Normal Cayley graphs of finite groups*

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Abstract Let G be a finite group and let S be a nonempty subset of G not containing the identity element 1. The Cayley (di) graph $X = \mathrm{Cay}(G, S)$ of G with respect to S is defined by $V(X) = G$, $E(X) = \{(g, sg) \mid g \in G, s \in S\}$. A Cayley (di)graph $X = \mathrm{Cay}(G, S)$ is said to be normal if $R(G) \lhd A = \mathrm{Aut}(X)$. A group G is said to have a normal Cayley (di)graph if G has a subset S such that the Cayley (di)graph $X = \mathrm{Cay}(G, S)$ is normal. It is proved that every finite group G has a normal Cayley graph unless $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ or $G \cong Q_8 \times \mathbb{Z}_2^r$ $(r \geqslant 0)$ and that every finite group has a normal Cayley digraph, where $\mathbb{Z}_m$ is the cyclic group of order m and Q_8 is the quaternion group of order 8.

Keywords: Cayley graph, normal Cayley (di)graph.

The symmetry and the classification of vertex-transitive graphs have been much studied[1,2]. Cayley graph is one of the most important classes of vertex-transitive graph. In these studies, we often need to determine the full automorphism groups of the corresponding Cayley graphs[3]. Normal Cayley graphs play an important role in determining their full automorphism groups. On the other hand, the question about the normality of Cayley graphs should be investigated further after the conclusion of GRR[4]. In this paper we just want to find out which kind of finite groups has normal Cayley graphs.

Throughout this paper symbol G always denotes a finite group and 1 denotes its identity. S stands for a nonempty subset of a group G not containing the identity element 1. Symbol X denotes a simple graph. Symbols $V(X)$, $E(X)$, $A(X)$ and $A_b(X)$ denote, respectively, its vertex set, edge set, automorphism group, and the stabilizer of the vertex b in $A(X)$. For any set T, 1_T indicates the identity permutation on T. Let $\mathbb{Z}_m$ denote the cyclic group of order m, Σ_n the symmetric group of degree n, D_n the dihedral group of order n, and Q_8 the quaternion group of order 8.

A group G is generalized to be dicyclic if it is non-abelian and has an abelian subgroup L and an element $b \in G \setminus L$ such that $|G:L| = 2$, $o(b) = 4$ and $b^{-1}xb = x^{-1}$ for each element x in L.

Definition 1. Let G be a finite group and let S be a nonempty subset of G not containing the identity element 1. The Cayley (di)graph $X = \mathrm{Cay}(G, S)$ of G with respect to S is defined

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by $V(X)=G$, $\{(g, sg) \mid g\in G$, $E(X)=s\in S\}$. A Cayley (di)graph $X=\mathrm{Cay}(G,S)$ is said to be nor-mal if $R(G)\lhd A=\mathrm{Aut}(X)$, where $R(G)$ is the right regular representation of G. A group G is said to have a normal Cayley (di)graph if G has a subset S such that the Cayley (di) graph $X=\mathrm{Cay}(G,S)$ is normal.

Let $X=\mathrm{Cay}(G,S)$ be a Cayley digraph of G with respect to S. Then $A(X)$ contains the right regular representation $R(G)$ of G, so X is vertex-transitive. Moreover, X is (strongly) connected if and only if $G=\langle S\rangle$, and X is undirected if and only if $S^{-1}=S$.

The main result of this paper is the following two theorems.

Theorem 1. *Let G be a finite group. Then G has a normal Cayley graph unless* $G\cong\mathbb{Z}_4\times\mathbb{Z}_2$ *or* $G\cong Q_8\times\mathbb{Z}_2^r$, $r\geqslant 0$.

Theorem 2. *Every finite group G has a normal Cayley digraph.*

For convenience we recall the definitions of DRR and GRR. Let $X=\mathrm{Cay}(G,S)$ be a Cayley (di)graph of G with respect to S. Then X is called a (di)graphical regular representation (DRR or GRR) of G if $R(G)=A(X)$. We mention the results of GRR and DRR as follows[5,6].

Theorem A. *Every finite group G admits a GRR unless G belongs to one of the following classes of groups:*

Class C: abelian groups of exponent greater than two.

Class D: the generalized dicyclic groups.

Class E: the following thirteen "exceptional groups":

(1) $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$.

(2) D_6, D_8, D_{10}.

(3) A_4.

(4) $\langle a,b,c \mid a^2=b^2=c^2=1,\ abc=bca=cab\rangle$.

(5) $\langle a,b \mid a^8=b^2=1, bab=b^5\rangle$.

(6) $\langle a,b,c \mid a^3=c^3=b^2=1, ac=ca, (ab)^2=(cb)^2=1\rangle$.

(7) $\langle a,b,c \mid a^3=b^3=c^3=1, ac=ca, bc=cb, c=a^{-1}b^{-1}ab\rangle$.

(8) $Q_8\times\mathbb{Z}_3$, $Q_8\times\mathbb{Z}_4$.

Theorem B. *With five exceptions, every finite group admits a DRR. The exceptions are the elementary abelian groups of order* 4, 8, 9, 16 *and the quaternion group* Q_8.

1 Preliminary results

Let $X=\mathrm{Cay}(G,S)$ be a Cayley (di)graph. It is easy to see that any automorphism of G fixing S setwise induces an automorphism of the Cayley (di)graph X. Denote this group of automorphisms by $\mathrm{Aut}(G,S)$; that is,

$$\mathrm{Aut}(G,S)=\{\varphi\in\mathrm{Aut}(G) \mid S^{\varphi}=S\}.$$

For convenience we denote $\Lambda_i=\{x\in V(X) \mid d(1,x)=i\}$, $\Lambda_i(b)=\{x\in V(X) \mid d(b,x)=i\}$ for $b\neq 1$ and $\Gamma_i(x)=\Lambda_i\cap\Lambda_1(x)=\{y\in\Lambda_i \mid d(x,y)=1\}$ for $x\neq 1$ and positive integer i, where $d(\cdot,\cdot)$ is the distance function of X. Sometimes we also use Λ_i to denote its induced subgraph.

Lemma A[4]. *Let G be a group and* $X=\mathrm{Cay}(G,S)$. *Then*

(1) $N_{A(X)}(R(G))=R(G)\mathrm{Aut}(G,S)$, *where* $R(G)$ *is the right regular representation of* G.

