Proof and generalization of Kaplan-Yorke's conjecture under the condition $f'(0) > 0$ on periodic solution of differential delay equations*

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Abstract Using the theory of existence of periodic solutions of Hamiltonian systems, it is shown that many periodic solutions of differential delay equations can be yielded from many families of periodic solutions of the coupled generalized Hamiltonian systems. Some sufficient conditions on the existence of periodic solutions of differential delay equations are obtained. As a corollary of our results, the conjecture of Kaplan-Yorke on the search for periodic solutions for certain special classes of scalar differential delay equations is shown to be true when $f'(0) = \omega > 0$.

Keywords: differential-delay equations, Hamiltonian systems, periodic solutions, conjecture of Kaplan-Yorke.

In 1974, Kaplan and Yorke[1] studied and introduced a new technique for establishing the existence of periodic solutions for certain special classes of scalar differential delay equations of the forms

$$y'(t) = -f(y(t-1)) \tag{0.1}$$

and

$$y'(t) = -[f(y(t-1)) + f(y(t-2))] \tag{0.2}$$

when f is an odd function. They have reduced the search for periodic solutions of (0.1) and (0.2) to a problem of finding periodic solutions of associated ordinary differential systems and obtained some precise conditions under which (0.1) and (0.2) have nonconstant periodic solutions of period 4 and 6, respectively (Theorems 1.1 and 1.2 of ref. [1]). When the scalar differential equations contain more than two delays, like (0.3) below, they made the following conjecture:

"Consider the differential delay equation

$$x'(t) = -[f(x(t-1)) + f(x(t-2)) + \cdots + f(x(t-(n-1)))], \tag{0.3}$$

where $f(0) = 0$, f is odd and $xf(x) > 0$ for $x \neq 0$. Let $X = (x_1, x_2, \cdots x_n)^{\mathrm{T}}$ and $\Phi(X) = (f(x_1), f(x_2), \cdots, f(x_n))^{\mathrm{T}}$, where the superscript T denotes the transpose. Let A_n denote the $n \times n$ antisymmetric matrix with all elements being 1 under the diagonal.

It is readily verified that given any periodic solution $x(t)$ of (0.3) with period $2n$, such that $x(t) = -x(t-n)$ for all t, it must satisfy the system of ordinary differential equations

$$X'(t) = A_n\Phi(X), \tag{0.4}$$

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where $x_1(t) = x(t)$, $x_2(t) = x(t-1), \cdots, x_n(t) = x(t-(n-1))$. Theorems 1.1 and 1.2 of ref. [1] show that if $n = 2$ or $n = 3$ and f suitably well behaves near 0 and ∞, then the converse is also true. Hopefully, such a converse should be true for (0.3) and (0.4)$\cdots$. Unfortunately, we have been unable to carry through the details of the proof in this more general setting".

The existence of periodic solutions of delay differential equations has also been investigated by many authors using different techniques. In 1978, using a different idea from the conjecture of Kaplan and Yorke[1] and some general fixed point principles of nonlinear functional analysis, Nussbaum[2] proved that there exists a periodic solution of (0.3) with period $2n$. Gopalsamy et al.[3], have shown how to construct the periodic solution of Kaplan-Yorke type for some more general differential delay equations with two or three delays. For other related works, the reader may refer to the references cited in refs. [4—6]. However, to our knowledge, the conjecture of Kaplan-Yorke "in this more general setting" for (0.3) and (0.4) is still open.

In this paper, we consider differential delay equations of the form

$$x'(t) = -\sum_{i=1}^{n-1} f(x(t-r_i)), \tag{0.5}$$

which is a bit more general than (0.3), where f is a suitable odd function and $r_i (i = 1, 2, \cdots, n-1)$ are constant delays. We show that the coupled system (0.4) of (0.5) is actually a classical Hamiltonian system (for $n = 2k$) or a generalized Hamiltonian system (for $n = 2k+1$). Using the relationship of periodic solutions between (0.4) and (0.5) and the theory of periodic solutions of Hamiltonian systems, we derive some interesting results on the existence of many periodic solutions of (0.5). As a corollary of our results, we show that the conjecture of Kaplan-Yorke is true when the condition $f'(0) = \omega > 0$ holds.

