A predict-correct projection method for monotone variant variational inequalities

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Abstract A predict-correct projection method is presented for solving monotone variant variational inequalities, which could exploit the advantages and overcome the difficulties of both explicit and implicit projection methods.

Keywords: variant variational inequality, explicit projection method, implicit projection method, predict-correct projection method.

SOME nonlinear programming problems can be translated in to a class of variant variational inequalities: Find a u^* such that

$$F(u^*) \in \Omega,\ (v - F(u^*))^T u^* \geqslant 0,\ \forall v \in \Omega, \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$ is a closed convex set and F is a continuous mapping from $\mathbb{R}^n$ into itself. The existence results on such problems have been investigated recently by Pang and Yao[1]. Throughout this note we assume that the solution set of (1) denoted by Ω^* is nonempty and the projection on Ω is simple to carry out. The Euclidean norm in this note will be denoted by $\|\cdot\|$.

It is easy to prove that the variant variational inequality (1) is equivalent to the projection equation: $F(u) = P_\Omega[F(u) - \beta u]$ with $\beta > 0$, i.e. to find a zero point of the continuous nonsmooth function:

$$e(u,\ \beta) := \frac{1}{\beta}(F(u) - P_\Omega[F(u) - \beta u]),$$

where $P_\Omega[\cdot]$ denotes the orthogonal projection on Ω.

For an arbitrary start point u^0, we denote $\Omega^1 = \{u \in \mathbb{R}^n \mid \|u - u^*\| \leqslant \|u^0 - u^*\|,\ u^* \in \Omega^*\}$ and $\Omega^0 = \{u \in \mathbb{R}^n \mid \|u - u^*\| \leqslant 2\|u^0 - u^*\|,\ u^* \in \Omega^*\}$.

Definition 1. The function F is said to be Lipschitz continuous on set Ω^0 if there is a constant $L > 0$ such that $F(u) - F(v)\| \leqslant L\|u - v\|$ for any $u,\ v \in \Omega^0$.

Definition 2. The function F is said to be

1) monotone on set Ω^0 if $(u - v)^T(F(u) - F(v)) \geqslant 0$ for any $u, v \in \Omega^0$;

2) strongly monotone on the set Ω^0 if there exists a constant $\alpha > 0$ such that $(u - v)^T(F(u) - F(v)) \geqslant \alpha\|u - v\|^2$ for any $u, v \in \Omega^0$.

There have already been explicit projection method and implicit projection method for solving the nonlinear variant variational inequalities.

1 Explicit projection mdthod1)

Given $u^0 \in \mathbb{R}^n$, $\beta > \frac{L^2}{2\alpha}$. For $k = 0, 1, \cdots$, if $u^k \notin \Omega^*$,

$$u^{k+1} = u^k - e(u^k,\ \beta). \tag{2}$$

The explicit projection method is simple, but it converges under strong assumptions. If the mapping F is Lipschitz continuous and strongly monotone on Ω^0, and the stepsize $\beta > \frac{L^2}{2\alpha}$, the explicit method is globally linear convergence.

2 Implicit projection method2)

Given $u^0 \in \mathbb{R}^n$, $\beta > 0$. For $k = 0, 1, \cdots$, if $u^k \notin \Omega^*$, u^{k+1} solves $G_k(u) = 0$, where

$$G_k(u) = u + \frac{1}{\beta}F(u) - u^k - \frac{1}{\beta}F(u^k) + e(u^k,\ \beta). \tag{3}$$

1) He, B., A Goldstein's type projection method for a class of variant variational inequalities, to appear in *Journal of Computational Mathematics*.

2) He, B., Inexact implicit methods for monotone general variational inequalities, *Technical Report* 95-60, *Faculty of Technical Mathematics and Informatics*, Delft University of Technology, 1995.

The implicit projection method is globally convergent under mild condition that F is monotone. However, one has to solve a system of nonlinear equations in each iteration.

We consider a predict-correct method, which uses the explicit method (2) to make a prediction:

$$\text{(P)} \quad u_{(0)}^{k+1} = u^k - e(u^k, \beta), \tag{4}$$

and then uses implicit scheme (3) to make a correction:

$$\text{(C)} \qquad u^{k+1} = u^k + \frac{1}{\beta}F(u^k) - e(u^k, \beta) - \frac{1}{\beta}F(u_{(0)}^{k+1}). \tag{5}$$

Combining formulas (4) and (5), we state our algorithm as follows.

3 Predict-Correct projection method.

Give $u^0 \in \mathbb{R}^n$ and $\beta \geqslant 3L$. For $k = 0, 1, \cdots$, if $u^k \notin \Omega^*$,

$$u^{k+1} = u^k - \frac{1}{\beta}\mathrm{d}(u^k, \beta),$$

where $\mathrm{d}(u, \beta) = F(u - e(u, \beta)) - P_\Omega[F(u) - \beta u]$.

