Exact solitary wave solutions of nonlinear wave equations

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Abstract The hyperbolic function method for nonlinear wave equations is presented. In support of a computer algebra system, many exact solitary wave solutions of a class of nonlinear wave equations are obtained via the method. The method is based on the fact that the solitary wave solutions are essentially of a localized nature. Writing the solitary wave solutions of a nonlinear wave equation as the polynomials of hyperbolic functions, the nonlinear wave equation can be changed into a nonlinear system of algebraic equations. The system can be solved via Wu Elimination or Gröbner base method. The exact solitary wave solutions of the nonlinear wave equation are obtained including many new exact solitary wave solutions.

Keywords: nonlinear wave equations, exact solitary wave solutions, travelling wave solutions, hyperbolic function method.

Nonlinear wave equations are applied in many fields of natural sciences. It is very important for us to obtain the exact solutions of these equations. Up to now, only a few methods for nonlinear wave equations proved successful, such as IST (inverse scattering transform) method, Hirota method, Bäcklund method and homogeneous balance method[1—4]. Shang also carried out a deep research on how to solve nonlinear wave equations[5]. In this paper, we present a hyperbolic function method which is based on the hyperbolic tangent method[6], changing a nonlinear wave equation into a nonlinear system of algebraic equations. Solving this system via Wu Elimination[7] or Gröbner base method[8], the exact solutions of the nonlinear wave equation can be obtained. We apply the method to some nonlinear wave equations. The exact solutions of these equations are obtained, which indicates that the method is feasible.

1 Hyperbolic function method

The hyperbolic function method is based on the fact that many solitary wave solutions have the format of hyperbolic functions. In this method, we assume that the nonlinear wave equations have solitary wave solutions, and the solutions can be expressed as the combination of hyperbolic functions.

We assume that PDE is a nonlinear wave equation, and it can be used to describe the dynamic evolution process of solitary wave $u(x,t)$. The steps of hyperbolic function method can be shown as follows:

1) Solitary wave is a kind of special travelling wave. That PDE has travelling wave solutions requires that PDE has only one argument $\xi = kx - ct + l$, where k (wave number), c (frequen-

cy) are constants to be determined and l is an arbitrary constant. Then $u(x,t) = u(\xi)$. PDE can be changed into an ordinary differential equation (ODE) via the following differential transformation:

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = k\frac{\partial}{\partial \xi}. \tag{1}$$

2) In order to obtain exact solitary wave solutions of ODE, we introduce two elementary solitary wave functions f and g defined as

$$f(\xi) = \frac{1}{\cosh\xi + r}, \quad g(\xi) = \frac{\sinh\xi}{\cosh\xi + r}, \tag{2}$$

where $r(\geqslant 0)$ is a constant to be determined. Functions $f(\xi)$ and $g(\xi)$ satisfy the coupled Riccati equations[9]

$$f'(\xi) = -f(\xi)g(\xi), \quad g'(\xi) = 1 - g^2(\xi) - rf(\xi) \tag{3}$$

and their first integral

$$g^2(\xi) = 1 - 2rf(\xi) + (r^2 - 1)f^2(\xi). \tag{4}$$

3) We assume that the solutions of ODE are polynomials of f and g which have the polynomial degree of m

$$\phi = \sum_{i=0}^{m} a_i f^i + \sum_{j=1}^{m} b_j f^{j-1} g, \tag{5}$$

where the coefficients $a_i(i = 0,1,2,\cdots,m)$ and $b_j(j = 1,2,\cdots,m)$ are constants to be determined and satisfy $a_m^2 + b_m^2 \neq 0$. The polynomial degree m can be determined via balancing the highest order derivative terms and the nonlinear terms in ODE.

4) Constructing this polynomial with the degree m and substituting the polynomial into ODE, eliminating any derivative of (f, g) and any power of g higher than one with eqs. (3,4) and setting the coefficients of the different powers of f and g to zero, we obtain a nonlinear system of algebraic equations (AES) with all parameters which are to be determined.

5) Solving the AES to obtain all parameters via Wu Elimination or Gröbner base method, we obtain the exact solitary wave solutions of PDE in support of computer algebra system maple 4 in this paper.

2 Exact solitary wave solutions of nonlinear wave equations

In this section we will apply the hyperbolic function method to some nonlinear wave equations to verify the correctness of the method. We wish that the exact solitary wave solutions can be obtained via the method.