1 Equivalence on the existence of periodic solutions

We assume that

(H_1) the function $f(x) \in C^1$ for $x \neq 0$, $f(-x) = -f(x)$, $f(0) = 0$, $xf(x) > 0$ for $x \neq 0$ and $0 < x < a$, where a is a constant.

The function

$$H(X) \equiv H(x_1, x_2, \cdots, x_n) = F(x_1) + F(x_2) + \cdots + F(x_n) \tag{1.1}$$

is called a Hamiltonian, where $F(x) = \int_0^x f(x) \mathrm{d}s$ for $x \in R$. Denote $\Phi(X) = \nabla H(X)$ in which $\nabla H(X)$ is the gradient of $H(X)$ and hence $\Phi(X) = \nabla H(X) = (f(x_1), f(x_2), \cdots, f(x_n))^{\mathrm{T}}$ and system (0.4) can be rewritten as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = A_n \nabla H(X), \tag{1.2}$$

where A_n is the $n \times n$ antisymmetric matrix defined in the previous section.

It is known from the theory of generalized Hamiltonian systems[7—9] that when $n = 2k$, (1.2) is a classical $2k$-dimensional Hamiltonian system since $A_n^{\mathrm{T}} J + JA_n = 0$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is a $2k \times 2k$ matrix and I is the $k \times k$ identity matrix; that is, A_n is a classical Hamiltonian matrix. When $n = 2k+1$, A_n is called a structure matrix of system (1.2) and system (1.2) is called a generalized Hamiltonian system. In this case, there is an invariant function

$$C(X) \equiv C(x_1, x_2, \cdots, x_n) = x_1 + \sum_{i=1}^{k} (x_{2i+1} - x_{2i}). \tag{1.3}$$

$C(X)$ is called a Casimir function of (1.2). The level set defined by $C(X) = c$ is called a symplectic leaf. According to the theory of the generalized Hamiltonian systems (1.2) can be reduced to a $2k$-dimensional Hamiltonian system on the symplectic leaf $x_1 + \sum_{i=1}^{k} (x_{2i+1} - x_{2i}) = c$. In fact, by choosing $c = 0$, the symplectic leaf is given by $x_{2k+1} = \sum_{i=1}^{k} (x_{2i} - x_{2i-1})$ and on this leaf we have

$$\begin{aligned} H^*(X) &\equiv H^*(x_1, x_2, \cdots, x_{2k}) \\ &= F(x_1) + F(x_2) + \cdots + F(x_{2k}) + F\left(\sum_{i=1}^{k} (x_{2i} - x_{2i-1}) \right) \end{aligned} \tag{1.4}$$

and

$$\frac{\mathrm{d}X}{\mathrm{d}t} = A_{2k} \nabla H^*(X), \quad X = (x_1, x_2, \cdots, x_{2k})^{\mathrm{T}}. \tag{1.5}$$

Clearly, A_{2k} is a $2k \times 2k$ Hamiltonian matrix, meaning that (1.5) is also a classical Hamiltonian system. Therefore we have

Proposition 1. *For $n = 2k$, (1.2) is a classical $2k$-dimensional Hamiltonian system. For $n = 2k + 1$, (1.2) can be reduced to (1.5), which is a classical $2k$-dimensional Hamiltonian system. Moreover, if $(x_1(t), \cdots, x_{2k}(t))$ is a periodic solution of period P of (1.5), then $(x_1(t), x_2(t), \cdots, x_{2k}(t), x_{2k+1}(t))$ with $x_{2k+1}(t) = \sum_{i=1}^{k} [x_{2i}(t) - x_{2i-1}(t)]$ is a periodic solution of (1.2) with period P; if $x_1(t), \cdots, x_{2k}(t), x_{2k+1}(t))$ is a periodic solution of period p of (1.2) with $x_{2k+1}(t_0) = \sum_{i=1}^{k} [x_{2i}(t_0) - x_{2k-1}(t_0)]$ for some $t_0 \in R$, then $(x_1(t), \cdots, x_{2k}(t))$ is a periodic solution of (1.5) with period P.*

Definition 1. Let G be a compact Lie group of transformations acting on R^n. The mapping Φ: $R^n \to R^n$ is called G-equivariant, if for all $g \in G$ and $X \in R^n$, $\Phi(gX) = g\Phi(X)$. A function H: $R^n \to R$ is called G-invariant function if for all $g \in G$ and $X \in R^n$, $H(gX) = H(X)$. Also, for $X_0 \in R^n$, the set $GX_0 = \{gX_0 \mid g \in G\}$ is called the group G orbit of X_0.