(2) $R(G)\lhd A(X)$ *if and only if* $A_1(X)\subseteq\mathrm{Aut}(G,S)$.

From this lemma and the vertex-transitivity of (di)graph Cay(G, S) we can obtain the following lemma.

Lemma 1. *Let* $X=\mathrm{Cay}(G,S)$ *be connected. Then* X *is a normal Cayley (di)graph of* G *if and only if the following conditions are satisfied:*

(i) *for each* $\varphi\in A_1(X)$ *there exists* $\sigma\in\mathrm{Aut}(G)$ *such that* $\varphi|_{\Lambda_1}=\sigma|_{\Lambda_1}$;

(ii) *for each* $\varphi\in A_1(X)$, $\varphi|_{\Lambda_1}=1_{\Lambda_1}$ *implies* $\varphi|_{\Lambda_2}=1_{\Lambda_2}$.

Proof. If X is normal then $A_1(X)=\mathrm{Aut}(G,S)$ by Lemma A, and conditions (i) and (ii) are obviously satisfied. Now we show that these two conditions are sufficient.

(1) If $\varphi\in A_1(X)$ and $\varphi|_{\Lambda_1}=1_{\Lambda_1}$, then $\varphi=1$.

Since $A(X)$ is transitive on $V(X)=G$, every vertex b meets hypothesis (ii). That is, for $b\in G$ and $\varphi\in A_b(X)$, $\varphi|_{\Lambda_1(b)}=1_{\Lambda_1(b)}$ implies $\varphi|_{\Lambda_2(b)}=1_{\Lambda_2(b)}$. If $\varphi\in A_1(X)$ and $\varphi|_{\Lambda_1}=1_{\Lambda_1}$, then $\varphi|_{\Lambda_2}=1_{\Lambda_2}$ by (ii). So $\varphi|_{\Lambda_1(s)}=1_{\Lambda_1(s)}$ for all $s\in\Lambda_1=S$. Therefore $\varphi|_{\Lambda_2(s)}=1_{\Lambda_2(s)}$ for all $s\in\Lambda_1$. Since for each $x\in\Lambda_3$ there exists some $s\in\Lambda_1$ such that $x\in\Lambda_2(s)$, $\varphi|_{\Lambda_3}=1_{\Lambda_3}$ follows. Since X is connected, by induction we can show that $\varphi|_{\Lambda_i}=1_{\Lambda_i}$ for all possible i. Hence $\varphi=1$.

(2) $A_1(X)\subseteq\mathrm{Aut}(G,S)$.

By hypothesis (i), for each $\varphi\in A_1(X)$, we may let $\sigma\in\mathrm{Aut}(G)$ such that $\varphi|_{\Lambda_1}=\sigma_{\Lambda_1}$. Then $\varphi\sigma^{-1}|_{\Lambda_1}=1_{\Lambda_1}$. By the proof above we have $\varphi\sigma^{-1}=1$ and $\varphi=\sigma\in\mathrm{Aut}(G)$.

Then (2) implies that X is a normal Cayley (di)graph of the group G.

It is well known that any (DRR) GRR must be (strongly) connected and must not be nontrivial lexicographic product. But for normal Cayley (di)graphs this is not true and we have the following results. We recall that a Cayley (di)graph $X=\mathrm{Cay}(G,S)$ is a nontrivial lexicograph product if and only if there is a proper nonidentity subgroup H of G such that $S\setminus H$ is a union of left cosets of H[7].

Theorem 3. *Let* $X=\mathrm{Cay}(G,S)$ *be a normal Cayley graph for group* G. *Then*

(i) *X is disconnected if and only if* $G\cong\mathbb{Z}_2^{r+1}$ *or* $\mathbb{Z}_4\times\mathbb{Z}_2^{r-1}$, *where* $r=1$ *or* $r\geqslant 5$; $H:=\langle S\rangle\cong\mathbb{Z}_2^r$, *and* $W:=\mathrm{Cay}(H,S)$ *is a GRR for* H.

(ii) *X is a nontrivial lexicographic product if and only if X or its complement X' is disconnected.*

Proof. (i) Let $X=\mathrm{Cay}(G,S)$ be normal and disconnected. Then $\langle S\rangle=H<G$ and $|G:H|=t>1$. Let $W=\mathrm{Cay}(H,S)$ and $G=H\cup Hb_1\cup\cdots\cup Hb_{t-1}$ be a coset-decomposition of G with respect to H. We use symbol W_i to denote the induced subgraph of X by the vertices in Hb_i, $i=1,2,\cdots,t-1$. It is easy to see that $W\cong W_i$.

For each $h\in H$ we define a map $\lambda_h:G\to G$ as

$$g^{\lambda_h}=\begin{cases}g, & g\in H;\\ gh, & g\notin H.\end{cases}$$

It is easy to verify that $\lambda_h\in A_1(X)$. By the normality of X we have $\lambda_h\in\mathrm{Aut}(G,S)$. If $t>2$ then $b_1b_2^{-1}\notin H$, and so $b_1b_2^{-1}h=(b_1b_2^{-1})^{\lambda_h}=b_1^{\lambda_h}(b_2^{-1})^{\lambda_h}=b_1hb_2^{-1}h$. This implies that $h=1$ for all $h\in H$; a contradiction. Hence $t=2$. We may let $G=H\cup Hb$. Notice that $Hb=bH$.

Let $h,k\in H$. Then $(bk)h=(bk)^{\lambda_h}=b^{\lambda_h}k^{\lambda_h}=bhk$ and so $hk=kh$. Therefore H is abelian.

Since $b^2 \in H$ we have $b^2 = (b^2)^{\lambda_h} = b^{\lambda_h} b^{\lambda_h} = bhbh$ and $b^{-1}hb = h^{-1}$ for all $h \in H$. Especially for $h = b^2$ we have $b^{-1}b^2 b = b^{-2}$; that is, $b^4 = 1$.

