SOME REMARKS ON WEAKLY COMPACTLY GENERATED BANACH SPACES

BY

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ABSTRACT

A simple example is given of a non WCG space whose dual is a WCG space with an unconditional basis. It is proved that if X^* is WCG and X is a subspace of a WCG space then X itself is WCG.

1. Introduction

A Banach space X is said to be weakly compactly generated (WCG in short) if there is a weakly compact subset K in X which generates X, that is, X is the closed linear span of K. In a survey paper on WCG spaces [8] (written in 1967) the second-named author raised several problems concerning this class of spaces. Among these problems there were the following permanence questions.

1. Assume that X is a subspace of a WCG space. Is X itself WCG?

2. Assume that X^* is WCG. Is X also WCG?

Problem 1 was recently answered negatively by H. P. Rosenthal [11]. In this paper we give a simple example which answers Problem 2 negatively, and show that a positive result can be obtained by combining both questions. We prove that a space X whose dual is WCG and which is a subspace of a WCG space is itself WCG.

The example mentioned above is presented in Section 2. It was obtained while studying the question of extending a WCG space by a WCG space, that is, by considering those spaces X such that $X \supset Y$ with Y and X/Y both WCG. The example (denoted by U), besides answering Problem 2, has other properties

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which are of some interest. For example, U^* is w^* separable without U being isomorphic to a subspace of $m = l_\infty$. This seems to be the first example in the literature of a space having this property. We would like to point out that a little after we found the example U, a different negative solution to Problem 2 was found. It turned out (refer to [9]) that the James tree J_0 [3] has the property that all its even conjugates (that is, J_0, $J_0^{**}, J_0^{(iv)}, \cdots$) are WCG while its odd conjugates ($J_0^*, J_0^{***}, \cdots$) are not WCG. This counterexample to Problem 2 is, however, more complicated than the space U given here and fails to have some of the additional interesting properties of U.

Section 3 is devoted to the proof of the positive result mentioned above. The proof uses the technique of long sequences of projections which is a common tool in the study of WCG spaces, as well as the recent theorem on the factorization of weakly compact operators [2]. Special cases of this result were proved earlier by John and Zizler [6]. (They showed, for example, that if X^* and X^{**} are both WCG the same is true for X.)

2. Extending WCG spaces by WCG spaces

In [8] it has already been observed that if $X \supset Y$ with Y and X/Y WCG then in general X need not be WCG. Here we want to investigate this situation a little further. Before we proceed we recall the following fact concerning WCG spaces which was proved in [8]: every WCG space is generated by a set which is in its weak topology homeomorphic to the one point compactification of a discrete set.

There are two special classes of spaces which are trivially WCG: separable spaces and reflexive spaces. These classes play a special role for the question in which we are interested.

Proposition 1. *Let $X \supset Y$ be Banach spaces such that Y is reflexive. Then X is* WCG *if and only if X/Y is* WCG.

Proof. Since every quotient space of a WCG space is trivially again WCG the *only if* part is evident. To prove the *if* part let $\{\hat{x}_\gamma\}_{\gamma \in \Gamma}$ be a set which spans X/Y so that $\|\hat{x}_\gamma\| < 1$ for every γ and so that $\{\hat{x}_\gamma\}_{\gamma \in \Gamma}$ is in its w topology homeomorphic to the one point compactification of a discrete set (with zero corresponding to the point at infinity). Let $T: X \to X/Y$ denote the quotient map and $J: X \to X^{**}$ the canonical embedding. Pick for every $\gamma \in \Gamma$ an $x_\gamma \in X$ so that $Tx_\gamma = \hat{x}_\gamma$ and $\|x_\gamma\| < 1$. The set $K = \{x_\gamma\}_{\gamma \in \Gamma} \cup \{y \in Y: \|y\| \leq 1\}$ is w compact and generates

X. Indeed, let $\{Jx_{\gamma_\tau}\}$ be a net of elements in $\{Jx_\gamma\}_{\gamma\in\Gamma}$ converging w^* to some $x^{**}\in X^{**}$. Since $\{Tx_{\gamma_\tau}\}$ tends weakly to zero we obtain that $T^{**}\,x^{**} = 0$. It follows that $x^{**}\in \ker T^{**} = JY$, by the reflexivity of Y. Hence $x^{**}\in JK$ and thus JK is w^* closed; that is, K is w compact. That K generates X follows from the following observation. Let $x^*\in X^*$ be such that $x^*(K) = 0$. Then $x^*_{|Y} = 0$ and thus $x^* = T^*\psi$ for some $\psi\in(X/Y)^*$. Since $\psi(\hat{x}_\gamma) = \psi(Tx_\gamma) = x^*(x_\gamma) = 0$ for every $\gamma\in\Gamma$, we obtain that $\psi = 0$ and hence $x^* = 0$.

