Continuous maps with the whole space being a scrambled set

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1 A problem of scrambled set

CHAOTIC behavior is a manifestation of complexity of nonlinear dynamical systems. Since the objects, methods, aims, or emphases of study are distinct, there are some variant definitions of chaos given by different authors, or given by the same author in his different works. The following definition mainly stems from Li and Yorke[1].

Definition 1. Let (X, d) be a metric space, and $f: X \to X$ be a continuous map. A subset S of X containing at least two points is called a scrambled set of f if

$$\liminf_{k\to\infty} d(f^k(x), f^k(y)) = 0, \qquad \limsup_{k\to\infty} d(f^k(x), f^k(y)) > 0,$$

for any two different points x, y in X. f is said to be chaotic in the sense of Li and Yorke if it has an uncountable scrambled set.

Many mathematicians and other scientists joined in the discussion on chaos, in which a question is, from the point of view of cardinal number, or topology, or measure: how large can the scrambled set S in Definition 1 be? When X is a compact interval or a circle, it is well known (see refs. [2—4]) that f has an uncountable scrambled set if and only if f has a scrambled set containing only two points. It is proved in ref. [5] that on the one-sided symbolic space Σ_N, the shift $\sigma: \Sigma_N \to \Sigma_N$ has a scrambled set C of which the Hausdorff dimension is 1 everywhere (this C has even stronger chaotic behavior stated in ref. [5]). On the other hand, it is pointed out in Theorem 2 of ref. [5] that any scrambled set C of σ can contain only a few points: under the metric given by ref. [5], the 1-dimensional Housdorff measure of Σ_N is 1, but that of C can only be 0.

We now raise a more general problem.

Problem. *Are there a metric space X and a continuous map $f: X \to X$ such that* (i) *the Hausdorff dimension of X is s with $0 < s < \infty$, and the s-dimensional Hausdorff measure of X is μ with $0 < \mu < \infty$;* (ii) *f has a scrambled set W of which the s-dimensional Hausdorff measure $H^s(W) > 0$?*

We will discuss this problem. Our main result is the following theorem.

Theorem 1. *Let X be a metric space uniformly homeomorphic to the n-dimensional open cube I^n, $n \geqslant 2$. Then there exists a homeomorphism $f: X \to X$ such that the whole space X is a scrambled set of f.*

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2 An expansive self-homeomorphism of the open cube I^n

In order to prove Theorem 1, we first introduce some conclusions concerned in reference [6]

Define an analytic function $h: \mathbb{R}^2 \to \mathbb{R}$ by, for any $(r, s) \in \mathbb{R}^2$,

$$h(r,s) = s[1 + \pi r^2 + \pi(1 + r^{126})^{1/64}] + r^2(1 + r^2)^{1/4}[\sin 2\pi(1 + r^2)^{1/4} - \sin 2\pi(1 + r^2)^{1/8}]. \quad (1)$$

Let $\mathbb{Z}_+$ denote the set of all nonnegative integers. We obtained the following key lemma in reference [6].

Lemma 1. *For any $N \in \mathbb{Z}_+$ and any two different points (λ_1, μ_1), $(\lambda_2, \mu_2) \in R^2$, there exists an integer $m \geqslant N$ such that*

$$h(\lambda_1 + m, \mu_1) < -1 < 1 < h(\lambda_2 + m, \mu_2),$$

or

$$h(\lambda_2 + m, \mu_2) < -1 < 1 < h(\lambda_1 + m, \mu_1).$$

Let the open interval $I = (-\pi/2, \pi/2)$, and $n \geqslant 2$ be a given integer. Define maps $\rho: R^n \to R^n$, $T_1: R^n \to R^n$ and $\psi: I^n \to R^n$ by

$$\rho(r, s_2, \cdots, s_n) = (r, h(r, s_2), \cdots, h(r, s_n)), \quad \text{for any } (r, s_2, \cdots, s_n) \in R^n, \quad (2)$$

$$T_1(r, s_2, \cdots, s_n) = (r + 1, s_2, \cdots, s_n), \quad \text{for any } (r, s_2, \cdots, s_n) \in R^n, \quad (3)$$

$$\psi(x, y_2, \cdots, y_n) = (\mathrm{tg} x, \mathrm{tg} y_2, \cdots, \mathrm{tg} y_n), \quad \text{for any } (x, y_2, \cdots, y_n) \in I^n. \quad (4)$$

Put $\varphi: R^n \to R^n$ and $f: I^n \to I^n$ by

$$\varphi = \rho T_1 \rho^{-1}, \qquad f = \psi^{-1} \varphi \psi. \quad (5)$$

Then it is easy to see that ρ, T_1, ψ, φ and f are all analytic homeomorphisms. In ref. [6] we obtained the following theorem.

