# Orbital angular momentum of the laser beam and the second order intensity moments

GAO Chunqing (高春清)[1], WEI Guanghui (魏光辉)[1] & Horst Weber[2]

1. Optical Engineering Department, Beijing Institute of Technology, Beijing 100081, China;
2. Optical Institute, Technical University Berlin, Strasse des 17. Juni 135, D-10623 Berlin, Germany
Correspondence should be addressed to Gao Chunqing (email: gao@bit.edu.cn)

Received December 23, 1999

**Abstract** From the wave equation of a generalized beam the orbital angular momentum is studied. It is shown that the orbital angular momentum exists not only in the Laguerre-Gaussian beam, but in any beam with an angular-dependent structure. By calculating the second order intensity moments of the beam the relation between the orbital angular momentum and the second order moments $\langle x\theta_y\rangle$, $\langle y\theta_x\rangle$ is given. As an example the orbital angular momentum of the general astigmatic Gaussian beam is studied.

**Keywords: orbital angular momentum, second order moments, beam characterization.**

The characterization of laser beams (beam radii, beam divergences, and beam quality factors, etc.) by using their second order intensity moments has been widely studied[1—3]. It is shown that any general astigmatic beam can be characterized by its ten independent second order intensity moments: $\langle x^2\rangle$, $\langle x\theta_x\rangle$, $\langle\theta_x^2\rangle$, $\langle y^2\rangle$, $\langle y\theta_y\rangle$, $\langle\theta_y^2\rangle$, $\langle xy\rangle$, $\langle\theta_x\theta_y\rangle$, $\langle x\theta_y\rangle$, $\langle y\theta_x\rangle$. Every second order moment has its specific physical meaning. $\langle x^2\rangle$ and $\langle y^2\rangle$ describe the beam dimensions in $x$ and $y$ directions, respectively. $\langle\theta_x^2\rangle$ and $\langle\theta_y^2\rangle$ give the far field divergences in $x$ and $y$ directions, respectively. $\langle xy\rangle$ and $\langle\theta_x\theta_y\rangle$ describe the orientations of the beam in the near field and the far field. $\langle x\theta_x\rangle$ and $\langle y\theta_y\rangle$ are related to the effective radii of curvature. A bit unknown parameters are $\langle x\theta_y\rangle$ and $\langle y\theta_x\rangle$. From their definitions we know that they are related to the coupling between the near field in one direction ($x$ or $y$) and the far field in the perpendicular direction ($y$ or $x$). In this paper we will show that they are related to the orbital angular momentum of the beam.

The orbital angular momentum of a beam was first described by Allen and his coworkers when they studied the properties of Laguerre-Gaussian beams[4,5]. They pointed out that the Laguerre-Gaussian beam has not only an angular momentum caused by the circular polarization, but also an orbital angular momentum. The angular momentum caused by the circular polarization was known long time ago and Beth proved its existence in experiment. But the orbital angular momentum was known only in the last few years.

In this paper we will study the orbital angular momentum of a generalized coherent beam. It is shown that the orbital angular momentum of the beam exists not only in Laguerre-Gaussian beams, but also in any beam with an angular-dependent phase structure. Furthermore, by calculating the second order moments of a generalized beam the relation between the orbital angular momentum and the second order moments is given. As an example the orbital angular momentum of a general astigmatic Gaussian beam is studied.

## 1 Angular momentum of the beam

### 1.1 The Poynting vector

Any beam carries energy and momentum. For a coherent beam propagating along $z$-direction, the electric field can be expressed by

$$\boldsymbol{E}(x,y,z) = E_0 \cdot \boldsymbol{e} \cdot u(x,y,z) \cdot \exp[\mathrm{i}(\omega t - kz)], \tag{1}$$

where $k$ is the wave number, which is related to the angular frequency $\omega$ by

$$k = \omega/c, \quad c = 1/\sqrt{\varepsilon_0 \mu_0}. \tag{2}$$

$\varepsilon_0$ and $\mu_0$ are the dielectric constant and the magnetic permeability in vacuum, respectively. In eq. (1) $E_0\, u(x,y,z)$ describes the amplitude of the electric field, where $u(x, y, z)$ is a normalized complex scalar function which describes the distribution of the amplitude. $u(x, y, z)$ satisfies the wave equation in the paraxial approximation. $\boldsymbol{e}$ represents the unit polarization vector of the beam.