We assert that $W = \mathrm{Cay}(H, S)$ is a GRR for H. Otherwise, let $\alpha \in A_1(W)$ with $\alpha \neq 1$. Then we can define a map $\sigma_\alpha : G \to G$ by

$$g^{\sigma_\alpha} = \begin{cases} g, & g \in H; \\ h^\alpha b, & g = hb,\ h \in H. \end{cases}$$

It is easy to verify that $\sigma_\alpha \in A_1(X)$. The normality of X implies $\sigma_\alpha \in \mathrm{Aut}(G, S)$. Then $(hb)^{\sigma_\alpha} = h^\alpha b = h^{\sigma_\alpha} b^{\sigma_\alpha} = hb$ and $h^\alpha = h$ for all $h \in H$. This contradicts the hypothesis that $\alpha \neq 1$. Hence $W = \mathrm{Cay}(H, S)$ is a GRR for H. By Theorem A, $H \cong \mathbb{Z}_2^r$, where $r = 1$ or $r \geqslant 5$. Therefore G must be abelian itself for $b^{-1}hb = h^{-1} = h$ for all $h \in H$ and $b \in G \setminus H$. If $b^2 = 1$ then $G \cong \mathbb{Z}_2^{r+1}$. If $b^2 \neq 1$ then $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2^{r-1}$. We have proved the necessity.

Now let $X = \mathrm{Cay}(G, S)$ be a Cayley graph which satisfies the conditions in the theorem above. Then X is obviously disconnected. So we only need to show that X is normal. It is easy to check that $\lambda_h \in \mathrm{Aut}(G, S)$ for all $h \in H$ in this case. Let $\sigma \in A_1(X)$. Then $\sigma|_W \in A_1(W) = 1$; that is, $\sigma|_W = 1_W$. Suppose that $b^\sigma = bh^{-1}$ for some $h \in H$. Then $b^{\sigma\lambda_h} = b$ and $\sigma\lambda_h|_{W_1} = 1_{W_1}$ for $W_1 \cong W$ and W is a GRR. Therefore $\sigma = \lambda_h$ for some $h \in H$ and X is normal.

(ii) First it is easy to see that both X and X' are nontrivial lexicographic products if $X = \mathrm{Cay}(G, S)$ is a disconnected normal Cayley graph. Now let normal Cayley graph $X = \mathrm{Cay}(G, S)$ be a connected nontrivial lexicographic product. We shall show that X' is disconnected. By our assumption we see that G has a nontrivial proper subgroup H such that $S = T \cup b_1 H \cup \cdots \cup b_r H$, where $T \subseteq H^*$, $b_i \notin H$, $b_i H \neq b_j H\ (i \neq j)$ and $r \geqslant 1$. For each $h \in H$ if we define a map λ_h as above, then still we have $\lambda_h \in A_1(X) = \mathrm{Aut}(G, S)$.

Let $G = H \cup b_1 H \cup \cdots \cup b_r H \cup \cdots \cup b_{t-1} H$ be a coset-decomposition of G with respect to H. If $t - 1 \geqslant 2$ then $b_1^{-1} b_2 \notin H$, and so $(b_1^{-1} b_2)^{\lambda_h} = b_1^{-1} b_2 h = (b_1^{-1})^{\lambda_h} b_2^{\lambda_h} = b_1^{-1} h b_2 h$. This implies that $h = 1$ for all $h \in H$; a contradiction. Hence $t - 1 = 1$ and $r = 1$. So we may let $G = H \cup Hb$. Notice that $Hb = bH$. In this case $X' = \mathrm{Cay}(G, H^* \setminus T)$ is obviously disconnected.

From this theorem we immediately have the following corollary.

Corollary 1. *If $X = \mathrm{Cay}(G, S)$ is a normal Cayley graph and $G \not\cong \mathbb{Z}_4$, $\mathbb{Z}_2^2$, $\mathbb{Z}_4 \times \mathbb{Z}_2^{r-1}$ and $\mathbb{Z}_2^{r+1} (r \geqslant 5)$, then both X and X' are connected.*

Theorem 4. *Let $X = \mathrm{Cay}(G, S)$ be a normal Cayley digraph for group G. Then*

1. *X is not strongly connected if and only if G has a subgroup H such that*

(1) *H is a nontrivial abelian subgroup of G and $|G : H| = 2$;*

(2) *for any $b \in G \setminus H$ and $h \in H$, $b^4 = 1$ and $b^{-1}hb = h^{-1}$;*

(3) *$W = \mathrm{Cay}(H, S)$ is a DRR for H.*

2. *X is a nontrivial lexicographic product if and only if X or X' is not strongly connected.*

As the proof of this lemma is similar to the undirected case we omit it. Notice that in this case, from Theorem 3, we know that $H \not\cong \mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$ or $\mathbb{Z}_3^2$. So group G is abelian or generalized dicyclic.

2 Normal Cayley graphs for the groups in class C

The Cayley index of a group G is defined by

$$c(G) = \min |A_1(\mathrm{Cay}(G, S))|,$$

where S runs over all Cayley subsets of G. It is easy to see that G admits a GRR if and only if $c(G)=1$. If $c(G)=2$ then G has a normal Cayley graph.

Lemma B[9]. *Let G be a finite abelian group. Then $c(G)\leqslant 2$ unless G is one of the following seven groups: $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2^2$, $\mathbb{Z}_3^2$, $\mathbb{Z}_3^3$ and $\mathbb{Z}_4^2$.*

Since $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$ are in Class E, by this lemma we only need to investigate the following five groups: $\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2^2$, $\mathbb{Z}_3^2$, $\mathbb{Z}_3^3$ and $\mathbb{Z}_4^2$.

Lemma 2. *Every group among $\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_2^2$, $\mathbb{Z}_3^2$, $\mathbb{Z}_3^3$ and $\mathbb{Z}_4^2$, except $\mathbb{Z}_4\times\mathbb{Z}_2$, has a normal Cayley graph.*

Proof. (i) $\mathbb{Z}_4\times\mathbb{Z}_2$ has no normal Cayley graph.

By Corollary 1, it is enough to show that G has no connected normal Cayley graph of degree 3. Suppose that $X=\mathrm{Cay}(G, S)$ is a connected normal Cayley graph for G and $|S|=3$. Then S must contain a pair of elements of order 4, say $a^{\pm 1}$, and an involution b; that is, $S=\{a^{\pm 1}, b\}$. Since $G=\langle a\rangle\times\langle b\rangle$, (Theorem 1 of ref. [10]), $X=\mathrm{Cay}(G, S)=\mathrm{Cay}(\langle a\rangle, \{a^{\pm 1}\})\times \mathrm{Cay}(\langle b\rangle, \{b\})\cong C_4\times K_2\cong K_2\times K_2\times K_2$. So $A(X)\cong\mathbb{Z}_2\mathrm{wr}\Sigma_3$, $|A(X)|=2^3\cdot 6=48$, and $|A_1(X)|=6$. But $\mathrm{Aut}(G, S)\cong\mathbb{Z}_2$. Hence $A_1(X)\neq\mathrm{Aut}(G, S)$, a contradiction.