PROPOSITION 2. *Let $X\supset Y$ be Banach spaces with X/Y separable. Then X is* WCG *if and only if Y is* WCG.

PROOF. Assume that Y is WCG and let K be a w compact subset of Y which generates Y. Let $\{x_n\}_{n=1}^\infty$ be a sequence in X tending in norm to zero so that $\{Tx_n\}_{n=1}^\infty$ spans X/Y ($T\colon X\to X/Y$ is the quotient map). It is trivial to verify that $K\cup\{x_n\}_{n=1}^\infty\cup\{0\}$ is a w compact set in X which generates X.

Assume, conversely, that X is WCG. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a set, homeomorphic in its w topology to the one point compactification of a discrete set, which generates X. Since X/Y is separable there is a subset Γ_0 of Γ with $\Gamma\sim\Gamma_0$ countable so that $x_\gamma\in Y$ for $\gamma\in\Gamma_0$. The closed linear span Z of $\{x_\gamma\}_{\gamma\in\Gamma_0}$ is WCG and Y/Z is separable. By the first part of the proof it follows that Y is WCG.

As Example 1 below shows, the roles of the reflexive and separable spaces in Propositions 1 and 2 cannot be interchanged. Before presenting this example we state another general proposition concerning WCG extensions of WCG spaces.

PROPOSITION 3. *Let $X\supset Y$ be Banach spaces with Y separable and X/Y* WCG. *Then the following three assertions are equivalent*:

(i) *X is* WCG;

(ii) *X is a subspace of a* WCG *space*;

(iii) *there is a* WCG *space Z containing Y and a bounded linear operator $S\colon X\to Z$ whose restriction to Y is the identity.*

PROOF. Trivially (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) so we have only to show that (iii) $\Rightarrow$ (i). Since whenever $Z\supset Y$ with Z WCG and Y separable there is a separable W so that $Z\supset W\supset Y$ and W is complemented in Z (see [1]), there is no loss of generality to assume in (iii) that Z is separable. Let $T\colon X\to X/Y$ be the quotient map. It is easily checked that $\tau\colon x\to(Sx, Tx)$ is an isomorphism from X into $Z\oplus X/Y$. Moreover, since Z is separable, $(Z\oplus X/Y)/\tau X$ is separable and hence by Proposition 2, τX (and thus X) is WCG.

EXAMPLE 1. *There exists a Banach space U having the following properties:*

(a) *U has a subspace V isometric to c_0 so that U/V is isometric to $l_2(\Gamma)$ (where Γ is a set of the cardinality of the continuum);*

(b) *U^* is w^* separable and thus U is not* WCG;

(c) *U^* is isomorphic to $l_1 \oplus l_2(\Gamma)$ and thus is a* WCG *space with an unconditional basis;*

(d) *U is not isomorphic to a subspace of l_∞; the same is true for every non-separable subspace of U;*

(e) *U has an equivalent Fréchet differentiable norm.*

PROOF. Let $\{N_\gamma\}_{\gamma \in \Gamma}$ be a collection of infinite subsets of the integers so that $N_{\gamma_1} \cap N_{\gamma_2}$ is finite for $\gamma_1 \neq \gamma_2$ and so that Γ has the cardinality of the continuum. For each $\gamma \in \Gamma$ let ϕ_γ be the element in l_∞ which is the characteristic function of the set N_γ. Let U_0 be the algebraic linear span in l_∞ of $V \cup \{\phi_\gamma : \gamma \in \Gamma\}$ where $V = c_0$ is the space of sequences tending to zero. We norm U_0 by