Theorem A. *Both $\varphi: R^n \to R^n$ and $f: I^n \to I^n$ defined above are doubly-expansive homeomorphisms. They can be imbedded in analytic flows on R^n and I^n respectively. An expansive constant of φ is 2, and that of f is $\pi/2$.*

(The definitions of expansive map and expansive constant can be found in reference [6] or [7]).

2 Homeomorphisms with the whole space being a scrambled set

Obviously, we have

Lemma 2. *Let (X, d) be a metric space, $\varepsilon > 0$, and $\xi: X \to X$ be an expansive continuous map of which ε is an expansive constant. If ξ is an injection, then*

$$\limsup_{k \to \infty} d(\xi^k(x), \xi^k(y)) \geqslant \varepsilon$$

for any two different points x, y in X.

Let $f: I^n \to I^n$ be as that in sec. 2. By Lemma 2 and Theorem A we know that for any two different points u and v in I^n,

$$\limsup_{k \to \infty} d(f^k(u), f^k(v)) \geqslant \pi/2, \quad (6)$$

where d is the Euclidean metric on R^n and on $I^n \subset R^n$.

Lemma 3. *Let $h: R^2 \to R$ be defined as in (1). For any $k \in \mathbb{Z}_+$, write $m_k = m(k) = 16k^4 + 2k^2$. Then*

$$\lim_{k \to \infty} h(r + m_k, s) = \infty, \qquad \textit{for any } (r, s) \in R^2.$$

Proof. Let g_1 be a function defined on some subset D of R^i, $i \geqslant 1$, and g_2 be a function on R. If there exists a positive number M such that $|g_1(x, y_2, \cdots, y_i)| < M \cdot g_2(x)$ for any $(x, y_2, \cdots, y_i) \in D$, and if we need only to estimate the value of $g_1(x, y_2, \cdots, y_i)$ but do not need to know its precise expression, then we simply write $O(g_2(x))$ for $g_1(x, y_2, \cdots, y_i)$.

Then, for any given point $(r, s) \in R^2$ and any integer $k > |r|$, we have

$$[1 + (r + m_k)^2]^{1/4} = [256k^8 + 64k^6 + O(k^4)]^{1/4}$$

$$= \left[\left(4k^2 + \frac{1}{4}\right)^4 + O(k^4)\right]^{1/4} = \left(4k^2 + \frac{1}{4}\right)\cdot [1 + O(k^{-4})]^{1/4}$$

$$= \left(4k^2 + \frac{1}{4}\right)\cdot [1 + O(k^{-4})] = 4k^2 + \frac{1}{4} + O(k^{-2}), \tag{7}$$

$$[1 + (r + m_k)^2]^{1/8} = \left[4k^2 + \frac{1}{4} + O(k^{-2})\right]^{1/2} = 2k \cdot [1 + O(k^{-2})]^{1/2}$$

$$= 2k \cdot [1 + O(k^{-2})] = 2k + O(k^{-1}), \tag{8}$$

$$(r + m_k)^2[1 + (r + m_k)^2]^{1/4} = [256k^8 + O(k^6)] \cdot [4k^2 + O(1)] = 1024k^{10} + O(k^8), \tag{9}$$

$$s\{1 + \pi(r + m_k)^2 + \pi[1 + (r + m_k)^{126}]^{1/64}\}$$

$$= s[256\pi k^8 + O(k^6) + O(k^{63/8})] = O(k^8). \tag{10}$$

It follows from (7) and (8) that

$$\lim_{k\to\infty} \sin 2\pi[1 + (r + m_k)^2]^{1/4} = \sin\frac{2\pi}{4} = 1, \tag{11}$$

$$\lim_{k\to\infty} \sin 2\pi[1 + (r + m_k)^2]^{1/8} = \sin 4\pi = 0. \tag{12}$$

By (1) and (9)—(12) we obtain

$$\lim_{k\to\infty} h(r + m_k, s) = \lim_{k\to\infty}[1\,024k^{10} + O(k^8)] = \infty.$$

Lemma 3 is proven.