(ii) Each of $\mathbb{Z}_4\times\mathbb{Z}_2^2$, $\mathbb{Z}_3^2$, $\mathbb{Z}_3^3$ has a normal Cayley graph.

We only prove it for $\mathbb{Z}_4\times\mathbb{Z}_2^2$. As for the other two cases, the proof is similar. Let $G=\mathbb{Z}_4\times\mathbb{Z}_2^2=\langle a\rangle\times\langle b\rangle\times\langle c\rangle$, where $o(a)=4$, $o(b)=o(c)=2$. If we take $S=\{a^{\pm 1}, b, bc, c\}$, $H=\langle a\rangle$ and $N=\langle b, c\rangle$, then by Theorem 1 of ref. [10], $X=\mathrm{Cay}(G, S)=\mathrm{Cay}(H\times N, S_1\cup S_2)=\mathrm{Cay}(H, S_1)\times\mathrm{Cay}(N, S_2)\cong C_4\times K_4$, where $S_1=\{a^{\pm 1}\}$, $S_2=\{b, bc, c\}$. By Lemma 2.1 of ref. [11], we have $A(X)=A(C_4)\times A(K_4)\cong D_8\times\Sigma_4$ and $|A_1(X)|=12$. It is easy to see that $\mathrm{Aut}(G, S)\cong\mathbb{Z}_2\times\Sigma_3$, and $|\mathrm{Aug}(G, S)|=12$. Hence $A_1(X)=\mathrm{Aut}(G, S)$ and $X=\mathrm{Cay}(G, S)$ is a normal Cayley graph of G.

(iii) $\mathbb{Z}_4^2$ has a normal Cayley graph.

Let $G=\langle a, b\mid a^4=b^4=1, ab=ba\rangle$ and take $S=\{a^{\pm 1}, b^{\pm 1}, ab, a^{-1}b^{-1}, b^2\}$. Consider the neighborhood subgraph Λ_1 induced by S (see figure 1).

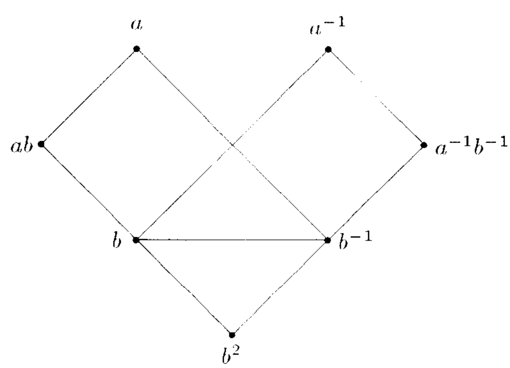

Fig. 1

Let $\varphi\in A_1(X)$. Considering the restriction of φ to Λ_1, one sees $a^{\varphi}\in\{a, a^{-1}, ab, a^{-1}b^{-1}\}$ since they are only 2-valent vertices of Λ_1 with only one 4-valent neighbor in Λ_1. We note that $(b^2)^{\varphi}=b^2$ since this is the only 2-valent vertex of Λ_1 with two 4-valent neighbors in Λ_1.

By simple calculation, we know that $\Gamma_1(x)\neq\Gamma_1(y)$ if $x, y\in G\setminus S$ and $x\neq y$. So $\varphi|_{\Lambda_1}=1_{\Lambda_1}$ implies $\varphi|_{\Lambda_2}=1_{\Lambda_2}$. Hence condition (ii) of Lemma 1 is satisfied.

Now we show that condition (i) also holds.

Let $\varphi\in A_1(X)$. By observation of Λ_1 we have permutation $\varphi|_S=1_S,(a,a^{-1})(b,b^{-1})(ab,a^{-1}b^{-1}),(a,ab)(b,b^{-1})(a^{-1},a^{-1}b^{-1})$, or $(a,a^{-1}b^{-1})(a^{-1},ab)$. In each case one can construct a suitable $\sigma\in\mathrm{Aut}(G)$ such that $X^\sigma=X^4$ for all $X\in S$. By Lemma 1, X is a normal Cayley graph of G.

3 Normal Cayley graphs for the groups in class D

In this section G always denotes a generalized dicyclic group generated by the abelian group L and an element b as in the definition in sec. 1 above. We define two functions $\alpha,\beta:G\to G$ given by $x^\alpha=x^{-1}$, $(xb)^\alpha=x^{-1}b$ and $x^\beta=x,(xb)^\beta=(xb)^{-1}=xb^{-1}$ for all $x\in L$. It is not difficult to check that $\alpha,\beta\in\mathrm{Aut}(G)$, $\alpha^2=\beta^2=1$, $\alpha\beta=\beta\alpha$. So $B=\{1,\alpha,\beta,\alpha\beta\}\leqslant\mathrm{Aut}(G)$ and $B\cong\mathbb{Z}_2^2$. In addition β is a graph-automorphism of Cay(G,S) for any Cayley subset S of G, and $x^{\alpha\beta}=x^{-1},(xb)^{\alpha\beta}=x^{-1}b^{-1}$ for all $x\in L$.

In ref. [11] Imrich and Watkins defined a graph X belonging to class $\mathscr{P}_{n,q}(n\geqslant2,q\geqslant1)$ if it is isovalent and the set $V(X)$ admits a partition $\{V_1,V_2,\cdots,V_p\}$ with $2\leqslant p\leqslant n$ such that every vertex in V_i is adjacent to at most q vertices in V_j for all $i\neq j$. It is shown in ref. [11] (Corollary 1B) that the following lemma holds.

Lemma C. *If $X\in\mathscr{P}_{n,q}$ and $|V(X)|>qn(n+2)$, then $X'\notin\mathscr{P}_{n,q}$.*

Lemma 3. *Let G be a generalized dicyclic group. If $G\not\cong Q_8\times\mathbb{Z}_2^m$ $(m\geqslant0)$ and $|G|>32$. Then G has a normal Cayley graph.*

Proof. Since $|L|>16$ and L has even order, by Lemmas B and 6, L has a normal Cayley graph $W=\mathrm{Cay}(L,J)$ such that $A_1(W)=\{1_L,\alpha|_L\}$. Set $W'=\mathrm{Cay}(L,J')$, $S=J\cup\{b,b^{-1}\}$, $T=J'\cup\{b,b^{-1}\}$, $X=\mathrm{Cay}(G,S)$, and $Y=\mathrm{Cay}(G,T)$.