$$\left|\left|\left| \sum_{i=1}^{k} a_{\gamma_i}\phi_{\gamma_i} + y \right|\right|\right| = \max\left(\left\| \sum_{i=1}^{k} a_{\gamma_i}\phi_{\gamma_i} + y \right\|_\infty , \left(\sum_{i=1}^{k} |a_{\gamma_i}|^2 \right)^{\frac{1}{2}} \right),$$

where $\| \ \|_\infty$ is the usual sup norm in l_∞, $y \in V$, and $\gamma_i \neq \gamma_j$ if $i \neq j$. It is easily seen that $|||\ |||$ is a well-defined norm (observe that the a_{γ_i} are determined by $x = y + \sum_{i=1}^{k} a_{\gamma_i}\phi_{\gamma_i}$ since $a_{\gamma_i} = \lim x(n)$ as n tends to infinity on N_{γ_i}). On V, $|||\ |||$ obviously coincides with $\| \ \|_\infty$. Let U be the completion of U_0 with respect to $|||\ |||$. We check now that U has all the desired properties.

(a) By definition,

$$\left|\left|\left| \sum_{i=1}^{k} a_{\gamma_i}\phi_{\gamma_i} + y \right|\right|\right| \geqq \left(\sum_{i=1}^{k} |a_{\gamma_i}|^2 \right)^{\frac{1}{2}}$$

for every $y \in V$. On the other hand given $\{a_{\gamma_i}\}_{i=1}^{k}$ it is possible to find a $y \in V$ so that

$$\left\| \sum_{i=1}^{k} a_{\gamma_i}\phi_{\gamma_i} + y \right\|_\infty = \max_{1 \leqq i \leqq k} |a_{\gamma_i}|$$

and thus

$$\left|\left|\left| \sum_{i=1}^{k} a_{\gamma_i}\phi_{\gamma_i} + y \right|\right|\right| \leqq \left(\sum_{i=1}^{k} |a_{\gamma_i}|^2 \right)^{\frac{1}{2}}.$$

(b) It is easily checked that U is a linear subspace of the space of all vectors z in l_∞ such that for every $\gamma \in \Gamma$, $z_\gamma = \lim\{z(n),\ n \to \infty \text{ on } N_\gamma\}$ exists and $||| z ||| = \max(\| z \|_\infty, (\sum_{\gamma \in \Gamma} |z_\gamma|^2)^{\frac{1}{2}}) < \infty$. Thus the coordinate functionals $z \to z(n)$, $n = 1, 2, \cdots$, form a total set of functionals in U^* (that is, $z(n) = 0$, all $n \Rightarrow z = 0$

and this in turn implies that U^* is w^* separable. Since U is nonseparable and every w compact subset of U is separable we infer that U is not WCG.

(c) Clearly $U^*/V^\perp$ is isometric to l_1. Since l_1 has the lifting property, $U^* \approx V^\perp \oplus l_1 \approx l_2(\Gamma) \oplus l_1$.

In order to verify (d) and (e) it is convenient to use the explicit natural embedding JU of U in $U^{**} = l_\infty \oplus l_2(\Gamma)$. It is easily seen that JU is the closed linear span of the vectors of the form $(y,0)$, $y \in c_0 \subset l_\infty$, and the vectors (ϕ_γ, e_γ), $\gamma \in \Gamma$ where $\phi_\gamma \in l_\infty$ is the characteristic function of N_γ and e_γ is the γth unit vector in $l_2(\Gamma)$.

(d) Let X be a non-separable subspace of U and assume first that X (considered as a subspace of JU) contains all the vectors of the form $(y,0)$ with $y \in c_0$. Let $u_i^* = (y_i^*, z_i^*)$ be a sequence of norm 1 functionals in $l_1 \oplus l_2(\Gamma) = U^*$. Let Γ_0 be the countable subset of Γ which is the union of the supports of $\{z_i\}_{i=1}^\infty$. Let k be an integer. Since X is non-separable it contains an element of the form (y_1, z_1) with $\| z_1 \| = 1$ and $(\Sigma_{\gamma \in \Gamma_0} |z_1(\gamma)|^2)^{\frac{1}{2}} \leqq k^{-2}$. Let $\Gamma_1 = \Gamma_0 \cup \{\text{support of } z_1\}$ and choose an element (y_2, z_2) in X with $\| z_2 \| = 1$ and $(\Sigma_{\gamma \in \Gamma_1} |z_2(\gamma)|^2)^{\frac{1}{2}} \leqq k^{-2}$. Continuing in this manner k steps we obtain that X contains an element (y,z) with $z = (z_1 + z_2 + \cdots + z_k)/\sqrt{k}$ so that $\| z \| \geqq 1 - k^{-1} \geqq \frac{1}{2}$, $(\Sigma_{\gamma \in \Gamma_0} |z(\gamma)|^2)^{\frac{1}{2}} \leqq k^{-1}$, and $\sup_{\gamma \in \Gamma} |z(\gamma)| \leqq k^{-1}$. By the description of JU given above it follows that there exists a $\tilde{y} \in c_0$ such that $\| y + \tilde{y} \|_\infty \leqq \max(k^{-1}, \sup_{\gamma \in \Gamma} |z(\gamma)|) = k^{-1}$. The element $(y + \tilde{y}, z)$ belongs to X, has norm greater than $\frac{1}{2}$ and for every integer i,