In the non-magnetic medium the magnetic field of the beam can be obtained from the Maxwell equations[6]

$$\boldsymbol{H}(x,y,z) = -\frac{1}{\mathrm{i}\omega\mu_0} \nabla \times \boldsymbol{E}(x,y,z). \tag{3}$$

From the electric field and the magnetic field the Poynting vector $\boldsymbol{S}$ is defined as

$$\boldsymbol{S} = \boldsymbol{E}_{\mathrm{re}} \times \boldsymbol{H}_{\mathrm{re}}, \tag{4}$$

where

$$\boldsymbol{E}_{\mathrm{re}} = \frac{1}{2}(\boldsymbol{E} + \boldsymbol{E}^*), \quad \boldsymbol{H}_{\mathrm{re}} = \frac{1}{2}(\boldsymbol{H} + \boldsymbol{H}^*).$$

The subscript "re" and superscript " * " mean the real part and the conjugate part.

The Poynting vector $\boldsymbol{S}$ describes the energy flow of the beam. The time averaged value of $\boldsymbol{S}$ reads

$$\langle \boldsymbol{S} \rangle_t = \frac{1}{4}(\boldsymbol{E} \times \boldsymbol{H}^* + \mathrm{c.c.}). \tag{5}$$

where "c.c." means the complex conjugate.

Substitution of eqs. (1) and (2) into eq. (5) yields

$$\begin{aligned}\langle \boldsymbol{S} \rangle_t =& \frac{k}{2\omega\mu_0} | E_0 |^2 \boldsymbol{e}_z | u |^2 - \frac{\mathrm{i}}{4\omega\mu_0} | E_0 |^2 [u \operatorname{grad} u^* - u^* \operatorname{grad} u] \\ & - \frac{\mathrm{i}}{4\omega\mu_0} | E_0 |^2 [-(\boldsymbol{e} \cdot \operatorname{grad} u^*) u \boldsymbol{e}^* + (\boldsymbol{e}^* \cdot \operatorname{grad} u) u^* \boldsymbol{e}]. \end{aligned} \tag{6}$$

In the right part of eq. (6) the second term is related to the beam structure and the third term is related to the beam polarization. For the sake of convenience these two terms are written as $\boldsymbol{T}_1$, $\boldsymbol{T}_2$:

$$\boldsymbol{T}_1 = -\frac{\mathrm{i}}{4\omega\mu_0} \cdot | E_0 |^2 (u \operatorname{grad} u^* - u^* \operatorname{grad} u), \tag{7}$$

$$\boldsymbol{T}_2 = -\frac{\mathrm{i}}{4\omega\mu_0} \cdot | E_0 |^2 [-(\boldsymbol{e} \cdot \operatorname{grad} u^*) u \boldsymbol{e}^* + (\boldsymbol{e}^* \cdot \operatorname{grad} u) u^* \boldsymbol{e}]. \tag{8}$$

### 1.2 Angular momentum of the beam

The angular momentum density of a beam is defined by[6]

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{p}, \tag{9}$$

where $\boldsymbol{r}$ is the position vector and $\boldsymbol{p}$ is the momentum density of the beam. $\boldsymbol{p}$ is defined by

$$\boldsymbol{p} = \varepsilon_0 \boldsymbol{E}_{\mathrm{re}} \times \boldsymbol{B}_{\mathrm{re}}, \tag{10}$$

where $\boldsymbol{B}$ is the magnetic induction of the beam. For the non-magnetic medium $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, therefore the angular momentum density is

$$\boldsymbol{M} = \varepsilon_0 \boldsymbol{r} \times (\boldsymbol{E}_{\mathrm{re}} \times \boldsymbol{B}_{\mathrm{re}}). \tag{11}$$

From eq. (4) the angular momentum density can be written as

$$\boldsymbol{M} = \frac{1}{c^2} \boldsymbol{r} \times \boldsymbol{S}. \tag{12}$$