We first prove either

$$L^\varphi = L \text{ for all } \varphi\in A_1(X) \tag{3.1}$$

or

$$L^\psi = L \text{ for all } \psi\in A_1(Y). \tag{3.2}$$

Otherwise, since $|L|>16$, by Lemma C, either W or W' is not in $\mathscr{P}_{2,2}$. If $L^\varphi\neq L$ then $(Lb)^\varphi\neq Lb$. Let $L=L_1\cup L_2$ and $Lb=L_3\cup L_4$ such that $L_1^\varphi\subseteq L$, $L_2^\varphi\subseteq Lb$, $L_3^\varphi\subseteq Lb$ and $L_4^\varphi\subset L$. Since every vertex in L_1^φ is adjacent to at most two vertices in L_2^φ, every vertex in L_1 is also adjacent to at most two vertices in L_2. So $W\in\mathscr{P}_{2,2}$. In the same way we have $W'\in\mathscr{P}_{2,2}$. This is a contradiction. Now without loss of generality we may assume that (3.1) holds. We shall show that $A_1(X)=\mathrm{Aut}(G,S)\leqslant B=\{1,\alpha,\beta,\alpha\beta\}$. So X is a normal Cayley graph of G. Now let $\varphi\in A_1(X)$. By (3.1) either $\varphi|_L=1_L$ or $\varphi|_L=\alpha|_L$.

First suppose $\varphi|_L=1_L$. Since $b^\varphi\in\{b,b^{-1}\}$, we begin by supposing $b^\varphi=b$. Notice that $A_b(X)=b^{-1}A_1(X)b$. Since $A_1(X)|_L\leqslant A_1(W)$, $A_b(X)|_{Lb}\leqslant b^{-1}A_1(W)b|_{Lb}$. By our assumption $\varphi\in A_b(X)$. So $\varphi|_{Lb}=1_{Lb}$ or $b^{-1}ab|_{Lb}$. In the former case $\varphi=1$; in the latter case $(xb)^\varphi=x^{-1}b$ for all $x\in L$. By our assumption on G, there exists an $x\in L$ such that $x^2\neq1$, b^2. In particular, if $L=\mathbb{Z}_4\times\mathbb{Z}_2^m=\langle a_1\rangle\times\mathbb{Z}_2^m$ for some $m\geqslant3$, where $\mathbb{Z}_4=\langle a_1\rangle$, then $b^2\neq a_1^2$, or else G would be isomorphic to $Q_8\times\mathbb{Z}_2^m$. Hence in this case we can take $x=a_1$. Since $(xb)^\varphi=x^{-1}b$ and $\varphi|_L=1_L$, the neighbors of xb in L, namely bxb and x^{-1}, must coincide with the neighbors of $x^{-1}b$ in L, namely $bx^{-1}b$ and x. But clearly $x\neq x^{-1}$, while $x=bxb$ implies x^2

$=b^2$; a contradiction. Therefore $\varphi|_L = 1_L$ and $b^\varphi = b$ include $\varphi = 1$. Now suppose that $b^\varphi = b^{-1}$. Then $(\varphi\beta)|_L = 1_L$ and $b^{\varphi\beta} = b$. It follows from what we have shown that $\varphi\beta = 1$. Since $\beta^2 = 1$ we have $\varphi = \beta$.

On the other hand, suppose $\varphi|_L = \alpha|_L$. Let $\psi = \alpha^{-1}\varphi$. Then $\psi|_L = 1_L$. From the proof above we conclude that $\psi = 1$ or $\psi = \beta$; that is, $\varphi = \alpha$ or $\varphi = \alpha\beta$. So $A_1(X) \leqslant B \leqslant \mathrm{Aut}(G)$ and X is a normal Cayley graph of G.

Proposition 1. *If G is a generalized dicyclic group with $L = \mathbb{Z}_{2m}$ ($m \geqslant 3$), then G has a normal Cayley graph.*

Proof. Let $L = \langle a \rangle$ and $J = \{a, a^{-1}\}$ whence $W = \mathrm{Cay}(L, J) \cong C_{2m}$. Then $A_1(W) = \{1_L, \alpha|_L\}$. Let $S = J \cup \{b, b^{-1}\}$ and $X = \mathrm{Cay}(G, S)$. We can check that each of the edges $(1, a^{\pm 1})$ lies on precisely two 4-cycles and each of the edges $(1, b^{\pm 1})$ lies on precisely three 4-cycles. Hence for any $\varphi \in A_1(X)$, $J^\varphi = J$ and $b^\varphi = b$ or b^{-1}. By Propsosition 2.1 of ref. [8], $L^\varphi = L$ and $(Lb)^\varphi = Lb$ for all $\varphi \in A_1(X)$. Since $a^2 \neq 1$, $a^2 \neq b^2$, by similar argument as in the proof of Lemma 3 we also have $A_1(X) = \mathrm{Aut}(G, S) \leqslant B = \{1, \alpha, \beta, \alpha\beta\}$ in this case. Hence X is a normal Cayley graph of G.

Proposition 2. *Let G be a group and $|G| > 8$. If G has a normal Cayley graph, then $G \times \mathbb{Z}_2$ has a normal Cayley graph.*

Proof. Let $W = \mathrm{Cay}(G, J)$ be a normal Cayley graph for G. Let $S = J \cup \{a\}$ and $X = \mathrm{Cay}(G \times \mathbb{Z}_2, S)$, where $\langle a \rangle = \mathbb{Z}_2$. We will show that X is a normal Cayley graph for $G \times \mathbb{Z}_2$.