Now let us consider a map T_n: $R^n \to R^n$ defined by the $n \times n$ matrix

$$T_n = \begin{pmatrix} 0 & I_{n-1} \\ -1 & 0 \end{pmatrix}, \tag{1.6}$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix.

Lemma 1. *If $n = 2k + 1$, then det $T_{2k+1} = -1$ and also, the matrix T_{2k+1} can be decomposed into the product of $2k$ elementary rotation matrices $(T_{2k+1})_{\mathrm{rot}}$ with $\theta = \pi/2$ and a reflection matrix as follows:*

$$T_{2k+1} = (T_{2k+1})_{\mathrm{rot}} (T_{2k+1})_{\mathrm{ref}}.$$

If $n = 2k$, then det $T_{2k} = 1$ and the matrix T_{2k} can be decomposed into the product of $2k - 1$ elementary rotation matrices with $\theta = \pi/2$; that is, $T_{2k} = (T_{2k})_{\mathrm{rot}}$.

Lemma 1 means that the action T_{2k+1} on R^n is a composition of a reflection $(T_{2k+1})_{\mathrm{ref}}$ and $2k$ clockwise rotations by $\theta = \pi/2$ on the plane $(\varepsilon_1, \varepsilon_2), (\varepsilon_2, \varepsilon_3), \cdots, (\varepsilon_{n-1}, \varepsilon_n)$, respectively. However, the action T_{2k} on R^n is only a composition of $2k - 1$ rotations.

Now we define a group $G^{(1)}$ as follows:

$$G^{(1)} = \{g \mid g = T_n^s, \ s = 1,2,\cdots,2n\}.$$

Obviously, $G^{(1)}$ is a closed subgroup of $O(n)$ with order $2n$ and $(T_n)_{\mathrm{rot}} \in SO(n)$.

Note that $T_n^{\mathrm{T}} J T_n = J$. This means that the compact Lie group $G^{(1)}$ is a generalized symplectic action on R^n (see ref. [10]). We can show that the following proposition holds.

Proposition 2. *Suppose* (H_1) *holds. Then the function* $\Phi(X) = A_n \nabla H(X)$ *is a* $G^{(1)}$*-equivariant and the Hamiltonian* $H(X)$ *is a* $G^{(1)}$*-invariant function. If* $X(t)$ *is a nonconstant periodic solution of* (1.2) *when* $n = 2k$ *or* (1.5) *when* $n = 2k+1$ *with period* P, *then for all* $g \in G^{(1)}$, $gX(t)$ *is also a periodic solution of* (1.2) *with period* P. *Furthermore*, $gX(t)$ *is oscillatory about zero in* R^n.

Let $X(t)$ be a nonconstant periodic solution of (1.2) (when $n = 2k$) or (1.5) (when $n = 2k+1$) with minimal period P. Suppose that there exists some l such that

$$X^{\mathrm{T}}(t) = T_n X^{\mathrm{T}}\left(t + \frac{lp}{2n}\right), \tag{1.7}$$

where $1 \leqslant l < n$ and $lm \neq 0 \pmod{2n}$ for $m = 1,2,\cdots,2n-1$.

In sec. 2 we will show that the integer l is dependent on the spectra of the linearized systems of (1.2) and (1.5) if $f'(0) = \omega > 0$.

