A conjecture on the Eichler cohomology of automorphic forms

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Abstract We prove partially a conjecture of Knopp about the Eichler cohomology of automorphic forms on H-groups.

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Knopp introduced the Eichler cohomology groups connected with automorphic forms of arbitrary real weight and with a suitable underlying space of functions and determined the structure of these groups (see Theorems 1 and 2 in ref. [1]).

For the detailed definitions of conceptions and notations in this paper, we follow ref. [1]. Now let r be an arbitrary real number and ν a multiplier system for H group Γ of weight r.

$P = \{$functions g holomorphic in H $\mid |g(z)| < K(|z|^{\rho} + y^{-\sigma})$ for $y = \mathrm{Im} z > 0\}$, where K, ρ, σ are positive constants.

Definition 1. Let $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{R})$ be a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Γ is called a hyperbolic subgroup if it satisfies

(1) Γ acts discontinuously on the upper-half plane $\mathrm{H} = \{x + \mathrm{i}y \in \mathbb{C} \mid y > 0\}$ but Γ is not discontinuous at any point on the real axis;

(2) there exist translation transformations in Γ, i.e. ∞ is a parabolic fixed point of Γ (cusp point);

(3) there is a fundamental domain D_0 of Γ with finite edges.

Definition 2. Let Γ be a hyperbolic subgroup. A complex-valued function $\nu = \nu[\Gamma, r]$ defined on Γ is called a multiplier system of group Γ with weight r if it satisfies

(1) $|\nu(M)| = 1$ for any $M \in \Gamma$;

(2) $\nu(M_1M_2)(c_{12}z + d_{12})^r = \nu(M_1)(c_1M_2z + d_1)^r\nu(M_2)(c_2z + d_2)^r$ for any M_1, $M_2 \in \Gamma$, $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $M_1M_2 = \begin{pmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{pmatrix}$.

Definition 3. Let ν be a multiplier system. A complex-valued function $F(z)$ is called an automorphic form of group Γ with weight r and multiplier system ν if it satisfies

(1) $F(z)$ is meromorphic on $\mathrm{H}^* = \mathrm{H} \cup \mathfrak{p}$, where $\mathfrak{p}$ is the set of all parabolic vertices (i.e. all cusp points);

(2) $F(Mz) = \nu(M)(cz + d)^r F(z)$ for any $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$.

Note. A function $F(z)$ is meromorphic at a cusp point p if $F(\rho^{-1}(z))j_{\rho^{-1}}(z)^{-r}$ is meromorphic at $z = \mathrm{i}\infty$, where $\rho \in \mathrm{PSL}(2, \mathbb{R})$ such that $\rho(p) = \mathrm{i}\infty$ and $j_M(z) = cz + d$ for $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.

Let Γ be a hyperbolic group. We always assume that $-I=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\in\Gamma$. If $A\in$ PSL(2,$\mathbb{R}$), $Ap=\infty$, then A maps H onto H. If $M\in\Gamma$ is parabolic and has a fixed point p, then AMA^{-1} is also parabolic and has the fixed point ∞. So $A\Gamma A^{-1}$ is a hyperbolic group.

Let $\mathfrak{p}$ be the set of all parabolic vertices of Γ. For any $p\in\mathfrak{p}\cap\mathbb{R}$ the isotropic subgroup $\Gamma_p=\{A\in\Gamma\mid Ap=p\}$ is cyclic. If P generates Γ_p, then so do $-P$, P^{-1}, $-P^{-1}$. We now choose a full representative set $\{p_0, p_1, \cdots, p_t\}$ of inequivalent parabolic vertices. If ∞ is one of these we assume $p_0=\infty$, otherwise p_0 disappears. Since Γ has only finite edges, we know that t is finite. Now let $\Gamma_{p_j}=\langle P_j\rangle$. We define

$$A_j=\begin{pmatrix}0 & -1\\ 1 & -p_j\end{pmatrix},\quad j>0,\ A_0=I.$$

It is clear that $A_jp_j=\infty$ for $j=0,1,\cdots,t$. $A_jP_jA_j^{-1}$ must be a translation because it is parabolic and has the fixed point ∞. So it must have the following forms:

$$A_jP_jA_j^{-1}=\begin{pmatrix}1 & \lambda_j\\ 0 & 1\end{pmatrix}=U^{\lambda_j},\ \lambda_j\in\mathbb{R}.$$

We can choose the generator P_j of Γ_{p_j} such that $\lambda_j>0$ for $j=0,1,\cdots,t$, especially, $\Gamma_\infty=\langle U^{\lambda_0}\rangle$, $\lambda_0>0$.

We introduce some notations as follows:

$$C_{jk}=\left\{x\,\middle|\begin{pmatrix}* & *\\ x & *\end{pmatrix}\in A_j\Gamma A_k^{-1}\right\},\quad j,\ k=0,1,\cdots,t;$$

$$D_c(j,k,\alpha,\beta)=\left\{d\,\middle|\begin{matrix}* & *\\ c & d\end{matrix}\in A_j\Gamma A_k^{-1},\ \alpha\leqslant-\frac{d}{c}<\beta\right\},\ c\neq0.$$

Lemma 1[2]**.** For every pair (j,k), C_{jk} is a discrete subset of $\mathbb{R}$.