Since $|G| > 8$, by Lemma C, either W or W' is not in $\mathscr{P}_{2,1}$. We may suppose that $W \notin \mathscr{P}_{2,1}$. Then $G^\varphi = G$ for all $\varphi \in A_1(X)$ and $a^\varphi = a$. Notice that $A_1(\mathrm{W}) = \mathrm{Aut}(G, J)$ and $\varphi|_G = \sigma \in A_1(W) = \mathrm{Aut}(G, J)$. Since $A_a(X \setminus W) = a^{-1}A_1(W)a$ and $\varphi|_{Ga} \in A_a(X \setminus W)$, we may let $\varphi|_{Ga} = a\tau a$, $\tau \in A_1(W)$. For any $x \in G$, $(x, xa) \in E(X)$. We have $(x, xa)^\varphi = (x^\sigma, x^\tau, a)$. But $x^\sigma a$ are the neighbors of x^σ in Ga. We must have $x^\sigma a = x^\tau a$; that is, $x^\sigma = x^\tau$ for all $x \in G$. So $\varphi|_{Ga} = a^{-1}\sigma a|_{Ga}$. It is easy to see that $\varphi \in \mathrm{Aut}(G \times \mathbb{Z}_2, S)$. Hence X is a normal Cayley graph of $G \times \mathbb{Z}_2$.

Lemma 4. *Let G be a generalized dicyclic group. If G is not of the form $Q_8 \times \mathbb{Z}_2^m$ for some $m \geqslant 0$ and $|G| \leqslant 32$, then G has a normal Cayley graph.*

Proof. The proof will fall into three cases according to the structure of the abelian group L. By Proposition 1 we only need to deal with the cases in which L is not a cyclic group. So we may let L be one of the following groups $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_6 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2^2$, $\mathbb{Z}_4^2$ or $\mathbb{Z}_8 \times \mathbb{Z}_2$.

Case 1. $L = \mathbb{Z}_6 \times \mathbb{Z}_2$ or $L = \mathbb{Z}_8 \times \mathbb{Z}_2$.

Let $L = \langle a_1 \rangle \times \langle a_2 \rangle$, where $\langle a_1 \rangle = \mathbb{Z}_6$ or $\mathbb{Z}_8$ and $\langle a_2 \rangle = \mathbb{Z}_2$. If $b^2 = a_1^m$ ($m = 3, 4$), then $G = \langle a_1, b \rangle \times \langle a_2 \rangle$. In this case, by Propositions 1 and 2 we know that G has a normal Cayley graph. Now suppose $b^2 = a_2$. (The case $b^2 = a_1^m a_2$ is equivalent under the automorphism $a_1 \mapsto a_1$, $a_2 \mapsto a_1^m a_2$ of group L because we have also $L = \langle a_1 \rangle \times \langle a_1^m a_2 \rangle$). Let $J = \{a_1, a_1^{-1}, a_2\}$. Then we can easily verify that $\mathrm{Cay}(L, J)$ is a normal Cayley graph of L such that $A_1(\mathrm{Cay}(L, J)) = \{1_L, \alpha|_L\}$. This time, however, let W be the complement of $\mathrm{Cay}(L, J)$, say, $W = \mathrm{Cay}(L, J')$. Let $X = \mathrm{Cay}(G, S)$, $S = J' \cup \{b, b^{-1}\}$. Notice that every edge with both vertices in L or both vertices in Lb lies on a 3-cycle. No edge with one vertex in L and one vertex in Lb has this property; for if (g, bg) were such an edge, then the other two edges on the 3-cycle would have to be (bg, b^2g) and (g, b^2g) or (b^{-1}, g, g) and $(b^{-1}g, bg)$. But (g, b^2g), $(b^{-1}g, bg) \notin E(X)$ since $b^2 = a_2 \notin S$. It is easy to see that in each instance W is a connected graph. So we have $L^{\varphi} = L$ for all $\varphi \in A_1(X)$.

Since $a_1^2 \neq 1$, b^2, by similar argument as in the proof of Lemma 3 we see that X is a normal Cayley graph of G.

Case 2. $L = \mathbb{Z}_4^2$.

Let $L = \langle a, c \mid a^4 = c^4 = 1, ac = ca \rangle$. We may assume that $b^2 = c^2$ without loss of generality. If $J = \{a^{\pm 1}, c^{\pm 1}, (ac)^{\pm 1}, c^2\}$, then by Lemma 2 $W = \mathrm{Cay}(L, J)$ is a normal Cayley graph of L. Let $S = J \cup \{b, b^{-1}\}$ and $X = \mathrm{Cay}(G, S)$. Then b and b^{-1} are only two vertices with valence 1 in the induced graph, then $b^{\varphi} = b$ or b^{-1} and $J^{\varphi} = J$ for all $\varphi \in A_1(X)$. By Propsosition 2.1 of ref. [8], $L^{\varphi} = L$ for all $\varphi \in A_1(X)$. Notice that $(b^2)^{\varphi} = b^2$ for all $\varphi \in A_1(X)$, especially for all $\sigma \in A_1(W)$.

Let $\varphi \in A_1(X)$ and $\varphi|_L = \sigma \in A_1(W) = \mathrm{Aut}(L, J)$. We suppose $b^{\varphi} = b$ first. Since $A_b(X \setminus W)$ is isomorphic as a permutation group to $A_1(W)$, and $A_b(X \setminus W)|_{Lb} = b^{-1}A_1(W)b$, we have $\varphi|_{Lb} = b^{-1}\tau b$ for some $\tau \in A_1(W)$. Then $(ab)^{\varphi} = a^{\tau}b$. The neighbors of $(ab)^{\varphi}$ in L, namely $ba^{\tau}b$ and $(a^{\tau})^{-1}$, must coincide with the images of the neighbors of ab in L, namely $ba^{\sigma}b$ and $(a^{\sigma})^{-1}$. If $(a^{\tau})^{-1} = (a^{\sigma})^{-1}$, then $a^{\sigma\tau^{-1}} = a$. This implies $\sigma = \tau$. If $(a^{\tau})^{-1} \neq (a^{\sigma})^{-1}$, then $(a^{\sigma})^{-1} = ba^{\tau}b$ and $a^{\sigma} = (ab^2)^{\tau}$ and $a^{\sigma\tau^{-1}} = ab^2 \notin V(\Lambda_1)$; a contradiction. Hence $\sigma = \tau$; that is, if $\varphi|_L = \sigma$, then $\varphi|_{Lb} = b^{-1}\sigma b$. For convenience, let δ_{σ} denote such a φ. Obviously $\delta_{\sigma} \in \mathrm{Aut}(G, S)$ for all $\sigma \in A_1(W)$. Now we suppose $b^{\varphi} = b^{-1}$. Then $\varphi\beta|_L = \sigma \in A_1(W)$ and $b^{\varphi\beta} = b$. By what we have just shown, $\varphi\beta = \delta_{\sigma}$ and $\varphi = \delta_{\sigma}\beta \in \mathrm{Aut}(G, S)$ since $\beta \in \mathrm{Aug}(G)$. Therefore X is a normal Cayley graph of G.