2.1 Burgers equation

Burgers equation is one of the important nonlinear wave equations in physics and mechanics. Its standard format is listed as follows[10]:

$$u_t + uu_x + pu_{xx} = 0. \tag{6}$$

In order to obtain the solitary wave solutions of eq. (6), using the hyperbolic function method we can determine the degree of the solitary wave solutions, and then we have $m = 1$. The corresponding AES is

$$\begin{cases} ka_1b_1 - k^2pa_1 + b_1cr - ka_0b_1r = 0, \\ 3ka_1b_1r - 3k^2pa_1r + b_1c(r^2-1) - ka_0b_1(r^2-1) = 0, \\ 2k^2pa_1(r^2-1) - 2ka_1b_1(r^2-1) = 0, \\ a_1c - ka_0a_1 + kb_1^2r - k^2pb_1r = 0, \\ 2k^2pb_1(r^2-1) - ka_1^2 - kb_1^2(r^2-1) = 0. \end{cases} \tag{7}$$

Using the Gröbner base method, in support of computer algebra system maple 4, three solitary wave solutions are obtained:

$$u_1 = \frac{c}{k} + 2pk\tanh(kx - ct + l), \tag{8}$$

where k, c, l are arbitrary constants;

$$u_2 = \frac{c}{k} + pk\frac{\sinh(kx - ct + l)}{\cosh(kx - ct + l) + 1}, \tag{9}$$

where k, c, l are arbitrary constants;

$$u_3 = \frac{c}{k} + pk\sqrt{r^2-1}\,\frac{1}{\cosh(kx - ct + l) + r} + pk\frac{\sinh(kx - ct + l)}{\cosh(kx - ct + l) + r}, \tag{10}$$

where $k, c, l, r(\geqslant 1)$ are arbitrary constants.

2.2 KdV equation

KdV equation[11]

$$u_t + uu_x + pu_{xxx} = 0 \tag{11}$$

is a very famous nonlinear wave equation. It is a fundamental model in nonlinear wave theory and it is a classical equation used to study the soliton phenomenon.

In order to obtain the solitary wave solutions of eq. (11), using the hyperbolic function method we can determine the degree of the solitary wave solutions, and then we have $m = 2$. The corresponding AES is

$$\begin{cases} b_2c - b_1cr - ka_1b_1 - ka_0b_2 + ka_0b_1r + k^3pb_1r - k^3pb_2 = 0, \\ b_1c(r^2-1) - 3b_2cr - ka_0b_1(r^2-1) - 2ka_2b_1 - 2ka_1b_2 + 3ka_1b_1r + 3ka_0b_2r \\ \quad + 4k^3pb_1 + 15k^3pb_2r - 7k^3pb_1r^2 = 0, \\ 2b_2c(r^2-1) - 2ka_1b_1(r^2-1) - 2ka_0b_2(r^2-1) - 3ka_2b_2 + 5ka_2b_1 + 5ka_1b_2r \\ \quad + 20k^3pb_2 - 50k^3pb_2r^2 + 12k^3pb_1r(r^2-1) = 0, \\ 7ka_2b_2r - 3ka_2b_1(r^2-1) - 3ka_1b_2(r^2-1) + 60k^3pb_2r(r^2-1) \\ \quad - 6k^3pb_1(r^2-1)^2 = 0, \\ 4ka_2b_2(r^2-1) + 24k^3pb_2(r^2-1)^2 = 0, \\ a_1c - ka_0a_1 - kb_1b_2 + kb_1^2r - k^3pa_1 = 0, \\ 2a_2c - ka_1^2 - 2ka_0a_2 - kb_2^2 - kb_1^2(r^2-1) + 4kb_1b_2r - 8k^3pa_2 + 6k^3pa_1r = 0, \\ 3ka_1a_2 + 3kb_1b_2(r^2-1) - 3kb_2^2r + 6k^3pa_1(r^2-1) - 30k^3pa_2r = 0, \\ 2ka_2^2 + 2kb_2^2(r^2-1) + 24k^3pa_2(r^2-1) = 0. \end{cases} \tag{12}$$

Using the same method as in sec. 2.1, two solitary wave solutions are obtained;

$$u_1 = \frac{c}{k} - 4k^2p + 12k^2p\,\mathrm{sech}^2(kx - ct + l), \tag{13}$$

where k, c, l are arbitrary constants;

$$u_2 = \frac{c}{k} + 6k^2pr\frac{1}{\cosh(kx - ct + l) + r} + \frac{6k^2p(1 - r^2) \pm 6k^2p\sqrt{r^2 - 1}\sinh(kx - ct + l)}{[\cosh(kx - ct + l) + r]^2}, \tag{14}$$

where $k, c, l, r(>1)$ are arbitrary constants.

