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Spherically symmetric solutions with heat flow in general relativity

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Abstract. Some new solutions of shear-free imperfect fluid spheres with heat flux in the radial direction are obtained They have isotropic pressure and could be the generalizations of earlier solutions of Nariai and of Banerjee and Banerji for perfect fluid without dissipation.

Keywords. Cosmology; imperfect fluid; general relativity, spherically symmetric solution

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1. Introduction

It is interesting to investigate the problem of gravitational collapse of stars with dissipation in the form of radial heat flow in the sense that it gives a more general picture of the collapse problem that what is described in a simple adiabatic fluid. Such models were proposed by Bergmann (1981) and later generalized by Modak (1984). The problem of matching the interior solutions with the exterior Vaidya (Vaidya 1953) metric representing a radial unpolarized radiation flux was studied by Santos (1985), as well as by de Oliveria *et al* (1986).

In the present paper we present some new interior solutions for an isotropic spherically symmetric dissipative fluid with heat flow in the radial direction which is an extension of the previous perfect fluid solutions of Buchdahl (1964), Nariai (1967) and Banerjee and Banerji (1976) to include the heat flow terms in the energy momentum tensor. The condition of fit at the boundary in one of the simple cases discussed. The density, pressure and heat flux terms are also calculated in this particular class of solutions. In other cases these expressions are rather complicated and definite conclusions cannot be derived without suitable assumptions.

2. Einstein's fixed equations and their solutions

We start with the well-known isotropic form of the spherically symmetric metric

$$dS^2 = A^2\, dt^2 - B^2(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2), \tag{1}$$

where A, B are functions of r and t. The energy momentum tensor for a fluid with heat flux is expressed in the standard form as

$$T^\mu_\nu = (\rho + p)v^\mu v_\nu - p\delta^\mu_\nu + q^\mu v_\nu + v^\mu q_\nu, \tag{2}$$

where q^μ represents the heat flow vector which is orthogonal to the velocity vector v^μ. In the present spherically symmetric case the radial component q^1 is nonvanishing. In comoving coordinates we have $v^\mu = \delta^\mu_4 A^{-1}$ and in Einstein's field equations the isotropy of pressure leads to the following equation (see Bergmann 1981)

$$A_{xx} + 2(F_x/F)A_x - (F_{xx}/F)A = 0, \tag{3}$$

where F is written for B^{-1} and x for r^2. For adiabatic motion, that is when there is no heat flux, the solutions must satisfy another field equation $G^1_4 = 0$ and this combined with (3) yields the perfect fluid solutions. The solutions of equation (3) for which $G^1_4 \neq 0$ allow heat flux through the fluid and were previously obtained by others as mentioned in §1. These solutions may be astrophysically interesting and more such solutions are worth investigating.

One of the groups of solutions is obtained by imposing simplifying assumptions regarding A and F in (3). Some such solutions are as follows:

(i) *Bergmann solution*:

$A = A(t)$, which results in $F_{xx} = 0$ and yields without any loss of generality

$$A = 1, \quad B = \frac{R(t)}{1 + \xi(t)r^2}, \tag{4}$$

$\xi(t)$ being an arbitrary function of time.

(ii) *Modak solution*:

$F_{xx} = 0$, so that one gets the solutions in the form

$$A = \left(1 + \frac{a(t)}{1 + \xi(t)r^2}\right), \quad B = \frac{R(t)}{1 + \xi(t)r^2}. \tag{5}$$

This solution is conformally flat and is the most general conformally flat solution of the above type for the metric (1). A special case of (5) was given by Maiti (1982) with $\xi(t) = \text{constant}$.

(iii) The assumption $A_{xx} = 0$ yields the solutions

$$A = [1 + \xi(t)r^2], \quad B = \frac{R(t)}{[1 + \xi(t)r^2]^3}, \tag{6}$$

where ξ and R are arbitrary functions of time.

(iv) Another class of solutions for the metric (1) representing fluid with heat flux in the radial direction is obtained by generalizing the perfect fluid solutions of Nariai and of Banerjee and Banerji.

These solutions are completely new. The metric chosen in this case is in the form originally given by Buchdahl (1964) for his static solutions. These are

$$A = (1 - f)/(1 + f), \quad B = S(t)(1 + f)^2. \tag{7}$$

The function f is assumed to be the form $f=[R(r)\cdot K(t)]$ suggested originally by Nariai. The isotropy of the pressure then demands that R and K must satisfy the following equation

$$2R^3R''K^4-2RR''K^2+6R'^2K^2-6R^2R'^2K^4+\frac{2RR'K^2}{r}-\frac{2R^3R'K^4}{r}=0. \quad (8)$$

