A $[k, k+1]$-factor containing given Hamiltonian cycle*

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Abstract Let $k \geqslant 2$ be an integer and let G be a graph of order n with minimum degree at least k, $n \geqslant 8k-16$ for even n and $n \geqslant 6k-13$ for odd n. If the degree sum of each pair of nonadjacent vertices of G is at least n, then for any given Hamiltonian cycle C, G has a $[k, k+1]$-factor containing C.

Keywords: graph, connected factor, Hamiltonian cycle.

All graphs under consideration are undirected, finite and simple. A graph, denoted by $G=(V, E)$, consists of a non-empty set $V(G)$ of vertices and a set $E(G)$ of edges. Let xy denote the edge joining vertices x and y. If X is a subset of $V(G)$, we write $G[X]$ for the subgraph of G induced by X, $E_G[X]=E(G[X])$ and $\bar{X}=V(G)-X$. Sometimes x is used for a singleton $\{x\}$. Given a graph $G=(V, E)$ and $x \in V(G)$, write $d_G(x)$ for the degree of x in G, which is the number of edges of G incident to x. For integers a and b, $b \geqslant a \geqslant 0$, an $[a, b]$-factor of G is defined as a spanning subgraph F of G such that

$$a \leqslant d_F(v) \leqslant b \text{ for all } v \in V(G),$$

and an $[a, a]$-factor is abbreviated to an a-factor. A subset M of $E(G)$ is called a matching if no two edges in M are adjacent in G. Other notations and definitions not defined here can be found in ref. [1].

We first mention some known results on k-factors or connected $[a, b]$-factors.

Theorem A[2]. *Let k be a positive integer, and let G be a graph of order n with $n \geqslant 4k-5$, kn even, and minimum degree at least k. Then G has a k-factor if the degree sum of each pair of nonadjacent vertices is at least n.*

Theorem B[3]. *Let $k \geqslant 3$ be an integer and let $G=(V, E)$ be a connected graph of order n with $n \geqslant 4k-3$, kn even, minimum degree at least k. If for each pair of nonadjacent vertices u and v of $V(G)$*

$$\max\{d_G(u),\ d_G(v)\} \geqslant \frac{n}{2},$$

G has a k-factor.

Theorem C[3]. *Let k be a positive integer and let G be a graph of order n such that $n \geqslant 4k-5$,*

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kn even, and minimum degree at least k. If the degree sum of each pair of nonadjacent vertices of G is at least n, then G has both a Hamiltonian cycle C and a k-factor F. Hence G has a connected $[k, k+2]$*-factor* $C+F$.

Theorem D[4]. *Let* $k \geqslant 2$ *be an integer and G be a connected graph of order n. If G has a k-factor F and, moreover, among any three independent vertices of G there are (at least) two with degree sum at least* $n-k$, *then G has a matching M such that M and F are edge-disjoint and* $M+F$ *is a connected* $[k, k+1]$*-factor of G.*

Theorem E[5]. *Let* $k \geqslant 3$ *be an odd integer, and G be a connected graph of odd order n with* $n \geqslant 4k-3$, *and minimum degree at least k. If for each pair of nonadjacent vertices u and v of G,*

$$\max\{d_G(u), d_G(v)\} \geqslant \frac{n}{2},$$

G has an almost $k^{\pm}$*-factor* $F^{\pm}$ *and a matching M such that* F^{-} *and M are edge-disjoint and* $F^{-}+M$ *is a connected* $[k, k+1]$*-factor of G (an almost* $k^{\pm}$*-factor is a factor whese every vertex has degree k except at most one with degree* $k \pm 1$).

Theorem F[6]. *Let* $k \geqslant 2$ *be an integer and let G be a graph of order n such that* $n \geqslant 8k-4$, *kn is even and minimum degree at least* $n/2$. *Then G has a k-factor containing a Hamiltonian cycle.*

The purpose of this paper is to extend "connected $[k, k+1]$-factor" in some of the above theorems to "$[k, k+1]$-factor containing a given Hamiltonian cycle", which is obviously a 2-connected $[k, k+1]$-factor under somewhat stronger conditions. Our main result is the following.