Lemma 2[2]**.** For every $c\in C_{jk}$ and every pair (α,β) of real numbers, the set $D_c(j,k,\alpha,\beta)$ is finite and $D_c(j,k,\alpha,\beta)=O(c)$.

Suppose that $\nu=\nu[\Gamma,r]$ is a multiplier system and define that $\kappa_j=\kappa_j(\Gamma)$ is the unique number satisfying the following conditions:

$$e(\kappa_j)=\nu(P_j),\ 0\leqslant\kappa_j<1,\quad j=0,\ 1,\cdots,t,$$

where $e(\alpha)=\exp(2\pi i\alpha)$.

Consider the group $\Gamma'=A_j\Gamma A_j^{-1}$. The multiplier system $\nu[\Gamma,r]$ deduces one $\nu'[\Gamma',r]$ on Γ':

$$\nu'(M')=\nu(M)\text{ if }M'=A_jMA_j^{-1}.$$

Since P_j generates Γ_{p_j} so $A_jP_jA_j^{-1}$ generates Γ'_∞, we see that

$$e(\kappa_j)=\nu(P_j)=\nu'(A_jP_jA_j^{-1})=e(\kappa_0(\Gamma')),$$

i.e.

$$\kappa_j(\Gamma)=\kappa_0(A_j\Gamma A_j^{-1}).$$

Similarly we have

$$\lambda_j(\Gamma)=\lambda_0(A_j\Gamma A_j^{-1}).$$

Definition 4[1]**.** If F is a function meromorphic in H such that

$$F\,|_\nu^r V-F\in P\text{ for }V\in\Gamma,$$

and for each j, $1\leqslant j\leqslant t$, there exists an integer m_j such that $\exp\{2\pi i(m_j+\kappa_j)/\lambda_j(z-p_j)\}\cdot$

$F(z)$ has a limit as $z \to p_j$ within D_Γ (a fundamental domain of Γ) and also there exists an integer m_0 such that $\exp\{-2\pi i(m_0+\kappa_0)/\lambda_0\}\cdot F(z)$ has a limit as $z\to i\infty$ within D_Γ, then we call F an automorphic integral of degree r with respect to Γ.

If r is an integer $\geqslant 0$ and P is replaced by P_r (the set of all polynomials with degree $\leqslant r$), then this definition coincides with the one of Eichler integral. We refer to the cocycle $\{F|_\nu^r V - F\}$ as the cocycle of period functions of automorphic integral F. A coboundary of degree r is a cocycle $\{F|_\nu^r V - F\}$ such that $F|_\nu^r V - F = f|_\nu^r V - f$ for all $V\in\Gamma$, with f a fixed function $f\in P$. The parabolic cocycles are the cocycles $\{F|_\nu^r V - F\}$ which satisfy the following condition: Let $Q_0, Q_1, \cdots, Q_t$ be a complete set of parabolic representatives for Γ, then for each j $(1\leqslant j\leqslant t)$, there exists a function $f_h\in P$ such that $F|_\nu^r Q_h - F = f_h|_\nu^r Q_h - f_h$.

Definition 5. (a) The Eichler cohomology group $\mathrm{H}^1_{r,\nu}(\Gamma, P)$ is defined to be the vector space of cocycles modulo coboundaries.

(b) Let $\widetilde{\mathrm{H}}^1_{r,\nu}(\Gamma, P)$ be the subgroup of $\mathrm{H}^1_{r,\nu}(\Gamma, P)$ defined as the space of parabolic cocycles modulo coboundaries.

In ref. [1], Knopp proved the following theorem.

Theorem 1. If $r\geqslant 0$ or $r\leqslant -2$ with ν a multiplier system of degree r, then $C^0(\Gamma, -r-2, \bar{\nu})$ is isomorphic to $\widetilde{\mathrm{H}}^1_{r,\nu}(\Gamma, P)$ under a canonical isomorphism α, where $C^0(\Gamma, -r-2, \bar{\nu})$ denotes the collection of cusp forms for group Γ, weight r and multiplier system $\bar{\nu}$.

And Knopp made the following

Conjecture. Theorem 1 is true in the range $-2<r<0$.

In this paper we shall partially prove this conjecture, i.e. we have

Theorem 2. If $-2<r<-1$ and $C^0(\Gamma, -r-2, \bar{\nu})=0$, then $\widetilde{\mathrm{H}}^1_{r,\nu}(\Gamma, P)=0$.