$$|u_i^*(y + \tilde{y}, z)| \leqq \max(\| y + \tilde{y} \|_\infty, \sum_{\gamma \in \Gamma_0} |z(\gamma)|^2)^{\frac{1}{2}}) = k^{-1}.$$

Thus X is not isomorphic to a subspace of l_∞. In order to prove the same without the assumption that X contains $V = c_0$ apply the following observation to $W = \overline{\text{span}}\{X, V\}$. Assume that $W \supset X$ are Banach spaces such that X and W/X are both isomorphic to subspaces of l_∞; then W is isomorphic to a subspace of l_∞. Indeed, let $T: X \to l_\infty$ and $S: W/X \to l_\infty$ be isomorphisms. Then $w \to (S\tau w, \tilde{T}w)$ is an isomorphism from W into $l_\infty = l_\infty \oplus l_\infty$ where $\tau: W \to W/X$ is the quotient map and $\tilde{T}$ an extension of T to W.

(e) Introduce in $U^* = l_1 \oplus l_2(\Gamma)$ the norm

$$\| (y^*, z^*) \|_0 = \sum_{n=1}^\infty |y^*(n)| + \left(\sum_{n=1}^\infty |y^*(n)|^2 + \sum_{\gamma \in \Gamma} |z^*(\gamma)|^2 \right)^{\frac{1}{2}}.$$

It is easily checked that $\| \quad \|_0$ is an equivalent locally uniformly convex norm in U^*

(for definition of locally uniformly convex see [8]). By using the explicit representation of JU in U^{**} it is easily checked that the unit ball of $\| \ \|_0$ is w^* closed and thus $\| \ \|_0$ is a dual to a norm in U. This norm in U must be Fréchet differentiable

REMARK 1. By Propositions 3 and parts (a), (b) and (e) above, we infer in particular that a smooth Banach space need not be isomorphic to a subspace of a WCG space. This answers negatively problem 9 in [8].

REMARK 2. The fact that U^* has an unconditional basis is of interest in connection with some recent results of the first named author and H. P. Rosenthal (cf. [11]) concerning WCG spaces which have an unconditional basis. It is shown, for example, in [11] that if $X^* \approx Y^*$ with Y WCG and X having an unconditional basis, then X is also WCG. Since $U^* \approx (c_0 \oplus l_2(\Gamma))^*$ we see that in the result above it is not possible to replace the assumption that X has an unconditional basis by assuming that Y and X^* have unconditional bases.

REMARK 3. As we already mentioned above it was shown in [1] that if X is WCG and Y a separable subspace of X then there is a separable Z with $X \supset Z \supset Y$ such that Z is complemented in X. It is also well known that for non WCG spaces X this assertion may fail to be true. For example, l_∞ has no complemented infinite-dimensional and separable subspaces. Pelczynski raised the question whether l_∞ is the worst example in the sense that whenever $X \supset Y$, with Y separable, there is a Z so that $X \supset Z \supset Y$, Z complemented in X and Z isomorphic to a subspace of l_∞. Example 1 shows that this is not the case. Indeed, if Z is such that $U \supset Z \supset V$ and Z isomorphic to a subspace of l_∞ then by (d) above Z is separable. By Sobczyk's theorem [12] V is complemented in Z. If thus, in addition, Z were to be complemented in U, it would follow that $U \approx V \oplus U/V$; that is, $U \approx c_0 \oplus l_2(\Gamma)$, and this contadicts (b).