Theorem 1. *Suppose that condition* (H_1) *holds and*

(H_2) *system* (1.2) (*when* $n = 2k$) *or* (1.5) (*when* $n = 2k+1$) *has a periodic solution* $X(t) = (x_1(t), x_1(t), \cdots, x_n(t))^{\mathrm{T}}$ *of period* $P = 2n\mu$, *satisfying* (1.7);

(H_3) *the delays of* (0.5) *satisfy*

$$r_i = (i + 2nm_i) l\mu, \ \text{for } i = 1,2,\cdots,n-1, \tag{1.8}$$

where m_i ($i = 1,2,\cdots,n-1$) *are some nonnegative integers* (*not necessarily distinct*), *and* l, μ *are given by* (H_2). *Then* $x(t) = x_1(t)$ *is a nonconstant periodic solution of* (0.5) *with period* $P = 2n\mu$.

Proof. By the conditions of Theorem, we know that, for the periodic solution $X(t)$ of period P of (1.2) (when $n = 2k$) of (1.5) (when $n = 2k+1$), $X^{\mathrm{T}}(t) = T_n X^{\mathrm{T}}(t + l\mu)$. This implies

$$X^{\mathrm{T}}(t) = (x_2(t), \cdots, x_n(t), -x_1(t))^{\mathrm{T}} = (x_1(t - l\mu), \cdots, x_n(t - l\mu))^{\mathrm{T}} = X^{\mathrm{T}}(t - l\mu).$$

Then, using condition (1.8), we have

$$x_2(t) = x_1(t - l\mu) = x_1\left(t - \frac{r_1}{1 + 2nm_1} - \frac{2m_1 n r_1}{1 + 2nm_1}\right) = x_1(t - r_1),$$

$$x_3(t) = x_2(t - l\mu) = x_1(t - 2\mu l) = x_1\left(t - \frac{2r_2}{2 + 2nm_1} - \frac{2m_2 n r_2}{2 + 2nm_2}\right) = x_1(t - r_2),$$

$$\cdots,$$

$$x_n(t) = x_{n-1}(t - l\mu) = x_1(t - (n-1)\mu l)$$
$$= x_1\left(t - \frac{(n-1) r_{n-1}}{(n-1) + 2nm_{n-1}} - \frac{2m_{n-1} n r_{n-1}}{(n-1) + 2nm_{n-1}}\right) = x_1(t - r_{n-1}).$$

Therefore, it follows from the first equation of system (1.2) and Proposition 1 that $x(t) = x_1(t)$ is a nonconstant periodic solution of 0.5 with period $P = 2n\mu$.

2 Periodic solutions of the coupled systems

We now turn to the existence of periodic solutions of the coupled systems (1.2) (for $n = 2k$) and (1.5) (for $n = 2k + 1$) of (0.5). We assume that the function f satisfies (H_1) and $f'(0) = \omega > 0$. Denote $X = (x_1, x_2, \cdots, x_{2k})^T$ in this section. It is easy to show the following two lemmas.

Lemma 2. *The coefficient matrices ωA_{2k} and ωA_{2k}^* of the linearized systems of (1.2) and (1.5) have respectively the following spectra*

$$\sigma(A_{2k}) = \left\{\pm \mathrm{i}\gamma_q = \pm \mathrm{i}\tan\frac{(2q+1)\pi}{2n},\ q = 0,1,2,\cdots,k-1\right\},\ \textit{for } n = 2k; \tag{2.1}$$

$$\sigma(A_{2k}^*) = \left\{\pm \mathrm{i}\tilde{\gamma}_q = \pm \mathrm{i}\tan\frac{q\pi}{n},\ q = 1,2,\cdots,k\right\},\ \textit{for } n = 2k+1. \tag{2.2}$$

Let $\xi_q = (\xi_q^{(1)}, \xi_q^{(2)}, \cdots, \xi_q^{(2k)})^T$ and $\eta_q = (\eta_q^{(1)}, \eta_q^{(2)}, \cdots, \eta_q^{(2k)})^T$ be the eigenvectors of $\mathrm{i}\tan[(2q+1)\pi/(2n)]$, *and* $\mathrm{i}\tan[(q\pi)/(2n)]$ *respectively.*

Then
$$\xi_q^j = (-1)^{j-1}\exp\frac{\mathrm{i}(j-1)(2q+1)\pi}{2k},\ \eta_q^{(j)} = (-1)^{j-1}\exp\frac{\mathrm{i}(j-1)2q\pi}{2k+1},\ j = 1,2,\cdots,2k.$$