It is easy to show that the above equation is satisfied if we have the following simple equation satisfied

$$(R''/R')-(3R'/R)-(1/r)=0. \quad (9)$$

The dash sign stands for the derivative with respect to r. The solutions of (9) will satisfy (8). Equation (9) with $R'\neq 0$ yields the solutions, in view of (7) as follows

$$A=\frac{(Tz^{1/2}-1)}{(Tz^{1/2}+1)},\quad B=\frac{(Tz^{1/2}+1)^2}{T_1 z}, \quad (10)$$

where $z=1+ar^2$, a being an arbitrary constant, $T(t)=K^{-1}$ and T_1 is another arbitrary function of time. In the special case $T_1=T$ we get Nariai's inhomogeneous perfect fluid solution, otherwise when $T_1\neq T$ there is heat flux. We calculate the pressure, density and heat flux for a simple case where $T_1=1$. These are

$$8\pi p=\frac{4a}{X^5Y}-\frac{4z^{1/2}}{Y^3}[\ddot{T}XY-4\dot{T}^2z^{1/2}+2T\dot{T}^2z], \quad (11)$$

$$8\pi\rho=(12a/X^5)+(12\dot{T}^2z/Y^2), \quad (12)$$

$$8\pi q^1B=-(4\dot{T}z^{1/2}ar/X^2Y^2), \quad (13)$$

where we have written $X=(Tz^{1/2}+1)$ and $Y=(Tz^{1/2}-1)$. The contraction takes place for $\dot{T}<0$. The constant a is chosen to be positive in order that ρ remains positive even when $\dot{T}=0$ at the turning point. One can refer to the matching conditions at the boundary where the interior metric can fit with the exterior radiating Vaidya metric. Here the interior spacetime V^- of the system is represented by a nonadiabatic spherically symmetric shear-free collapsing fluid undergoing dissipation in the form of a radial heat flow. While the fluid collapses it produces unpolarized radiation and hence the exterior space-time V^+ of the sphere is described by the Vaidya metric. We have therefore in the interior the metric (1), but the exterior is represented by the Vaidya metric

$$dS^2=\left(1-\frac{2m(v)}{\bar{r}}\right)dv^2-2\,dv\,d\bar{r}+\bar{r}^2(d\theta^2+\sin^2\theta\,d\phi^2),$$

where m, the total energy of the system, is a function of the retarded time v. The intrinsic metric to the hyper-surface Σ as a part of their boundaries is given by

$$dS^2_\Sigma=g_{ij}\,d\xi^i\,d\xi^j=d\tau^2-A^2(\tau)(d\theta^2+\sin^2\theta\,d\phi^2).$$

The conditions of fit need the continuity of the first and second fundamental forms at the boundary hypersurface (Santos 1985) which in turn lead to a relation

$$(p + Bq^1)_\Sigma = 0,$$

where the subscript Σ indicates the value at the boundary surface $r = r_\Sigma$.

It is evident from the above boundary condition that the pressures does not vanish at the boundary as long as there is a heat flux across it. The interior matches with the Schwarzschild's exterior solution only in the absence of the heat flow. In the present case the exterior is the Vaidya metric which includes a time-dependent mass function $m(v)$. This function, however, decreases with time as the energy flows across the boundary in the radial direction.

The behaviour of the sphere now depends on the choice of the function $T(t)$. Consider the instant $\dot{T} = 0$, when the pressure-heat flux relation at the boundary mentioned above reduces in view of (11) and (13), to

$$\ddot{T} = aY_\Sigma/(X_\Sigma^6 Z_\Sigma^{1/2}). \tag{14}$$

Since the relation (14) corresponds to the situation $\dot{T} = 0$, which means a turning point in the motion of the sphere, we get a maximum or a minimum to the dimension according as $\ddot{T} < 0$ or $\ddot{T} > 0$. Again we note that for $\ddot{T} < 0$ the constant a turns out to be negative and as a result the density ρ can attain only negative magnitude at this turning point, which however, does not give a physically realistic model. So for a reasonable model one should choose the function T in such a way that it must possess the property $\ddot{T} > 0$ at the moment $\dot{T} = 0$ (if it has at all a turning point). It can, therefore, represent only a minimum putting a lower limit to the dimension of a contracting sphere. One further notes that apart from a singularity of the type $X = 0$, there may be another kind of singularity given by

$$Y = 0 \text{ that is } Tz^{1/2} = 1.$$

The other classes of solutions of equation (3) we wish to present, are the direct generalizations of the perfect fluid solutions given previously by Nariai and Banerjee and Banerji. In the first case the solutions of (3) are

$$A = \frac{Tz^{1/2} - \alpha}{Tz^{1/2} + \alpha}, \quad B = \frac{(Tz^{1/2} + \alpha)^2}{Tz}. \tag{15}$$

In the second case the appropriate solutions are given by

$$A = \frac{1}{T}\frac{z}{(z + a/T)}, \quad B = \frac{T^2}{z^3}(z + b/T)^2, \tag{16}$$

where in both cases $z = 1 + \eta(t)\, r^2$, with $\eta(t)$ an arbitrary function of time and b, α are constants. Putting $\eta(t) = \text{constant}$ one gets back Nariai's perfect fluid solution from (15) and that of Banerjee and Banerji from (16). We have not investigated in detail the scalars like pressure and density which we hope to do in a subsequent paper.

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