To conclude this section we consider briefly the space U_0 of Example 1 but in its natural sup norm induced by l_∞.

EXAMPLE 2. *There is a compact Hausdorff space K so that the Banach space $C(K)$ of all the continuous real-valued functions on K has the following properties:*

(a) *$C(K)$ is not a subspace of a* WCG *space;*

(b) *$C(K)$ has an equivalent Frechét differentiable norm;*

(c) *every separable subspace of $C(K)$ is isomorphic to a subspace of c_0.*

PROOF. As in Example 1, let U_0 be the span of c_0 and $\{\phi_\gamma\}_{\gamma \in \Gamma}$ in l_∞. Since U_0 is a subalgebra of l_∞ its completion (after adding the function identically equal to 1) is a $C(K)$ space. Clearly $C(K)/V$ is isomorphic to $c_0(\Gamma)$ where V denotes as above the space c_0. Assertions (a) and (c) are evident (for (a) use Proposition 3). It is also clear that $C(K)^*$ is isomorphic to $l_1 \oplus l_1(\Gamma)$ (that is, to $l_1(\Gamma)$). The following is a norm on $l_1 \oplus l_1(\Gamma)$ which is locally uniformly convex and whose unit ball is w^* closed:

$$\|(y^*, z^*)\|_0 = 2 \sum_{n=1}^{\infty} |y^*(n)| + \sum_{\gamma \in \Gamma} |z^*(\gamma)| + \left(\sum_{n=1}^{\infty} |y^*(n)|^2 + \sum_{\gamma \in \Gamma} |z^*(\gamma)|^2 \right)^{\frac{1}{2}}.$$

We omit the easy verification (the factor 2 in the expression for $\| \ \|_0$ is needed for ensuring w^* closedness of the unit ball). Thus $\| \ \|_0$ induces an equivalent norm on $C(K)$ which is Fréchet differentiable.

REMARK 4. Example 2 shows that even for $C(K)$ spaces the answer to Problem 9 of [8] is negative.

REMARK 5. In [7] it is proved that if $2 < p < \infty$ and if X is a Banach space such that every separable subspace of X is isomorphic to a subspace of l_p, then X is isomorphic to a subspace of $l_p(\tilde{\Gamma})$ for some set $\tilde{\Gamma}$. Example 2 shows that a similar result does not hold if l_p is replaced by c_0.

3. The main result

In this section we shall prove the following theorem.

THEOREM. *Let X be a Banach space such that X^* is* WCG *and X is a subspace of a* WCG *space Y. Then X itself is* WCG.

The proof will be by transfinite induction using long sequences of projections. We shall actually prove that X has a normalized shrinking Markuschevich basis, that is, that there is a biorthogonal family $\{(x_\gamma, f_\gamma)\}_{\gamma \in \Gamma}$ for some index set Γ with $\|x_\gamma\| = 1$ so that X is the closed linear span of $\{x_\gamma\}_{\gamma \in \Gamma}$ and X^* is the (norm) closed linear span of $\{f_\gamma\}_{\gamma \in \Gamma}$. The set $\{x_\gamma\}_{\gamma \in \Gamma} \cup \{0\}$ is thus w compact and spans X We should point out however that once it is known that X is WCG, the existence of a shrinking Markuschevich basis in X follows from known results of Troyanski [14] and John and Zizler [6] (since X^* is by our assumption WCG).

The following lemma ensures the existence of the projections needed in the proof of the theorem.

LEMMA 1. *Let Y be a Banach space generated by a weakly compact symmetric and convex set K. Let X be a subspace of Y and let $T: X \to Z$ be a bounded linear operator into a reflexive space Z so that $T^*: Z^* \to X^*$ is one-to-one* *and has dense range. Let $\mathscr{M}$ be a cardinal number and let Y_0, Y_0', X_0', Z_0 and Z_0' be subspaces of Y, Y^*, X^*, Z and Z^* respectively each having density character no larger than $\mathscr{M}$. Then there are projections P on Y and Q on Z satisfying the following:*

(i) $PK \subseteq K$;

(ii) $PX \subseteq X$;

(iii) $PY \supseteq Y_0$, $P^*Y^* \supseteq Y_0'$, $(P_{|X})^*X^* \supseteq X_0'$;

(iv) $\overline{TPX} = QZ$;

(v) $QZ \supseteq Z_0$, $Q^*Z^* \supseteq Z_0'$;

(vi) $\overline{T^*Q^*Z^*} = (P_{|X})^*X^*$;

(vii) dens $PY \leqq \mathscr{M}$;

(viii) dens $QZ \leqq \mathscr{M}$.