Lemma 3. *There exists a nonsingular matrix B such that, by the transformation $Y^T = BX^T$, system (1.2) with $n = 2k$ and (1.5) with $n = 2k+1$ are transformed into the canonical forms*

$$\frac{\mathrm{d}Y^T}{\mathrm{d}t} = J\nabla\tilde{H}(Y) \tag{2.3}$$

and

$$\frac{\mathrm{d}Y^T}{\mathrm{d}t} = J\nabla\tilde{H}^*(Y), \tag{2.4}$$

respectively, where $Y = (y_1, y_2, \cdots, y_{2k})$, $\tilde{H}(Y) = H(B^{-1}Y^T)$, $\tilde{H}^*(Y) = H^*(B^{-1}Y^T)$, and $J = BA_{2k}B^T$.

Note that $\nabla\tilde{H}(Y) = (B^{-1})^T\nabla H(X)$, $\nabla\tilde{H}^*(Y) = (B^{-1})^T\nabla H^*(X)$ and $X^T = B^{-1}Y^T$, we have $\nabla\tilde{H}(Y) = \omega(B^{-1})^TB^{-1}Y^T + o(|Y|)$, $\nabla\tilde{H}^*(Y) = \omega(B^{-1})^TMB^{-1}Y^T + o(|Y|)$, as $|Y| \to 0$, where $\omega M = H_{xx}^*(0)$. Let $\tilde{B} \equiv (B^{-1})^TB^{-1}$ and $\tilde{M} \equiv (B^{-1})^TMB^{-1}$. By Lemma 3, $BA_{2k}B^T = J$. It follows that $BA_{2k}B^{-1} = J(B^{-1})^TB^{-1} = J\tilde{B}$ and hence $\sigma(J\tilde{B}) = \sigma(BA_{2k}B^{-1}) = \sigma(A_{2k})$ since similar matrices have the same characteristic equations. Similarly, we know from $J\tilde{M} = J(B^{-1})^TMB^{-1} = BA_{2k}B^T(B^{-1})^TMB^{-1} = BA_{2k}MB^{-1}$ that $\sigma(J\tilde{M}) = \sigma(BA_{2k}MB^{-1}) = \sigma(A_{2k}M)$. Note that the matrix $\omega A_{2k}M$ is the linearized matrix of (1.5) at the origin. Thus $\sigma(J\tilde{M}) = \sigma(A_{2k}^*)$.

We now consider the canonical Hamiltonian system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\frac{\partial H}{\partial y_i},\ \frac{\mathrm{d}y_i}{\mathrm{d}t} = \frac{\partial H}{\partial x_i},\ (i = 1,2,\cdots,k), \tag{2.5}$$

which can be written as

$$\dot{z} = J\nabla H(z),\ z = \begin{pmatrix} x \\ y \end{pmatrix},\ J = J_{2k} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.6}$$

where 0 is $k \times k$ zero matrix, I is the $k \times k$ identity matrix and $H(z) = H(x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k) \equiv H(x, y)$ is independent of t and vanishes together with its first partial derivatives at $x =$

0, $y=0$; that is, $H(0)=H_z(0)=0$.

Lemma 4 (**Lyapunov Center Theorem**)[11,12]. *Suppose that* $H(z)\in C^2(R^{2k},R)$ *in* (2.6) *satisfies*

(i) $H(0)=0$, $\nabla H(0)=0$ *and* $H''(0)>0$ (*that is*, $H''(0)$ *is positive definite*);

(ii) $JH''(0)$ *has* k *pairs of purely imaginary simple eigenvalues* $\pm i\omega_q$, $q=1,2,\cdots,k$, *such that* ω_i/ω_j *is not an integer for all* $i\neq j$.