Theorem 1. Let $u^* \in \Omega^*$, F be Lipschitz continuous and monotone on Ω^0. Then

$$(u - u^*)^{\mathrm{T}}\mathrm{d}(u, \beta) \geqslant e(u, \beta)^{\mathrm{T}} d(\mathrm{u}, \beta)$$

for any $u \in \Omega^1$ and $\beta > L$.

Proof. Setting $v = P_\Omega[F(u) - \beta u] \in \Omega$ in (1), we get

$$(u^*)^{\mathrm{T}}(P_\Omega[F(u) - \beta u] - F(u^*)) \geqslant 0. \tag{6}$$

It follows from the property of projection on a closed convex set (see Appendix B in ref. [2]) that

$$(F(u) - \beta u - P_\Omega[F(u) - \beta u])^{\mathrm{T}}(P_\Omega[F(u) - \beta u] - F(u^*)) \geqslant 0,$$

i.e.

$$(e(u, \beta) - u)^{\mathrm{T}}(P_\Omega[F(u) - \beta(u)] - F(u^*)) \geqslant 0. \tag{7}$$

Adding (6) and (7), we obtain

$$(e(u, \beta) - (u - u^*))^{\mathrm{T}}(P_\Omega[F(u) - \beta u] - F(u^*)) \geqslant 0. \tag{8}$$

It follows from $u \in \Omega^1$, lemma 1[1)], and the Lipschitz continuation and monotone on Ω^0 of F that

$$\begin{aligned}
\| u - e(u,\beta) - u^* \|^2 &= \| u - e(u,\beta) - (u^* - e(u^*,\beta)) \|^2 \\
&= \frac{1}{\beta^2} \| F(u) - \beta u - P_\Omega[F(u) - \beta u] \\
&\quad - (F(u^*) - \beta u^* - P_\Omega[F(u^*) - \beta u^*]) \|^2 \\
&\leqslant \frac{1}{\beta^2} \| F(u) - F(u^*) - \beta(u - u^*) \|^2 \\
&\leqslant \left(1 + \frac{L^2}{\beta^2}\right) \| u - u^* \|^2,
\end{aligned} \tag{9}$$

i.e. $u - e(u,\beta) \in \Omega^0$, and that

$$(u^* - (u - e(u,\beta)))^{\mathrm{T}}(F(u^*) - F(u - e(u,\beta))) \geqslant 0. \tag{10}$$

Adding (8) and (10) yields

1) See footnote 1) on page 1265.

$$(e(u,\beta)-(u-u^*))^{\mathrm{T}}(P_{\Omega}[F(u)-\beta u]-F(u-e(u,\beta)))\geqslant 0.$$

Then Theorem 1 is proved. Q.E.D.

Theorem 2. If F is Lipschitz continuous and monotone on Ω^0 and $\beta\geqslant 3L$, the sequence $\{u^k\}$ is generated by the predict-correct method, then

$$\| u^{k+1}-u^* \|^2 \leqslant \| u^k - u^* \|^2 - 0.22 \| e(u^k,\ \beta) \|^2,$$

for any $u^*\in\Omega^*$.

Proof. If $u^k\in\Omega^1$, $u^k-e(u^k,\ \beta)\in\Omega^0$ from (9), F being Lipschitz continuous on Ω^0, we get

$$\begin{aligned}\| \mathrm{d}(u^k,\ \beta) \|^2 &= \| F(u^k-e(u^k,\beta)) - F(u^k) + \beta e(u^k,\ \beta) \|^2\\ &\leqslant (L^2+\beta^2) \| e(u^k,\beta) \|^2,\end{aligned}$$

and

$$\begin{aligned}e(u^k,\beta)^{\mathrm{T}}\mathrm{d}(u^k,\beta) &= e(u^k,\beta)^{\mathrm{T}}(F(u^k-e(u^k,\beta)) - F(u^k) + \beta e(u^k,\beta))\\ &\geqslant (\beta - L) \| e(u^k,\beta) \|^2.\end{aligned}$$

Then it follows from Theorem 1 that

$$\begin{aligned}\| u^{k+1}-u^* \|^2 &= \| u^k-u^* \|^2 - \frac{2}{\beta}(u^k-u^*)^{\mathrm{T}}\mathrm{d}(u^k,\beta) + \frac{1}{\beta^2}\| \mathrm{d}(u^k,\beta) \|^2\\ &\leqslant \| u^k-u^* \|^2 - \left(1-\frac{2L}{\beta}-\frac{L^2}{\beta^2}\right) \| e(u^k,\beta) \|^2\\ &\leqslant \| u^k-u^* \|^2 - 0.22 \| e(u^k,\beta) \|^2.\end{aligned} \tag{11}$$

From $u^0\in\Omega^1$ and (11), we get $u^1\in\Omega^1$, and then $u^2\in\Omega^1\cdots$. And the theorem is proved.

Q.E.D.

Theorem 3. If the assumption of Theorem 2 is satisfied, the predict-correct method is globally convergent.

Proof. From Theorem 2 and Theorem 3 of ref. [3], the convergent conclusion is obtained.

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