PROOF. We construct P and Q to be limit points in the weak operator topologies of sequences of projections $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ on Y and Z respectively.

Let $P_0 = Q_0 = 0$. We construct inductively for $n \geqq 1$ the projections P_n and Q_n so that the following hold:

(a) $\|P_n\| = 1$;

(b) $P_nK \subset K$;

(c) $P_nX \subset X$;

(d) $P_nY \supseteq P_{n-1}Y \cup Y_0$, $P_n^*Y^* \supseteq P_{n-1}^*Y^* \cup Y_0'$;

(e) $(P_{n|X})^*X^* \supseteq T^*Q_{n-1}^*Z^* \cup X_0'$;

(f) $\overline{TP_nX} \supseteq Q_{n-1}Z \cup Z_0$;

(g) dens $P_nY \leqq \mathscr{M}$;

(h) $\|Q_n\| = 1$;

(i) $Q_nZ \supset TP_nX$;

(j) $Q_n^*Z^* \supset Q_{n-1}^*Z^* \cup Z_0'$;

(k) dens $Q_nZ \leqq \mathscr{M}$.

To see that this is possible, assume that $P_0, \cdots, P_{n-1}$ and $Q_0, \cdots, Q_{n-1}$ have been defined so that all the relevant conditions in (a)-(k) are satisfied. Since the density character of the span of $Q_{n-1}Z \cup Z_0$ is $\leqq \mathscr{M}$ by (k) and since T has dense range (for T^* is assumed to be one-to-one) there is a subspace W of X with dens $W \leqq \mathscr{M}$ and $\overline{TW} \supseteq Q_{n-1}Z \cup Z_0$. By an extension of a result of [1] due to John and Zizler [5] there is a projection P_n on Y satisfying (a), (b), (c), (d), (e), (g) and $P_nX \supset W$ and thus also (f). (Observe that w^* dens $P_{n-1}^*Y^* \leqq$ dens $P_{n-1}Y \leqq \mathscr{M}$ and dens span$[T^*Q_{n-1}^*Z^* \cup X_0'] \leqq \mathscr{M}$). Having chosen P_n we construct Q_n using the reflexivity of Z and the result of [1] so that (h), (i), (j) and (k) hold.

Since K is weakly compact and since Z is reflexive it follows from (b) and (h) that there are subnets $\{P_{n_\alpha}\}$ and $\{Q_{n_\alpha}\}$ of $\{P_n\}$ and $\{Q_n\}$ respectively so that $P_{n_\alpha}y$ converges weakly for every y in K to Py, say, and $Q_{n_\alpha}z$ converges weakly for every $z \in Z$ to a limit Qz. Since K generates Y and $\|P_n\| \leqq 1$ for all n we obtain that $P_{n_\alpha}y \to Py$ for every $y \in Y$. Since X is closed, and thus weakly closed in Y, it follows that $PX \subseteq X$. It is also immediate that P and Q are projections and that (i), (ii), (iii), (v), (vii) and (viii) hold. That (iv) holds follows from (f) and (c). Finally (e) implies that $(P_{|X})^*\, X^* \supseteq T^*Q^*Z^*$ and since, by (iv), $(P_{|X})^*T^*(I - Q^*) = 0$ we obtain, using the fact that T^* has dense range, that $\overline{T^*Q^*Z^*} = (P_{|X})^*X^*$ which is requirement (vi) of the lemma.

Having proved the lemma we pass to the proof of the theorem itself. Let α_0 be the smallest ordinal whose cardinality $|\alpha_0|$ is the density character of X and let ω denote, as usual, the first infinite ordinal. For every ordinal, α, $\omega \leqq \alpha \leqq \alpha_0$, we shall construct a projection S_α on X so that:

(A) $S_\alpha S_\beta = S_{\min(\alpha,\beta)}$;

(B) $S_\beta x \to S_\alpha x$ as $\beta \uparrow \alpha$ for every $x \in X$;

(C) $S_{\alpha_0} =$ identity on X;

(D) dens $S_\alpha X \leqq |\alpha|$;

(E) $S_\beta^* x^* \to S_\alpha^* x^*$ as $\beta \uparrow \alpha$ for every $x^* \in X^*$.