Then for all $\varepsilon>0$ *small enough system* (2.6) *has* k (*geometrically*) *distinct periodic orbits on the surface* $H(z)=\varepsilon$. *More precisely, the surface* $H(z)=\varepsilon$ *carries* k *distinct periodic orbits* Z_{q,ε_q} *whose periods* T_{q,ε_q} *tend to* $2\pi/\omega_q$, $q=1,2,\cdots,k$ *as* $\varepsilon_q\to 0$. *Moreover*, Z_{q,ε_q} *depends in a* C^1 *fashion on* ε_q *and* $\lim_{\varepsilon_q\to 0}\|Z_{q,\varepsilon_q}\|_{L^\infty}\to 0$,

$$\lim_{\varepsilon_q\to 0}\frac{Z_{q,\varepsilon_q}}{\varepsilon_q}=v_q=\xi_q e^{i\omega_q t}+\bar{\xi}_q e^{-i\omega_q t},\ q=1,2,\cdots,k, \tag{2.7}$$

where $A\xi_q=i\omega_q\xi_q$, $A=JH''(0)$.

By using Theorems 9.2, 9.5 and Remark 9.6 in ref. [2], we know

Lemma 5. *Suppose that* H *in* (2.6) *satisfies* (i), (ii) *of Lemma* 4 *and* (iii) H: $R^{2k}\to R$ *is invariant under a generalized symplectic action of a compact Lie group* G *on* R^{2k}. *Then for each* $T_{q,\varepsilon_q}<2\pi/\omega_q$ *and* T_{q,ε_q} *sufficiently close to* $2\pi/\omega_q$, *there exist at least one G-orbit of closed trajectories of* (2.6) *with period* $T_{q,\varepsilon_q}<2\pi/\omega_q$, *which lies on the surface* $H(z)=\varepsilon$ *for sufficiently small* $\varepsilon>0$. *This trajectory converges towards the origin as* $T_{q,\varepsilon_q}\to 2\pi/\omega_q$ ($q=1,2,\cdots,k$).

Lemma 6. *Suppose that* $f'(0)=\omega>0$. *Then for the linearized systems* $\dot{x}=\omega A_{2k}x$ *and* $x=\omega A_{2k}^*x$ *of* (1.2) *and* (1.5), *corresponding respectively to the pairs* $\pm i\gamma_q$ *and* $\pm i\tilde{\gamma}_q$ *of eigenvalue of* ωA_{2k} *and* ωA_{2k}^*, *there exist* k *families of periodic solutions as follows*:

$$\begin{aligned}X_q^{\mathrm T}(t)&=a\xi_q e^{i\gamma_q t}+\overline{a\xi_q}e^{-i\gamma_q t}\\&=(\alpha Re\xi_q-\beta Im\xi_q)\cos\gamma_q t-(\alpha Im\xi_q+\beta Re\xi_q)\sin\gamma_q t\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\tilde{X}_q^{\mathrm T}(t)&=(x_1^{(q)}(t),x_2^{(q)}(t),\cdots,x_{2k}^{(q)}(t))^{\mathrm T}=a\eta_q e^{i\tilde{\gamma}_q t}+\overline{a\eta_q}e^{-i\tilde{\gamma}_q t}\\&=(\alpha Re\eta_q-\beta Im\eta_q)\cos\tilde{\gamma}_q t-(\alpha Im\eta_q+\beta Re\eta_q)\sin\tilde{\gamma}_q t,\end{aligned} \tag{2.9}$$

where $a=(\alpha+i\beta)/2$, $q=0,1,2,\cdots,k-1$ *for* $n=2k$ *and* $q=1,2,\cdots,k$, *for* $n=2k+1$. *Moreover, we have for* $n=2k$

$$X_q^{\mathrm T}(t)=T_{2k}X_q^{\mathrm T}\left(t+\frac{2k-(2q+1)}{4k}T_q\right)=T_{2k}X_q^{\mathrm T}\left(t+\frac{l}{2n}T_q\right), \tag{2.10}$$

where $T_q=2\pi/\gamma_q$, $l=2k-(2q+1)$; *and for* $n=2k+1$ *letting* $x_{2k+1}^{(q)}=\sum_{j=1}^k[x_{2j}^{(q)}(t)-x_{2j-1}^{(q)}(t)]$, $\hat{X}_q(t)=(\tilde{X}_q(t),x_{2k+1}^{(q)}(t))$, *we have*

$$\hat{X}_q^{\mathrm T}(t)=T_{2k+1}\hat{X}_q^{\mathrm T}\left(t+\frac{(2k+1)-2q}{2(2k+1)}T_q^*\right)=T_{2k+1}\hat{X}_q^{\mathrm T}\left(t+\frac{l}{2n}T_q^*\right), \tag{2.11}$$

where $T_q^*=2\pi/\tilde{\gamma}_q$, $l=(2k+1)-2q$.