2.3 Chaffee-Infante equation

The standard format of Chaffee-Infante equation[12] is

$$u_t - u_{xx} + \lambda(u^3 - u) = 0. \tag{15}$$

In order to obtain the solitary wave solutions of eq. (15), using the hyperbolic function method we can determine the degree of the solitary wave solutions, then we have $m = 1$. The corresponding AES is

$$\begin{cases} \lambda a_0^3 + 3\lambda a_0 b_1^2 - \lambda a_0 = 0, \\ 3\lambda a_0^2 b_1 + 3\lambda a_1 b_1^2 - 6\lambda a_0 b_1^2 r - \lambda a_1 - b_1 cr - k^2 a_1 = 0, \\ b_1 c(r^2 - 1) + 3k^2 a_1 r + 3\lambda a_0 a_1^2 + 3\lambda a_0 b_1^2(r^2 - 1) - 6\lambda a_1 b_1^2 r = 0, \\ \lambda a_1^3 + 3\lambda a_1 b_1^2(r^2 - 1) - 2k^2 a_1(r^2 - 1) = 0, \\ 3\lambda a_0^2 b_1 + \lambda b_1^3 - \lambda b_1 = 0, \\ a_1 c + k^2 b_1 r + 6\lambda a_0 a_1 b_1 - 2\lambda b_1^3 r = 0, \\ 3\lambda a_1^2 b_1 + \lambda b_1^3(r^2 - 1) - 2k^2 b_1(r^2 - 1) = 0. \end{cases} \tag{16}$$

Using the same method as in sec. 2.1, some solitary wave solutions are obtained.

If $\lambda < 0$, eq. (15) admits two exact solitary wave solutions:

$$u_1 = \pm\sqrt{2}\operatorname{sech}(kx + l), \tag{17}$$

where $k^2 = -\lambda$, and l is an arbitrary constant;

$$u_2 = \frac{\sqrt{6}}{3}\frac{1}{\cosh(kx + l) + \frac{\sqrt{6}}{3}} \pm 1, \tag{18}$$

where $k^2 = -\lambda$, and l is an arbitrary constant.

If $\lambda > 0$, eq. (15) admits four exact solitary wave solutions:

$$u_3 = \pm\tanh(kx + l), \tag{19}$$

where $k^2 = \lambda/2$, and l is an arbitrary constant;

$$u_4 = a_1\frac{1}{\cosh(kx + l) + r} \pm \frac{\sinh(kx + l)}{\cosh(kx + l) + r}, \tag{20}$$

where $k^2 = 2\lambda$, $a_1^2 = r^2 - 1$, l and $r(\geqslant 1)$ are arbitrary constants;

$$u_5 = \pm\frac{1}{2} + b_1\frac{\sinh\left[kx \mp (3\lambda b_1 \mp \frac{3}{4}\lambda)t + l\right]}{\cosh\left[kx \mp (3\lambda b_1 \mp \frac{3}{4}\lambda)t + l\right] + 1}, \tag{21}$$

where $k^2 = \lambda/2$, $b_1^2 = 1/4$, and l is an arbitrary constant;

$$u_6 = a_0 \pm \frac{1}{2}\frac{1}{\cosh(kx \mp 3\lambda a_0 t + l) + \sqrt{2}} \pm \frac{1}{2}\frac{\sinh(kx \mp 3\lambda a_0 t + l)}{\cosh(kx \mp 3\lambda a_0 t + l) + \sqrt{2}}, \tag{22}$$

where $k^2 = \lambda/2$, $a_0^2 = 1/4$, and l is an arbitrary constant.

2.4 NLS$^+$ equation

NLS$^+$ equation is a very important equation in quantum mechanics. It has the following format[13]:

$$iu_t - u_{xx} + 2(|u|^2 - \rho^2)u = 0, \tag{23}$$

where $u(x, t)$ is a complex wave function, ρ is a constant.

In order to solve eq.(23), we must change eq.(23) into a real function and eliminate the symbol of absolute value. We introduce the following travelling wave transformation:

$$u(x, t) = \phi(kx - ct + l)\exp\left[-i\left(\frac{c}{2k}x + \lambda t\right)\right], \tag{24}$$

where l is an arbitrary constant and k, c, λ are constants to be determined. Let $\xi = kx - ct + l$. Substituting (24) into eq.(23), an ordinary differential equation of one argument ϕ is obtained,

$$\phi'' + p\phi - q\phi^3 = 0, \tag{25}$$

where

$$p = \frac{1}{k^2}\left(2\rho^2 - \lambda - \frac{c^2}{4k^2}\right), \quad q = \frac{2}{k^2}. \tag{26}$$