Theorem 1. *Let* $k \geqslant 2$ *be an integer and let G be a graph of order* $n \geqslant 3$ *with minimum degree at least k,* $n \geqslant 8k-16$ *for even n and* $n \geqslant 6k-13$ *for odd n. If for each pair of nonadjacent vertices u and v of G,*

$$d_G(u) + d_G(v) \geqslant n, \tag{1}$$

then for any given Hamiltonian cycle C, G has a $[k, k+1]$*-factor containing C.*

Remark 1. The conditions $n \geqslant 8k-16$ for even n and $n \geqslant 6k-13$ for odd n are best possible. To see this, for even n such that $2k \leqslant n < 8k-16$, write $m = (n/2)+2$; for odd n such that $2k-1 \leqslant n < 6k-13$, write $m = (n+3)/2$. Let $C' = v_1 v_2 \cdots v_m$ be a cycle and let $P = v_{m+1} \cdot v_{m+2} \cdots v_n$ be a path. Set $G = C' \vee P$, where $\vee$ denotes join union. Then it is easy to check that G has no $[k, k+1]$-factor containing Hamiltonian cycle $C = v_1 v_2 \cdots v_n$ even if the minimum degree is at least $n/2$.

Remark 2. For a graph G of order n, the condition that the minimum degree $\geqslant n/2$ cannot guarantee the existence of a k-factor containing a given Hamiltonian cycle in G. For instance, suppose $n \geqslant 5$ and $k \geqslant 3$. Write

$$m = \begin{cases} \dfrac{n}{2}+2 & \text{for even } n, \\ \dfrac{n+3}{2} & \text{for odd } n. \end{cases}$$

Let $C' = v_1 v_2 \cdots v_m$ be a cycle and let $P = v_{m+1} v_{m+2} \cdots v_n$ be a path. Set $G = C' \vee P$. Then the minimum degree $\geqslant n/2$ and G has no k-factor containing Hamiltonian cycle $C = v_1 v_2 \cdots v_n$.

Proof of Theorem 1. We may suppose $k \geqslant 3$ as G contains C for $k=2$. Write

$$H := G - C,\ U := \left\{ v \in V(G) \mid d_G(v) \geqslant \frac{n}{2} \right\},\ L := V(G)/U,\ \rho := k - 2.$$

Then $V(H) = V(G)$, $\rho \geqslant 1$, $d_H(v) = d_G(v) - 2 \geqslant \rho$ for all $v \in V(G)$, $n \geqslant 8\rho$ for even n and $n \geqslant 6\rho - 1$ for odd n.

Obviously, G has a required factor if and only if H has a $[\rho, \rho+1]$-factor. Suppose, to the contrary, that H has no such factor. Then, by Lovász's $[g, f]$-factor theorem[7], there exists an ordered pair $B = (S, T)$ of disjoint subsets S and T of $V(H)$ such that

$$\delta(B) := -(\rho+1)s + \rho t - \sum_{v \in T} d_{H-S}(v) \geqslant 1, \tag{2}$$

where $s = |S|$ and $t = |T|$.

We may assume

$$d_{H-S}(v) \leqslant \rho - 1 \quad \text{for all } v \in T. \tag{3}$$

Otherwise, say $d_{H-S}(u) \geqslant \rho$ for some $u \in T$, and put $B' = (S, T \setminus u)$. Then $\delta(B') \geqslant \delta(B)$, (2) still holds for B'.

Assertion 1. *$G[L]$ is a complete graph.*

Indeed, for any two vertices $u, v \in L$, $d_G(u) + d_G(v) < n$ by the definition of L. Thus $uv \in E(G)$ by (1).

Assertion 2. $s \geqslant 1$.

Otherwise $\delta(B) = \rho t - \sum_{v \in T} d_H(v) \leqslant 0$, a contradiction.

Assertion 3. $t \geqslant \rho + 2$.