Case 3. $L = \mathbb{Z}_4 \times \mathbb{Z}_2$.

Let $L = \langle a, c \mid a^4 = c^2 = 1, ac = ca \rangle$. By our assumption on G, $b^2 \neq a^2$ or else $G \cong Q_8 \times \mathbb{Z}_2$. So we may assume that $b^2 = c$ without loss of generality. In this case $G = \langle a, b \mid a^4 = b^4 = 1, b^{-1}ab = a^{-1} \rangle$.

Let $S = \{a^{\pm 1}, b^{\pm 1}, (ba)^{\pm 1}, a^2\}$. Observe the neighborhood graph Λ_1 induced by S (see figure 2).

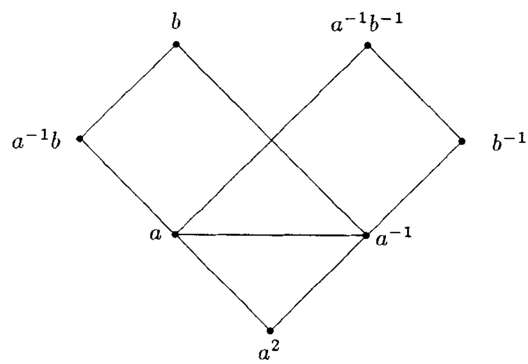

Fig. 2

Looking at the graph we see that for any $\varphi \in A_1(X)$, $\varphi|_S = 1_S$ or $(b, b^{-1})(ba, a^{-1}b^{-1})$ or $(a, a^{-1})(b, ba)(b^{-1}, a^{-1}b^{-1})$ or $(a, a^{-1})(b, a^{-1}b^{-1})(b^{-1}, ba)$. So $\varphi|_S$ can always be regarded as a restriction of a group-automorphism on S. By computation we have $\Gamma_1(x) \neq \Gamma_1(y)$ when $x, y \in \Lambda_2$ and $s \neq y$. So $\varphi|_S = 1_S$ implies $\varphi = 1$ since $G = S \cup S^2 \cup S^3$. By Lemma 1 $X = \mathrm{Cay}(G, S)$ is a normal Cayley graph of G.

Case 4. $L = \mathbb{Z}_4 \times \mathbb{Z}_2^2$.

Let $L = \langle a \rangle \times \langle c \rangle \times \langle d \rangle$, where $o(a) = 4$, and $o(c) = o(d) = 2$. Since $G \ncong Q_8 \times \mathbb{Z}_2^2$, we may assume that $b^2 = c^2$ without loss of generality. Then $G = \langle a, b \rangle \times \langle d \rangle$. By Proposition 2 and the above Case 3 we see that G has a normal Cayley graph.

Lemma 5. *If $G = Q_8 \times \mathbb{Z}_2^m$ ($m \geqslant 0$), then G has no normal Cayley graph.*

Proof. Obviously $G = Q_8 \times \mathbb{Z}_2^m$ is a generalized dicyclic group with $L = \mathbb{Z}_4 \times \mathbb{Z}_2^m$ and $b^2 \in \mathbb{Z}_4$. For any Cayley subset of G we shall construct a graph-automorphism γ of $X = \mathrm{Cay}(G, S)$ such that $\gamma \in A_1(X)$ but $\gamma \notin \mathrm{Aut}(G, S)$. This will imply that G has no normal Cayley graph.

We define γ: $G \to G$ by $g^\gamma = g^{-1}$, $g \in G$. Since G is not abelian, γ is not a group-automorphism of G. Notice that for any $x \in L$, $x^2 = 1$ or $x^2 = b^2$. Now we show that $\gamma \in A_1(X)$. Let $e = (u, v) \in E(X)$. We only need to verify that $e_1 = (u, v)^\gamma = (u^{-1}, v^{-1}) \in E(X)$. If u, v are both in L or Lb, it is easy to see $e_1 \in E(X)$. Now let $u \in L$ and $v = xb \in Lb$, $x \in L$. Since $e \in E(X)$, $vu^{-1} = xbu^{-1} = xub \in S$ and $(xub)^{-1} = xub^{-1} \in S$. While $v^{-1}(u^{-1})^{-1} = v^{-1}u = xu^{-1}b^{-1}$, if $u^2 = 1$, then $xu^{-1}b^{-1} = xub^{-1} \in S$; if $u^2 \neq 1$, then $xu^{-1}b^{-1} = xuu^2b^{-1} = xub^2b^{-1} = xub \in S$. Hence in each case we have $e_1 \in E(X)$. Therefore, $\gamma \in A_1(X)$.

4 Normal Cayley graphs for the groups in class E

In this section we shall prove that each exceptional group has a normal Cayley graph. We choose some appropriate generating set S for each exceptional group G, and verify the conditions in Lemma 1.

Lemma 6. *The Cayley graphs of the following exceptional groups G with respect to the corresponding generating set are all normal:*

(1) $G = \langle a_1 \rangle \times \cdots \times \langle a_r \rangle$, $S = \{a_1, \cdots, a_r\}$.

(2) $G = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$, $S = \{a, b\}$.

(3) $G = A_4$, $S = \{a, b, b^{-1}\}$, *where* $a = (12)(34)$, $b = (123)$.

(4) $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$, $S = \{a, b, c, s^2\}$, *where* $s = abc$.

(5) $G = \langle a, b \mid a^8 = b^2 = 1, bab = a^5 \rangle$, $S = \{a, a^{-1}, b, a^4, a^4b\}$.

(6) $G = \langle a, b, c \mid a^3 = c^3 = b^2 = 1, ac = ca, (ab)^2 = (cb)^2 = 1 \rangle$, $S = \{a^{\pm 1}, b, c^{\pm 1}\}$.

(7) $G = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, ac = ca, bc = cb, c = a^{-1}b^{-1}ab \rangle$, $S = \{a^{\pm 1}, b^{\pm 1}\}$.