Integrating $\boldsymbol{M}$ yields the total angular momentum

$$\boldsymbol{L} = \frac{1}{c^2} \iiint \boldsymbol{r} \times \boldsymbol{S} \mathrm{d}x \mathrm{d}y \mathrm{d}z. \tag{13}$$

The angular momentum flux $\boldsymbol{J}$ of the field is defined by

$$\boldsymbol{J} = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{L}. \tag{14}$$

The component of $\boldsymbol{J}$ in the direction of propagation (the $z$-axis) is of importance because it can be transferred to optical elements. From eqs. (13) and (14) the angular momentum flux in $z$-direction reads

$$J_z = \frac{1}{c} \iint \boldsymbol{r} \times \boldsymbol{S} \mathrm{d}x \mathrm{d}y. \tag{15}$$

Substitution of eq. (6) into eq. (15) shows that $J_z$ can be separated into two parts: $J_z = J_{z,L} + J_{z,S}$, where $J_{z,L}$ is related to the structural term of the Poynting vector and $J_{z,S}$ to the beam polarization.

1.2.1 Orbital angular momentum. The angular momentum related to the beam structure is called the orbital angular momentum, which is defined by

$$J_{z,L} = \frac{1}{c} \iint [\boldsymbol{r} \times \boldsymbol{T}_1] \mathrm{d}x \mathrm{d}y = -\frac{\mathrm{i}\varepsilon_0 c}{4\omega} |E_0|^2 \iint \left( xu \frac{\partial u^*}{\partial y} - yu \frac{\partial u^*}{\partial x} - \mathrm{c.c.} \right) \mathrm{d}x \mathrm{d}y. \tag{16}$$

The orbital angular momentum of light was first predicted for Laguerre-Gaussian beams[4]. Actually it exists in any beam with an angular-dependent structure. For an arbitrary beam with an azimuthal structure of the phase $u(r, \phi) = f(r) \cdot \exp[\mathrm{i}g(\phi)]$, the orbital angular momentum flux reads

$$J_{z,L} = -\frac{2}{\omega} P \frac{\partial g}{\partial \phi}, \tag{17}$$

where $P$ is the power of the beam:

$$P = \frac{1}{2} c \varepsilon_0 |E_0|^2 \iint f^2 \mathrm{d}x \mathrm{d}y. \tag{18}$$

Therefore when $\partial g / \partial \phi \neq 0$, the beam will have orbital angular momentum. The Laguerre-Gaussian beam is a special case of eq. (17) with $g(\phi) = l\phi$, where $l$ is the azimuthal mode index of the Laguerre-Gaussian mode. Then the orbital angular momentum flux of a Laguerre-Gaussian mode reads

$$J_{z,L} = -\frac{2l}{\omega}P. \tag{19}$$

The beams with orbital angular momentum have many applications. They can be used to rotate or manipulate micro-particles. For example, the ring-shaped mode with orbital angular momentum has been used as optical spanners in biological and medical fields[7].

1.2.2 Angular momentum related to polarization. The angular momentum flux related to the beam polarization is given by

$$J_{z,S} = \frac{1}{c}\iint [\boldsymbol{r} \times \boldsymbol{T}_2]\mathrm{d}x\mathrm{d}y = -\frac{\mathrm{i}\varepsilon_0 c}{4\omega} \mid E_0 \mid^2 \iint [(xe_y^* - ye_x^*)(\boldsymbol{e}\,\mathrm{grad}u^*)u - \mathrm{c.c.}]\mathrm{d}x\mathrm{d}y. \tag{20}$$

The unit-vector of the polarization can be written as $\boldsymbol{e} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\alpha} \end{pmatrix}$, where $\alpha = 0, \ \pm \pi/2$, represent the linear and the circular polarization, respectively. Substitution of $\boldsymbol{e}$ into eq. (20) yields

$$J_{z,S} = -\frac{\mathrm{i}}{4\pi}P\iint \left[ u\left(\mathrm{e}^{-\mathrm{i}\alpha}x\frac{\partial u^*}{\partial x} - \mathrm{e}^{\mathrm{i}\alpha}y\frac{\partial u^*}{\partial y}\right) - \mathrm{c.c.}\right]\mathrm{d}x\mathrm{d}y. \tag{21}$$

From eq. (21) it is known that the circularly-polarized beam has the angular momentum while the linearly-polarized beam does not. For the fundamental Gaussian beam with circular polarization the angular momentum flux is $J_{z,S} = P/\omega$.