Indeed, assume $t \leqslant \rho + 1$, then using $d_H(v) \geqslant \rho$ for all $v \in V(H)$, we have

$$\begin{aligned}\delta(B) &\leqslant -(\rho+1)s + \rho t - \sum_{v \in T}(d_H(v) - s) \\ &\leqslant -(\rho+1)s + \rho t - t(\rho - s) \\ &= s(t - \rho - 1) \leqslant 0,\end{aligned}$$

which contradicts (2).

Assertion 4. $s \leqslant \left[\frac{n}{2}\right] - 3$.

Write $X = \{v \in \bar{S} \mid d_G(v) \geqslant n/2\}$, $Y = \bar{S} \setminus X$. We consider two cases according to whether n is even or odd.

For even n, assume $s \geqslant (n/2) - 2$, and put $q := s - (n/2) + 2 (\geqslant 0)$, $r := n - s - t (\geqslant 0)$. Then

$$\begin{aligned}\delta(B) &= -(\rho+1)s + \rho(n - s - r) - \sum_{v \in T} d_{H-S}(v) \\ &= -s(2\rho+1) + \rho(n - r) - \sum_{v \in T} d_{H-S}(v) \\ &= -\left(\frac{n}{2} - 2 + q\right)(2\rho+1) + \rho(n - r) - \sum_{v \in T} d_{H-S}(v) \\ &= 4\rho + 2 - \frac{n}{2} - (\rho+1)q - \rho(r + q) - \sum_{v \in T} d_{H-S}(v) \\ &\leqslant 0,\end{aligned}$$

unless $q = 0$, $r \leqslant 1$ and $\sum_{v \in T} d_{H-S}(v) = 0$.

Indeed, suppose $\sum_{v \in T} d_{H-S}(v) \geqslant 1$. If $r \geqslant 1$, obviously $\delta(B) \leqslant 0$. And if $r = 0$, then

$\sum_{v\in T} d_{H-S}(v) = 2\mid E_G[\bar{S}]\setminus C\mid \equiv 0 \pmod 2$, yielding $\delta(B)\leqslant 0$. So it suffices to show $\sum_{v\in T} d_{H-S}(v) \geqslant 1$. $q=0$ yields $s=(n/2)-2$. Assuming $\sum_{v\in T} d_{H-S}(v)=0$, as $r\leqslant 1$, it follows that

$$E_G[\bar{S}]\subset C.$$

Hence for each $v\in\bar{S}$, $d_G(v)\leqslant d_{H-S}(v)+s+2=n/2$, implying

$$d_G(v)=\frac{n}{2}\qquad \text{for all } v\in X.$$

Therefore, all the edges of C incident to vertices in X are contained in $E_G[\bar{S}]$. We have

$$\begin{aligned}\mid X\mid+\mid Y\mid-1 &= \mid\bar{S}\mid-1\geqslant\mid E_G[\bar{S}]\cap C\mid\\ &\geqslant\mid X\mid+1+\mid E_G[Y]\mid=\mid X\mid+1+\frac{\mid Y\mid(\mid Y\mid-1)}{2},\end{aligned}$$

implying

$$\mid Y\mid\geqslant 2+\frac{\mid Y\mid(\mid Y\mid-1)}{2},$$

a contradiction.

For odd n, assume $s\geqslant(n-3)/2$, and put $q:=s-(n-3)/2(\geqslant 0)$, $r:=n-s-t(\geqslant 0)$. Then

$$\begin{aligned}\delta(B) &= -(\rho+1)s+\rho t-\sum_{v\in T} d_{H-S}(v)\\ &= 3\rho+\frac{3}{2}-\frac{n}{2}-(\rho+1)q-\rho(r+q)-\sum_{v\in T} d_{H-S}(v)\\ &\leqslant 0,\end{aligned}$$

unless $q=0$ and $\sum_{v\in T} d_{H-S}(v)=0$. Similarly we have $E_G[\bar{S}]\subset C$, and

$$d_G(v)=\frac{n+1}{2}\text{for all } v\in X.$$

All the edges of C incident to vertices in X are contained in $E_G[\bar{S}]$. We derive a contradiction $|Y|\geqslant 2+|Y|(|Y|-1)/2$.