The precise meaning of (B) is that for every $x \in X$, the function $\alpha \to S_\alpha x$ is a continuous function from the ordinals $\{\alpha \colon \omega \leqq \alpha \leqq \alpha_0\}$ with their order topology into X with its norm topology. In a similar manner (E) should be understood, noting that in X^* we take again the *norm topology.*

Once the existence of the projections S_α satisfying (A)-(E) is proved, a simple transfinite induction shows that X has a normalized shrinking Markuschevich basis and thus is WCG. Indeed, by a result of Mackey [10], this is the case if X is separable (that is, $\alpha_0 = \omega$) since X^* being WCG must also be separable. Since for $\alpha < \alpha_0$, dens $(S_{\alpha+1} - S_\alpha)X < |\alpha_0|$ there is by the induction hypothesis a normalized shrinking Markuschevich basis $\{(x^\alpha_\gamma, f^\alpha_\gamma)\}_{\gamma<\alpha}$ for $(S_{\alpha+1}-S_\alpha)x$. It is easily checked that $\{(x^\alpha_\gamma, (S^*_{\alpha+1} - S^*_\alpha)f^\alpha_\gamma)\}_{\gamma<\alpha;\omega\leqq\alpha<\alpha_0}$ is a shrinking Markuschevich basis for X.

Let $Y \supset X$ be the WCG space whose existence is assumed in the statement of the theorem and let K be a convex symmetric, weakly compact set which generates Y. Since X^* is WCG there exists by [2] a reflexive Banach space Z and a one-to-one operator $T^*: Z^* \to X^*$ with dense range. Since Z is reflexive T^* (like every operator on Z^*) is the adjoint of an operator $T: X \to Z$. With α_0 having the same meaning as above, observe that $|\alpha_0| \geqq \text{dens } Z = \text{dens } Z^*$. Let $\{x_\alpha : \omega \leqq \alpha < \alpha_0\}$ and $\{z^*_\alpha : \omega \leqq \alpha < \alpha_0\}$ be dense sets in X and Z^* respectively. (We assume that $|\alpha_0| > \aleph_0$ since otherwise there is nothing to prove.) We construct now inductively norm 1 projections $\{P_\alpha\}_{\omega\leqq\alpha<\alpha_0}$ on Y and $\{Q_\alpha\}_{\omega\leqq\alpha<\alpha_0}$ on Z so that:

(1) $P_\alpha K \subseteq K$;

(2) $P_\alpha X \subseteq X$;

(3) $P_{\alpha+1} Y \supset P_\alpha Y \cup \{x_\beta : \omega \leqq \beta \leqq \alpha\}$;

(4) $P^*_{\alpha+1} Y^* \supset P^*_\alpha Y^*$;

(5) $P_\beta y \to P_\alpha y$ as $\beta \uparrow \alpha$ for every $y \in Y$;

(6) $\text{dens } P_\alpha Y \leqq |\alpha|$;

(7) $\overline{TP_\alpha X} = Q_\alpha Z$;

(8) $Q_{\alpha+1} Z \supset Q_\alpha Z$;

(9) $Q^*_{\alpha+1} Z^* \supset Q^*_\alpha Z^* \cup \{z^*_\beta : \omega \leqq \beta \leqq \alpha\}$;

(10) $\text{dens } Q_\alpha Z \leqq |\alpha|$;

(11) $\overline{T^* Q_\alpha Z^*} = (P_{\alpha|X})^* X^*$;

(12) $Q_\beta z \to Q_\alpha z$ as $\beta \uparrow \alpha$ for each $z \in Z$.