Hence Lemmas 5,6, Proposition 2 and relation (2.7) can be used to derive the following result.

Theorem 2. *Suppose that condition* (H_1) *holds and* $f'(0)=\omega>0$. *Then system* (1.2)

((1.5)) has k distinct $G^{(1)}$-orbit families $\{\Gamma^q\}$ of periodic solutions in a neighborhood of the origin and each familiy of periodic solutions depends on one parameter ε_q. If $\varepsilon_q \to 0$, then the corresponding orbits tend to the origin and the period $T_{q,\varepsilon_q} < \frac{2\pi}{\gamma_q}$, $q = 0,1,2,\cdots,k-1$ for $n = 2k$ ($T_{q,\varepsilon_q} < 2\pi/\tilde{\gamma}_q$, $q = 1,2,\cdots,k$ for $n = 2k+1$), as $\varepsilon_q \to 0$, $T_{q,\varepsilon_q} \to 2\pi/\gamma_q (2\pi/\tilde{\gamma}_q)$. Under the generalized symplectic action T_n, these distinct families $\{\Gamma^q\}$ of periodic solutions keep relations of (2.10) and (2.11), where $X_q^{\mathrm{T}}(t)$ and $\hat{X}_q^{\mathrm{T}}(t)$ are $G^{(1)}$-orbits of periodic solutions of (1.2) and (1.5), respectively.

3 Periodic solution of differential delay equations

This section is devoted to the study on the existence of periodic solutions of differential delay equation (0.5). For every q defined by Lemma 2, let

$$\begin{cases} l = 2k - (2q+1) \text{ or } q = (2k - l - 1)/2, \text{ for } n = 2k, \\ l = (2k+1) - 2q \text{ or } q = (2k - l + 1)/2, \text{ for } n = 2k+1. \end{cases} \tag{3.1}$$

Theorem 3. *Suppose that condition (H_1) holds and $f'(0) = \omega > 0$.*

(i) *When $n \neq jl$ (for $2 \leqslant j \leqslant n/3$, l is odd and $3 \leqslant l \leqslant k$), for a real μ satisfying $T_{q,\varepsilon_q} \equiv 2n\mu < 2\pi/\gamma_q (2\pi/\tilde{\gamma}_q)$, and T_{q,ε_q} is sufficiently close to $2\pi/\gamma_q (2\pi/\tilde{\gamma}_q)$, with $r_i = (i + 2nm_i) l\mu$ where l is defined by (3.1), i.e. the following holds:*

(H_3) *there exist nonnegative integers $m_1, m_2, \cdots, m_{n-1}$ (not necessarily distinct) such that the delays $r_i (i = 1,2,\cdots,n-1)$ of (0.5) satisfy*

$$\frac{r_1}{1 + 2nm_1} = \cdots = \frac{r_i}{i + 2nm_i} = \cdots = \frac{r_{n-1}}{(n-1) + 2nm_{n-1}} = l\mu. \tag{3.2}$$

Then every family $\{\Gamma^q\}$ of periodic solutions of (1.2) ((1.5)) gives a periodic solution of period $2n\mu$ of (0.5).