Assertion 5. $T\cap U\neq\emptyset$.

Indeed, if $T\subseteq L$, then $|E_G[T]|=t(t-1)/2$ by Assertion 1. As C is a Hamiltonian cycle, $|E_G[T]\cap C|\leqslant t-1$. Hence

$$\sum_{v\in T} d_{H-S}(v)\geqslant 2\mid E_G[T]\setminus C\mid\geqslant t(t-1)-2(t-1)=(t-1)(t-2),$$

$$\begin{aligned}\delta(B) &\leqslant -(\rho+1)s+\rho t-(t-1)(t-2),\\ &\leqslant -(\rho+1)s+\rho t-(t-1)\rho \qquad \text{(by Assertion 3)},\\ &= -(\rho+1)s+\rho<0 \qquad \text{(by Assertion 2)},\end{aligned}$$

a contradiction.

Assertion 6. $T\cap L\neq\emptyset$

Indeed, suppose $T\subseteq U$. Then

$$\left\lceil\frac{n}{2}\right\rceil\leqslant d_G(v)\leqslant d_{H-S}(v)+s+2\leqslant\rho+s+1 \text{ for all } v\in T,$$

yielding $d_{H-S}(v)\geqslant\lceil n/2\rceil-s-2$ and $\rho+s+2-\lceil n/2\rceil\geqslant 1$. Hence

$$\delta(B)\leqslant-(\rho+1)s+\rho t-t\left(\left\lceil\frac{n}{2}\right\rceil-s-2\right)$$

$$= t\left(\rho + s + 2 - \left\lceil \frac{n}{2} \right\rceil\right) - (\rho + 1)s$$
$$\leqslant (n - s)\left(\rho + s + 2 - \left\lceil \frac{n}{2} \right\rceil\right) - (\rho + 1)s.$$

Put $f(s) = (n - s)(\rho + s + 2 - [n/2]) - (\rho + 1)s$. Then

$$f'(s) = -2\rho - 3 + n + \left\lceil \frac{n}{2} \right\rceil - 2s$$
$$\geqslant -2\rho - 3 + n + \left\lceil \frac{n}{2} \right\rceil - 2\left\lceil \frac{n}{2} \right\rceil + 6 \qquad \text{(by Assertion 4)}$$
$$= -2\rho + 3 + \left\lceil \frac{n}{2} \right\rceil \geqslant 0, \qquad (\text{as } n \geqslant 6\rho - 1)$$

implying

$$f(s) \leqslant f\left(\left\lceil \frac{n}{2} \right\rceil - 3\right)$$
$$= \left(\left\lceil \frac{n}{2} \right\rceil + 3\right)(\rho - 1) - (\rho - 1)\left(\left[\frac{n}{2} - 3\right]\right)$$
$$= \rho\left(\left\lceil \frac{n}{2} \right\rceil - \left\lceil \frac{n}{2} \right\rceil + 6\right) - n \leqslant 0.$$

The last inequality follows from the condition that $n \geqslant 8\rho$ for even n and $n \geqslant 6\rho - 1$ for odd n. Therefore $\delta(B) \leqslant 0$, a contradiction. Hence Assertion 6 is true.

Now put

$$T_1 := T \cap U,\ T_2 := T \cap L,\ t_1 := |T_1|,\ t_2 := |T_2|.$$

Clearly, $t_1 \geqslant 1$, $t_2 \geqslant 1$ and $d_{H-S}(v) \geqslant d_G(v) - s - 2$ for all $v \in T$. Hence for all $v \in T_1$,

$$d_{H-S}(v) \geqslant \begin{cases} \frac{n}{2} - s - 2 & \text{if } n \text{ is even,} \\ \frac{n}{2} - s - \frac{3}{2} & \text{if } n \text{ is odd.} \end{cases} \qquad (4)$$

It follows from (3) that

$$\rho + s + 2 - \frac{n}{2} \geqslant 1 \text{ if } n \text{ is even and } \rho + s + \frac{3}{2} - \frac{n}{2} \geqslant 1 \text{ if } n \text{ is odd.}$$

Assertion 4 yields

$$\rho \geqslant 2. \qquad (5)$$

We claim

$$t_2 \leqslant \rho + 2. \qquad (6)$$

Indeed, by Assertion 1, $d_{H-S}(v) \geqslant t_2 - 3$ for all $v \in T_2$. (6) derives from (3).