Suppose that the homeomorphisms ρ, T_1, ψ and φ are also as in sec. 2. For any given point $u \in I^n$, let $w = \psi(u)$ and let $(r, s_2, \cdots, s_n) = \rho^{-1}(w)$. Then by (5), (3) and (2) we have

$$\varphi^{m(k)}(w) = \rho T_1^{m(k)}\rho^{-1}(w) = \rho(r + m_k, s_2, \cdots, s_n)$$

$$= (r + m_k, h(r + m_k, s_2), \cdots, h(r + m_k, s_n)). \tag{13}$$

For $i = 1, \cdots, n$, define the projection $p_i : R^n \to R$ by $p_i(t_1, t_2, \cdots, t_n) = t_i$, (for any $(t_1, t_2, \cdots, t_n) \in R^n$). By (13) and Lemma 3 we get

$$\lim_{k\to\infty} p_i\varphi^{m(k)}(w) = \infty, \qquad i = 1, \cdots, n. \tag{14}$$

Put $z_0 = \left(\frac{\pi}{2}, \frac{\pi}{2}, \cdots, \frac{\pi}{2}\right) \in R^n$, and $J = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then z_0 is a vertex of the n-dimensional closed cube J^n. From (5), (4) and (14) it follows that

$$\lim_{k\to\infty} f^{m(k)}(u) = \lim_{k\to\infty} \psi^{-1}\ \varphi^{m(k)}(w) = z_0. \tag{15}$$

From (15) we see that for any u and $v \in I^n$,

$$\liminf_{k\to\infty} d(f^k(u), f^k(v)) = 0. \tag{16}$$

By (6) and (16) we obtain at once the following theorem.

Theorem 2. *Let $f : I^n \to I^n$ be defined as in sec. 2. Then the whole space I^n is a scrambled set of f.*

Definition 2. Let (X, d) and (X', d') be metric spaces, and let $\eta : X \to X'$ be a homeomorphism. X and X' are said to be uniformly homeomorphic (under η) if both η and η^{-1} are uniformly continuous. X and X' are said to be Lipschitz homeomorphic (under η) if both η and η^{-1} satisfy the Lipschitz condition.

Example. (i) Two Lipschitz homeomorphic metric spaces must be uniformly homeomorphic.

(ii) If X and X' are subspaces of compact metric spaces Y and Y' respectively, and there exists a homeomorphism $\eta' : Y \to Y'$ such that $\eta'(X) = X'$, then X and X' must be uniformly homeomorphic. Particularly, the interior $\mathring{B}^n$ of every n dimensional ball B^n must be uniformly homeomorphic to the n-dimensional open cube I^n.

Proposition 1. *Let $\eta : X \to X'$ be a uniformly homeomorphism, $V \subset X$, and $V' = \eta(V)$. Suppose that $g : X \to X$ is a continuous map, and $g' = \eta g \eta^{-1}$. Then V' is a scrambled set of g' if and only if V is a scrambled set of g.*

Proposition 1 is evident. From Proposition 1 and Theorem 2 we obtain Theorem 1 immediately.

Finally, we have the following conjecture, which yet remains to be proved.

Conjecture. *If X is a compact metric space, then the whole space X cannot be a scrambled set of any continuous map from X to X.*

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References

1 Li, T. Y., Yorke, J., Period 3 implies chaos, *Amer. Math. Monthly*, 1975, 82:985.

2 Fedorenko, V. V., Sarkovskii, A. N., Smital, J., Characterizations of weakly chaotic map of the interval, *Proc. Amer. Math. Soc.*, 1990, 110:141.

3 Liao Gongfu, ω-limit sets and chaos for maps of the interval, *Northeastern Math. J.*, 1990, 6:127.

4 Mai Jiehua, Some dynamical properties of maps of the circle and their equivalent condition, *Advances in Math.* (in Chinese), 1997, 26(1):1.

5 Xiong Jincheng, The Hausdorff dimension of a scrambled set of the shift on symbolic space, *Science in China*, Ser. A (in Chinese), 1995, 25(2):97.

6 Mai Jiehua, Ouyang Yiru, Yang Yanchang, An analytic doubly-expansive self-homeomorphism of the open n-cube, *Acta. Math. Sinica*, New Series, 1985, 1:335.

7 Nitecki, Z., *Differentiable Dynamics*, London: M.I.T. Press, 1971.

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