## 2 Orbital angular momentum and the second order moments

In this sector we will show that the orbital angular momentum is related to the second order intensity moments $\langle x\theta_y\rangle$, $\langle y\theta_x\rangle$. For an arbitrary optical field $E(x, y)$ the second order moments $\langle x\theta_y\rangle$, $\langle y\theta_x\rangle$ are defined as[3]

$$\langle x\theta_y\rangle = \frac{1}{2P(\mathrm{i}k)}\iint xE(x,y)\frac{\partial}{\partial y}E^*(x,y)\mathrm{d}x\mathrm{d}y + \mathrm{c.c.}, \tag{22}$$

$$\langle y\theta_x\rangle = \frac{1}{2P(\mathrm{i}k)}\iint yE^*(x,y)\frac{\partial}{\partial x}E(x,y)\mathrm{d}x\mathrm{d}y + \mathrm{c.c.} \tag{23}$$

Substitution of eq. (1) into eqs. (22) and (23) yields

$$\langle x\theta_y\rangle = \frac{-\mathrm{i}\lambda}{4\pi}\iint xu\frac{\partial u^*}{\partial y}\mathrm{d}x\mathrm{d}y + \mathrm{c.c.}, \tag{24}$$

$$\langle y\theta_x\rangle = \frac{-\mathrm{i}\lambda}{4\pi}\iint yu\frac{\partial u^*}{\partial x}\mathrm{d}x\mathrm{d}y + \mathrm{c.c.} \tag{25}$$

The difference of eqs. (24) and (25) delivers

$$\langle x\theta_y\rangle - \langle y\theta_x\rangle = \frac{-\mathrm{i}\lambda}{4\pi}\iint\left(xu\frac{\partial u^*}{\partial y} - yu\frac{\partial u^*}{\partial x}\right)\mathrm{d}x\mathrm{d}y + \mathrm{c.c.} \tag{26}$$

Comparison of eq. (16) with eq. (26) delivers the orbital angular momentum flux of the beam

$$J_{z,L} = \frac{P}{2c}(\langle x\theta_y\rangle - \langle y\theta_x\rangle), \tag{27}$$

where $P$ is given by eq. (18). Because the orbital angular momentum of the beam means a rotation of the phase structure, $\langle x\theta_y\rangle$ and $\langle y\theta_x\rangle$ are called the twist parameters of the beam. By calculating the second order intensity moments $\langle x\theta_y\rangle$ and $\langle y\theta_x\rangle$, the orbital angular momentum of an arbitrary beam can be obtained.

We calculate the orbital angular momentum of the general astigmatic Gaussian beam as an example. The wave function of a general astigmatic Gaussian beam reads

$$E(x,y) = E_0\exp\left\{-\frac{x^2}{\omega_x^2}-\frac{2xy}{\omega_{xy}^2}-\frac{y^2}{\omega_y^2}-\frac{ik}{2}\left[\frac{x^2}{R_x}+\frac{2xy}{R_{xy}}+\frac{y^2}{R_y}\right]\right\}, \tag{28}$$

where $\omega_x$ and $\omega_y$ describe the beam radii along $x$-axis and $y$-axis, and $R_x$ and $R_y$ describe the radii of curvature along $x$-axis and $y$-axis, respectively. $\omega_{xy}$ and $R_{xy}$ are used to describe the orientations of the intensity ellipse and phase ellipse, respectively. They are related to the angles of the principal axes of the intensity and phase ellipses $\alpha$, $\theta$ by