(8) $G = Q_8 \times \mathbb{Z}_3$, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, $\mathbb{Z}_3 = \langle a \rangle$, $S = \{ai, -a^{-1}i, j, -j\}$.

(9) $G = Q_8 \times \mathbb{Z}_4$, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, $\mathbb{Z}_4 = \langle a \rangle$, $S = \{a^{\pm 1}, \pm i, (ai)^{\pm 1}, \pm j, -1\}$.

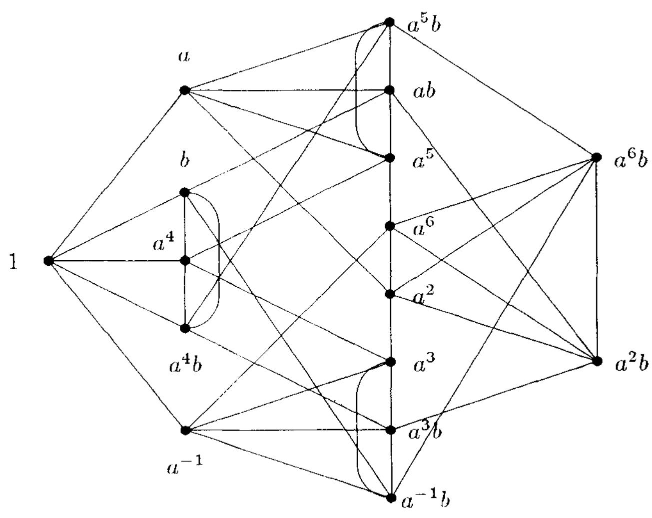

Fig. 3

Proof. The normality of the Cayley graphs in (1) and (2) can be shown easily, and the normality of those in (6), (8) and (9) can be proved similarly. The proof for the remainder is similar, and we only prove case (5).

Set $S = \{a, a^{-1}, b, a^4, a^4b\}$. Then $A_1 = S$, $A_2 = \{a^5b, ab, a^5, a^6, a^2, a^3, a^{-1}b, a^3b\}$, $A_3 = \{a^2b,$

$a^6b\}$ (see fig. 3). We can examine whether $\Gamma_1(x)\neq\Gamma_1(y)$ for any $x\neq y\in\Lambda_2$. So if $\varphi\in A_1(X)$ and $\varphi|_S=1|_S$, then $\varphi|_{\Lambda_2}=1|_{\Lambda_2}$. Now we consider the action of any $\varphi\in A_1(X)$ on $\Lambda_1=S$. φ stabilizes $\{a, a^{-1}\}$ and $\{a^4, a^4b, b\}$, and since φ stabilizes $\{x\in\Lambda_2\mid\Gamma_3(x)=\emptyset\}=\{a^3, a^5\}$ it also stabilizes $\Gamma_1(a^3)\cap\Gamma_1(a^5)=\{a^4\}$. Hence φ stabilizes $\{a, a^{-1}\}$, $\{b, a^4b\}$ and $\{a^4\}$. It follows that $A_1|_S\leqslant\langle(a, a^{-1}), (b, a^4b)\rangle$. It is easy to see that $\langle(a, a^{-1}), (b, a^4b)\rangle\leqslant\mathrm{Aut}(G, S)|_S$, so we have $A_1(X)|_S\leqslant\mathrm{Aut}(G, S)|_S$. Therefore, by Lemma 1, X is a normal Cayley graph and $A_1(X)=\mathrm{Aut}(G, S)\cong\mathbb{Z}_2\times\mathbb{Z}_2$.

5 Normal Cayley digraphs of finite groups

Now we give the proof of Theorem 2.

By Theorems 1 and B we need only to show that Q_8 has a normal Cayley digraph. Let $G=Q_8=\{\pm1, \pm i, \pm j, \pm k\}$ and $S=\{i\}$. Then $X=\mathrm{Cay}(G, S)\cong\vec{C}_4\cup\vec{C}_4$ (see figure 4).

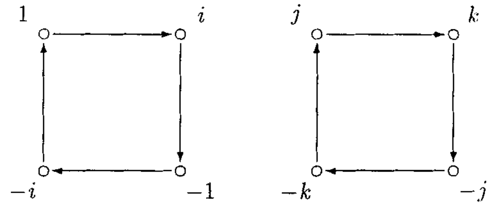

Fig. 4

If $\varphi\in A_1(X)$, then φ fixes $\{\pm1, \pm i\}$ pointwise, and so it stabilizes the directed cycle $(j, k, -j, -k)$. Let $\sigma=(j, k, -j, -k)$ be a permutation on G. It is easy to verify that $\sigma\in\mathrm{Aut}(G, S)$, and $\varphi\in\langle\sigma\rangle$. Hence X is a normal Cayley digraph of G by Lemma A.

References

1 Wang, R. J., Xu, M. Y., A classification of symmetric graphs of order $3p$, *J. Combin. Theory*, Ser. B, 1993, 58: 197.
2 Wang, R. J., 1/2-transtive graphs of order a product of two distinct primes, *Communication in Algebra*, 1994, 22: 915.
3 Xu, M. Y., Half-transitive graphs of prime-cube order, *Journal of Algebraic Combinatorics*, 1992, 1: 275.
4 Godsil, C. D., On the full automorphism group of a graph, *Combinatorica*, 1981, 1: 243.
5 Godsil, C. D., GRR's for non-solvable groups, *Colloq. Math. Soc. Janos. Bolyai*, 1978, 25: 221.
6 Babai, L., Finite digraphs with given regular automorphism groups, *Periodica. Math. Hung.*, 1980, 11: 257.
7 Godsil, C. D., The automorphism groups of some cubic Cayley graphs, *Europ. J. Combinatorics*, 1983, 4: 25.
8 Nowitz, L. A., Watkins, M. E., Graphical regular representation of non-abelian groups I, *Canadian J. Math.*, 1972, 24: 993.
9 Imrich, W., Watkins, M. E., On automorphism groups of Cayley graphs, *Periodica Mathematica Hungarica*, 1976, 7: 234.
10 Wang Dianjun, Wang Jianzhong, Hamiltonian decomposition of Cayley graphs of abelian groups(in Chinese), *Advance in Mathematics*, 1994, 4(23): 551.
11 Imrich, W., Watkins, M. E., On graphical regular representation of cyclic extensions of groups, *Pacific J. Math.*, 1974, 55: 461.