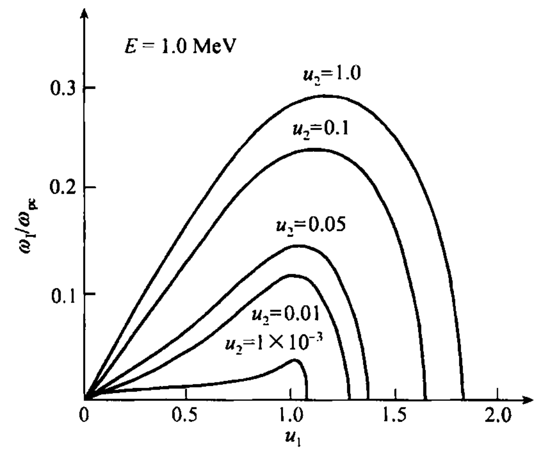

Fig. 3. The curves for ω_I/ω_{pe} vs. u_1 (as E=1.0 MeV, u_2 varies).

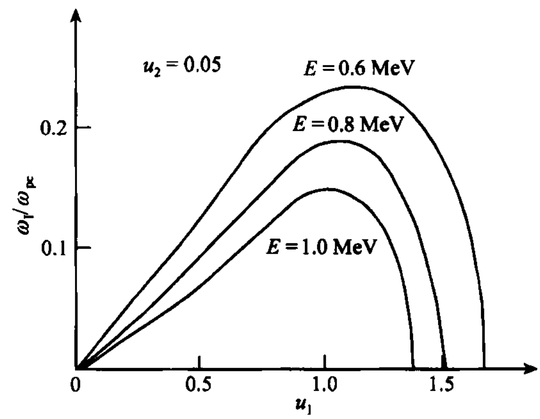

Fig. 4. The curves for ω_I/ω_{pe} vs. u_1 (as u_2=0.05, E varies).

4 Conclusion and discussion

Stability criteria for a weak relativistic beam-plasma interaction in a strong magnetic field are found. Two beam modes occur, $\omega \approx k_z c$ and $\omega \approx k_z c - \omega_c$. The former mode is stabilized if the k_x is sufficiently large. For electrostatic two-beam mode we have obtained analytic expressive forms (24) and (25). These solutions are general. The result of numerical calculation is expressed as figs. 2—4. The maximum of growth rate ω_I shifts to the region of $ku > \omega_{pe}$, the growth rate and the region of instability became larger with increasing density ratio n_{Be}/n_e, also became less with increasing energy, and effect of relativity can suppress electrostatic two beam instability, too. It can be seen that the higher the energy of relativistic electron beam, the larger the application range of approximation of weak stream.

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