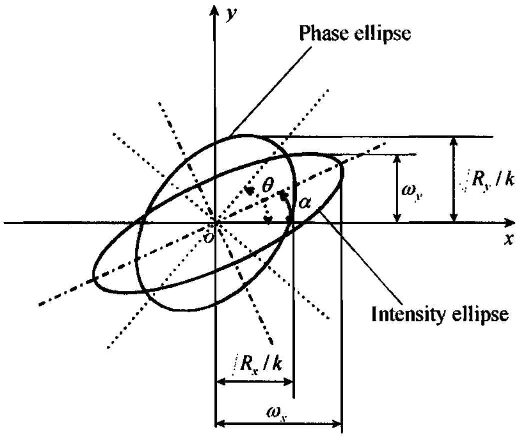


Fig. 1. Intensity ellipse and phase ellipse of a general astigmatic Gaussian beam.

$$\tan 2\alpha = \frac{1/\omega_{xy}^2}{1/\omega_x^2 - 1/\omega_y^2}, \tag{29}$$

$$\tan 2\theta = \frac{1/R_{xy}}{1/R_x - 1/R_y}. \tag{30}$$

Fig. 1 shows the physical meanings of respective beam parameters $\omega_x$, $\omega_y$, $R_x$, $R_y$. Substituting eq. (28) into eqs. (22) and (23) yields the second order intensity moments $\langle x\theta_y\rangle$, $\langle y\theta_x\rangle$ of the general astigmatic Gaussian beam:

$$\langle x\theta_y\rangle = \frac{1}{4}\cdot\frac{\omega_x^2\omega_{xy}^2}{\omega_{xy}^4-\omega_x^2\omega_y^2}\left[\frac{\omega_{xy}^2}{R_{xy}}-\frac{\omega_y^2}{R_y}\right], \tag{31}$$

$$\langle y\theta_x\rangle = \frac{1}{4}\cdot\frac{\omega_y^2\omega_{xy}^2}{\omega_{xy}^4-\omega_x^2\omega_y^2}\left[\frac{\omega_{xy}^2}{R_{xy}}-\frac{\omega_x^2}{R_x}\right]. \tag{32}$$

The orbital angular momentum of the general astigmatic Gaussian beam is

$$J_{z,L} = \frac{P}{8c}\cdot\frac{\omega_{xy}^2}{\omega_{xy}^4-\omega_x^2\omega_y^2}\left[\frac{\omega_{xy}^2(\omega_x^2-\omega_y^2)}{R_{xy}}-\omega_x^2\omega_y^2\left(\frac{1}{R_x}-\frac{1}{R_y}\right)\right]. \tag{33}$$

## 3 Conclusion

From a generalized wave function we show that the Poynting vector and the angular momentum consist of two parts: one related to the beam polarization and the other related to the beam structure. The angular momentum related to the beam structure is named the orbital angular momentum, which exists in any beam with an angular-dependent structure. By calculating the second order intensity moments of the beam the relation between the orbital angular momentum and the second moments is given. As an example the orbital angular momentum of the general astigmatic Gaussian beam is calculated.

Since the ten second order intensity moments of an arbitrary beam can be determined experimentally[8], the orbital angular momentum of the beam can be obtained from eq. (27). It gives a simple method for determining the orbital angular momentum.

## References


1. Siegman, A. E., New developments in laser resonators, SPIE, 1990, 1224: 2.
2. Nemes, G., Siegman, A. E., Measurement of all ten second-order moments of an astigmatic beam by use of rotating simple astigmatic optics, J. Opt. Soc. Am. A, 1994, 11: 2257.
3. Weber, H., Propagation of higher-order intensity moments in quadratic-index media, Opt. and Quan. Elec., 1992, 24: S1027.
4. Allen, L., Beijersbergen, M. W., Spreeuv, R. J. C. et al., Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes, Phys. Rev., 1992, A45: 8185.
5. Baennet, S. M., Allen, L., Orbital angular momentum and nonparaxial light beams, Opt. Comm., 1994, 110: 670.
6. Jackson, J. D., Classical Electrodynamics, New York: John Wiley & Sons, Inc., 1962.
7. Simpson, N. B., Dholakia, K., Allen, L. et al., Mechanical equivalence of spin and orbital angular momentum of light: an optical spanner, Opt. Lett., 1997, 22: 52.
8. Eppich, B., Gao, C., Weber, H., Determination of the ten second order moments, Opt. & Laser Tech., 1998, 30: 337.