(ii) *When $n = lj_0$ (for $2 \leqslant j \leqslant n/3$, $l_0 = 2k - 2q_0 + 1$) or $l_0 = (2k+1) - q_0$ fixed), for a real μ satisfying $T_{q_0,\varepsilon_q} \equiv 2j\mu < 2\pi/\gamma_{q_0} (2\pi/\tilde{\gamma}_{q_0})$ and T_{q_0,ε_q} is sufficiently close to $2\pi/\gamma_{q_0} (2\pi/\tilde{\gamma}_{q_0})$, taking $r_i = (i + 2jm_i) l\mu$ for some nonnegative integers $m_1, m_2, \cdots, m_{n-1}$ (not necessarily distinct). This means that the delays of (1) satisfy*

$$\frac{r_1}{1 + 2jm_1} = \frac{r_2}{2 + 2jm_2} = \cdots = \frac{r_{n-1}}{(n-1) + 2jm_{n-1}} = \mu. \tag{3.3}$$

Then the family $\{\Gamma^{q_0}\}$ of periodic solutions of (1.2) ((1.5)) yields a periodic solution of period $2j\mu$ of (0.5)

Proof. It follows from the assumptions, Lemmas 4, 5 and Theorem 2 that system (1.2) (for $n = 2k$) or (1.5) (for $n = 2k+1$) has at least k distinct families (say $\{\Gamma^q\}$) of periodic solutions and the period $T_{q,\varepsilon \setminus - q}$ of each family of periodic orbits is close to $2\pi/\gamma_q$ for $n = 2k (q = 0,1,\cdots, k-1)$ or $2\pi/\tilde{\gamma}_q$ for $n = 2k+1 (q, = 1,2,\cdots,k)$. By our assumption, the period $T_{q,\varepsilon_q} = 2n\mu$ of the qth family of the periodic solutions of (1.2) or (1.5) satisfies condition (i). Hence when $n \neq jl$, for $l = 2k - (2q+1) (n = 2k)$ or $l = (2q+1) - 2k (n = 2k+1)$, Hamiltonian system (1.2) or (1.5) has a periodic solution of period $2n\mu$ in every family $\{\Gamma^q\}$. Thus the conclusion of (i) follows from Theorem 1.

When $n = jl_0$, condition (H_1), $f'(0) = \omega > 0$ and condition (ii) imply that (1.2) ((1.5)) has a $T_{q_0,\varepsilon_{q_0}} = 2j\mu$-periodic solutions in $\{\Gamma^{q_0}\}$. Taking into account

$$X_{q_0}^{\mathrm{T}} = T_n X_{q_0}^{\mathrm{T}}\left(t + \frac{l_0}{2n}T_{q_0}\right) = T_n X_{q_0}^{\mathrm{T}}\left(t + \frac{1}{2j}T_{q_0}\right) = T_n X_q^{\mathrm{T}}(t + \mu)$$

instead of n by j and by similar discussion as Theorem 1, we know from (3.3) that (0.5) has a $2j\mu$-periodic solution.

Remark. (i) When $n = jl_0 = j(2m_0 + 1), q_0 = 0.5\ (2k - l_0 - 1)$, for $n = 2k$ and $q_0 = 0.5\ (2k - l_0 + 1)$ for $n = 2k + 1$, the linearized systems $\dot{x} = \omega A_{2k}x$ and $\dot{x} = \omega A_{2k}^{*}x$ have an eigenvalue $\gamma_{q_0} = \pm\, \omega i$. The corresponding eigenvector is $\tilde{\xi} = (0, -\beta, 0, \beta, 0, -\beta, \cdots, 0, (-1)^{k+1}\beta)^{\mathrm{T}}$ for ωA_{2k}. It is easy to see that $T_{2k}^{2j}\tilde{\xi} = \tilde{\xi}$, i.e. $\tilde{\xi}$ is an eigenvector of T_{2k}^{2j} corresponding to the eigenvalue 1.

(ii) We know from Theorem 3 that, except the family $\{\Gamma^{q_0}\}$ of periodic solutions of (1.2) ((1.5)), all families $\{\Gamma^{q}\}$ of periodic solutions do yield $2n\mu$-periodic solutions of (0.5).

(iii) Under the asymptotically linear conditions:

$$\lim_{x\to 0} f(x)/x = f'(0) = \omega,\ \lim_{x\to\infty} f(x)/x = \omega_\infty,$$

we can give a concrete estimation for μ in Theorem 3. We will discuss this problem in another paper.

Let $r_i = i$, $l\mu = 1$ and $m_i = 0$ in (1.8). Then we have

Corollary 1. *Suppose the assumption of Theorem 3 holds. Then (0.3) has a periodic solution of period $2n$.*

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