In order to obtain the solitary wave solutions of eq.(25), using the hyperbolic function method we can determine the degree of the solitary wave solutions, and then we have $m = 1$. The corresponding AES is

$$\begin{cases} a_0 p - (a_0^3 + 3a_0 b_1^2)q = 0, \\ a_1 + a_1 p - (3a_0^2 a_1 + 3a_1 b_1^2 - 6a_0 b_1^2 r)q = 0, \\ 3a_1 r + (3a_0 a_1^2 + 3a_0 b_1^2(r^2 - 1) - 6a_1 b_1^2 r)q = 0, \\ 2a_1(r^2 - 1) - (a_1^3 + 3a_1 b_1^2(r^2 - 1))q = 0, \\ b_1 p - (3a_0^2 b_1 + b_1^3)q = 0, \\ b_1 r + (6a_0 a_1 b_1 - 2b_1^3 r)q = 0, \\ 2b_1(r^2 - 1) - (3a_1^2 b_1 + b_1^3(r^2 - 1))q = 0. \end{cases} \tag{27}$$

Using the same method as in sec. 2.1, two solitary wave solutions of eq.(23) are obtained,

$$u_1 = \left[a_1 \frac{1}{\cosh(kx - ct + l) + r} + b_1 \frac{\sinh(kx - ct + l)}{\cosh(kx - ct + l) + r}\right] \times \exp\left[-i\left(\frac{c}{2k}x - \left(2\rho^2 - \frac{k^2}{2} - \frac{c^2}{4k^2}\right)t\right)\right], \tag{28}$$

where a_1, b_1 satisfy

$$a_1 = \pm\frac{k}{2}\sqrt{r^2 - 1}, \quad b_1 = \pm\frac{k}{2}, \tag{29}$$

and $k, c, r(\geqslant 1)$ are arbitrary constants.

$$u_2 = \pm k\,\mathrm{sech}(kx - ct + l) \cdot \exp\left[-i\left(\frac{c}{2k}x - \left(2\rho^2 - 2k^2 - \frac{c^2}{4k^2}\right)t\right)\right], \tag{30}$$

where k, c are arbitrary constants.

3 Conclusion

Using the hyperbolic function method to solve the nonlinear wave equations, many exact solitary wave solutions are obtained, including some new exact solitary wave solutions. It indicates that the method is really a very simple and effective method. The method can be used to solve a large class of nonlinear wave equations. We hope this method can do some work in research of nonlinear wave equations.

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References

1. Ablowitz, M. J., Carkson, P. A., Nonlinear Evolution and Inverse Scattering., New York: Cambridge University Press, 1991, 47—350.
2. Miura, M. R., Bäcklund Transformation, Berlin: Springer-Verlag, 1978, 4—156.
3. Hirota, R., Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons, Phys. Rev. Lett., 1971, 27: 1192—1194.
4. Wang, M. L., Zhou, Y. B., Li, Z. B., Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Phys. Lett. A, 1996, 213: 67—75.
5. Shang, Y. D., Explicit and exact solutions for a class of nonlinear wave equations, Acta Appl. Math. Sinica (in Chinese), 2000, 23(1): 21—30.
6. Li, Z. B., Zhang, S. Q., Exact solitary wave equations for nonlinear wave equations using symbolic computation, Acta Math. Phys. Sinica (in Chinese), 1997, 17(1): 81—89.
7. Wu Wenjun, On zeros of algebraic equations: An application of Ritt principle, Kexue Tongbao (Chinese Science Bulletin), 1986, 31(1): 1—5.
8. Heegard, C., Little, J., Saints, K., Systematic encoding via gröbner bases fro a class of algebraic geometric codes, IEEE Trans. Inform. Theory, 1995, IT-41: 1752—1761.
9. Conte, R., Musette, M., Link between solitary waves and projective Riccati equation, J. Phys. A: Math. Gen., 1992, 25: 2609—2612.
10. Wahlquist, H. D., Estabrook, F. B., Prolongation structures and nonlinear evolution equations, J. Math. Phys., 1975, 16: 1—7.
11. Whitham, G. B., Linear and Nonlinear Waves, New York: Wiley, 1974, 44.
12. Constantin, P., Foias, C., Nicolaenko, B., Integral Manifolds and Inertial Manifolds for Dissipative Partial Differtial Equations, New York: Springer-Verlag, 1981, 111—118.
13. Chen, S. R., Chen, X. J., Completeness relation of squared Jost functions to the NLS equation, Acta Phys. Sinica (in Chinese), 1999, 48(5): 882—886.