Suppose that the P_β and Q_β have been defined for all $\beta < \alpha$ so that (1)–(12) hold. If α is not a limit ordinal we apply the lemma, with $\mathcal{M} = |\alpha|$, $Y_0 = \text{span } [P_{\alpha-1} Y \cup \{x_{\alpha-1}\}]$, $Y_0' = P^*_{\alpha-1} Y^*$, $X_0' = (P_{\alpha-1\ X})^* X^*$, $Z_0 = Q_{\alpha-1} Z$ and $Z_0' = \text{span } [Q^*_{\alpha-1} Z^* \cup \{z_{\alpha-1}\}]$, to obtain projections $P - P_\alpha$ and $Q = Q_\alpha$ on Y and Z respectively. Properties (i)–(viii) ensured by the lemma easily imply that (1)–(12) are valid.

Suppose next that α is a limit ordinal. Since $P_\beta K \subset K$ for every $\beta < \alpha$ we have, as in the proof of the lemma, that some subnet of $\{P_\beta\}_{\omega\leqq\beta<\alpha}$ converges in the weak operator topology to a norm 1 projection P_α on Y. Since the ranges of $\{P_\beta\}_{\omega \leqq \beta < \alpha}$ and $\{P_\beta^*\}_{\omega \leqq \beta < \alpha}$ are both increasing (by (3) and (4)) we obtain in fact that $\| P_\beta y - P_\alpha y \| \to 0$ as $\beta \uparrow \alpha$ for every $y \in Y$. Similarly $\{Q_\beta\}_{\omega \leqq \beta < \alpha}$ converges in the strong operator topology to a projection Q_α on Z. It is clear that (1)–(12) are satisfied and this completes the inductive construction of the P_α and Q_α.

Now we simply set $S_\alpha = P_{\alpha|X}$ for $\omega \leqq \alpha < \alpha_0$ and $S_{\alpha_0} = I$. It is easily checked that (A), (B), (C) and (D) hold. It remains to verify that the crucial condition (E) is also satisfied. Let α be a limit ordinal. By (12) and the reflexivity of Z it follows that $Q_\beta^* z^*$ tends w to $Q_\alpha^* z^*$ as $\beta \uparrow \alpha$ for every $z^* \in Z^*$. Since $Q_{\beta_1}^* Q_{\beta_2}^* = Q_{\min(\beta_1, \beta_2)}$ we deduce from this that $\| Q_\beta^* z^* - Q_\alpha^* z^* \| \to 0$ as $\beta \uparrow \alpha$ for every $z^* \in Z^*$. Thus by (11),

$$\begin{aligned}(P_{\alpha|X})^* X^* &= \overline{T^* Q_\alpha^* Z^*} = \overline{T^* \bigcup_{\beta < \alpha} Q_\beta^* Z^*} \\ &= \overline{\bigcup_{\beta < \alpha} T^* Q_\beta^* Z^*} = \overline{\bigcup_{\beta < \alpha} (P_{\beta|X})^* X^*}\,.\end{aligned}$$

Since, by (4), $(P_{\beta|X})^* X^*$ is an increasing net of subspaces as $\beta \uparrow \alpha$ it follows that for every $x^* \in (P_{\alpha|X})^* X^*$, $\| (P_{\beta|X})^* x^* - x^* \| \to 0$ as $\beta \uparrow \alpha$. This statement is the same as (E) and thus we have completed the proof of the theorem.

We conclude by mentioning four questions related to the subject of this paper whose answers we do not know.

(1) Assume that X^{**} is WCG. Is X also WCG? James's tree J_0 mentioned in the introduction shows that X^* need not be WCG.

(2) Assume that X^* is WCG. Does X have an equivalent Fréchet differentiable norm? Troyanski proved [13] that X^* has an equivalent locally uniformly convex norm. John and Zizler [4] observed that if, in addition, X is also WCG then Troyanski's construction can be made in such a manner that the unit ball of the locally uniformly convex norm in X^* is also w^* closed and thus it induces an equivalent Fréchet differentiable norm on X. In the two examples we considered in Section 2 we could do the same even in cases where X is not WCG. It is however unclear whether this can be done in general without any further assumption on X.

(3) Characterize those compact Hausdorff spaces K for which $C(K)$ admits an equivalent Fréchet differentiable norm. Example 2 seems to indicate that the answer may not be simple.

(4) Let $X \supset Y$ be Banach spaces, with Y separable. Does there exist a space Z with $X \supset Z \supset Y$, Z complemented in X and the density character of Z is less than or equal to that of the continuum? In this connection see Remark 3.

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