To complete the proof it suffices to consider two cases according to whether n is even or odd. For even n, using (4)—(6) we have

$$\delta(B) \leqslant -(\rho + 1)s + \rho t - t_1\left(\frac{n}{2} - s - 2\right)$$
$$= t_1\left(\rho + s + 2 - \frac{n}{2}\right) - (\rho + 1)s + \rho t_2$$
$$\leqslant (n - s - t_2)\left(\rho + s + 2 - \frac{n}{2}\right) - (\rho + 1)s + \rho t_2$$
$$= -\left(s - \frac{n}{2} + 3\right)^2 + \left(s - \frac{n}{2} + 3\right)\left(\frac{n}{2} + 3 - 2\rho - t_2\right) + 6\rho + t_2 - n$$

$$\leqslant -2\rho + t_2 \leqslant 0.$$

For odd n, write $r := n - s - t(\geqslant 0)$. It is easy to see

$$\sum_{v \in T_2} d_{H-S}(v) \geqslant 2 \mid E_G[T_2] \setminus C \mid \geqslant (t_2 - 1)(t_2 - 2). \tag{7}$$

Using (4), (5) and (6) we have

$$\begin{aligned}
\delta(B) &\leqslant -(\rho + 1)s + \rho t - t_1\left(\frac{n}{2} - s - \frac{3}{2}\right) - (t_2 - 1)(t_2 - 2) \\
&= t_1\left(\rho + s + \frac{3}{2} - \frac{n}{2}\right) - (\rho + 1)s + \rho t_2 - (t_2 - 1)(t_2 - 2) \\
&= (n - s - t_2 - r)\left(\rho + s + \frac{3}{2} - \frac{n}{2}\right) - (\rho + 1)s + \rho t_2 - (t_2 - 1)(t_2 - 2) \\
&= -\left(s - \frac{n}{2} + \frac{5}{2}\right)^2 + \left(s - \frac{n}{2} + \frac{5}{2}\right)\left(\frac{n}{2} + \frac{5}{2} - 2\rho - t_2\right) \\
&\quad + 5\rho - n + t_2 - (t_2 - 1)(t_2 - 2) - r\left(\rho + s + \frac{3}{2} - \frac{n}{2}\right) \\
&\leqslant 0,
\end{aligned}$$

unless $s = \frac{n}{2} - \frac{5}{2}$, $t_2 = 2$, $r = 0$, $\rho = 2$ and (7) holds throughout with equality. If (7) holds throughout with equality, $|E_G[T_2] \cap C| = 1$. As $s = \frac{n}{2} - \frac{5}{2}$ and $\rho = 2$, it follows from (3) and (4) that for all $v \in T_1$,

$$d_{H-S}(v) = 1 \text{ and } d_G(v) = \frac{n+1}{2},$$

implying all the edges of C incident to vertices in T_1 are contained in $E_G[T] \setminus E[T_2]$ and thus the number of such edges is at least $t_1 + 1$. Therefore, $|E_G[T] \cap C| \geqslant t_1 + 1 + 1 = t$, contradicting the assumption that C is a Hamiltonian cycle. Consequently the theorem is proved.

Slightly modifying the proof of Theorem 1, one can prove the following.

Theorem 2. *In Theorem* 1 *with* $n \geqslant 8k - 12$ *in place of* $n \geqslant 8k - 16$ *and* $n \geqslant 6k - 9$ *in place of* $n \geqslant 6k - 13$, *the other conditions being the same, for any given* 2-*factor* F, G *has a* $[k, k+1]$-*factor containing* F.

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