1 Non-analytic and analytic Poincaré series

Let n be a negative integer and Γ, ν, κ_j, λ_j as above and r a positive real number. We introduce non-analytic Poincaré series $P_{nr}(z,s;\ \nu,\ \Gamma,\ \kappa_0)$ as follows:

$$P_{nr}(z,s;\ \nu,\ \Gamma,\ \kappa_0) = \sum_{T\in(D)} \frac{\exp(2\pi i(n+\kappa_0)Tz/\lambda_0)}{\nu(T)(cz+d)^r\,|cz+d|^s},$$

where $T=\begin{pmatrix} * & * \\ c & d\end{pmatrix}$, $D=\Gamma_\infty\setminus\Gamma$.

We first have to show the convergence of the above series. We see that

$$\begin{aligned}|\exp(2\pi i(n+\kappa_j)Tz/\lambda_0)| &\leqslant \exp\{2\pi(|n|+1)\mathrm{Im}(Tz)/\lambda_0\}\\ &= \exp\left\{\frac{2\pi(|n|+1)y}{\lambda_0}((cx+d)^2+c^2y^2)^{-1}\right\}\\ &\leqslant \exp\left\{\frac{2\pi(|n|+1)y}{\lambda_0 c^2y^2}\right\}\leqslant \exp\left\{\frac{2\pi(|n|+1)y}{\lambda_0\tilde{c}_{00}^2\alpha}\right\}\end{aligned}$$

if $y\geqslant\alpha>0$ and $0\neq c\in C_{00}$, where $\tilde{c}_{0j}=\min\{|c|\,|\,0\neq c\in C_{0j}\}>0$ by Lemma 1.

On the other hand, the series

$$\sum_{T\in(D)}\frac{1}{|cz+d|^l},\ l>2,\ T=\begin{pmatrix} * & * \\ c & d\end{pmatrix}$$

is uniformly convergent with respect to z in the domain $\bar{D}_\alpha=\{z\in\mathrm{H}\,|\,|x|\leqslant\alpha^{-1},\ y\geqslant\alpha>0\}$ which implies that the non-analytical Poincaré series is absolutely and uniformly convergent with respect to z in the domain $\bar{D}_\alpha$, and hence it defines a holomorphic function in H if $\mathrm{Re}(s) > 2 - r$.

A straightforward computation shows that

$$P_{nr}(Lz, s; \nu, \Gamma, \kappa_0) = \nu(L)(cz+d)^r |cz+d|^s P_{nr}(z, s; \nu, \Gamma, \kappa_0)$$

for any $L = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$. In particular, we see that

$$P_{nr}(U^{\lambda_0}z, s; \nu, \Gamma, \kappa_0) = \nu(U^{\lambda_0}) P_{nr}(z, s; \nu, \Gamma, \kappa_0) = e^{2\pi i \kappa_0} P_{nr}(z, s; \nu, \Gamma, \kappa_0),$$

which implies that $P_{nr}(z, s; \nu, \Gamma, \kappa_0)$ has a Fourier expansion with respect to variable $e^{2\pi i x/\lambda_0}$, where $z = x + iy \in$ H.

In order to discuss properties of Poincaré series at cusp points we introduce generalized non-analytic Poincaré series which are A^{-1}-transformations of $P_{nr}(z, s; \nu, \Gamma, \kappa_0)$:

$$P_{nr}(z, s; \nu, \Gamma, \kappa_j) = P_{nr}(z, s; \nu', \Gamma', \kappa'_0) |_{A_j^{-1}}.$$

Lemma 3. For $j = 0, 1, \cdots, t$, we have

$$P_{nr}(z, s; \nu, \Gamma, \kappa_j) = \sum_{L \in (S)} \frac{e((n+\kappa_j)Lz/\lambda_j)}{\nu(A_j^{-1}L)(cz+d)^r |cz+d|^s},$$

where $L = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma'_\infty \setminus A_j\Gamma = (S)$.

Similarly as above we can prove that $P_{nr}(z, s; \nu, \Gamma, \kappa_j)$ is holomorphic with respect to $z \in$ H.

Lemma 4. For $j = 0, 1, \cdots, t$, we have the following transformation formulae:

$$P_{nr}(Mz, s; \nu, \Gamma, \kappa_j) = \nu(M)(cz+d)^r |cz+d|^s P_{nr}(z, s; \nu, \Gamma, \kappa_j),$$

where $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$.

We now want to find the Fourier expansions of $P_{nr}(z, s; \nu, \Gamma, \kappa_j)$ with respect to variable $e^{2\pi i x/\lambda_j}$, which makes an analytic continuation of $P_{nr}(z, s; \nu, \Gamma, \kappa_j)$ become possible.

For $\mathrm{Re}(s) > 2 - r$,

$$P_{nr}(z, s; \nu, \Gamma, \kappa_j) = \sum_1 + \sum_2,$$

where $\sum_1$ and $\sum_2$ sum over $L = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in (S)$ with $c = 0$ and $c \neq 0$, respectively. Since the values of summands at L and $-L$ are equal we see that

$$\sum_2 = 2 \sum_{\substack{L \in (S) \\ c > 0}} \frac{e((n+\kappa_j)Lz/\lambda_j)}{\nu(A_j^{-1}L)\,(cz+d)^r |cz+d|^s}.$$

Noting that $S = \bigcup_{q=-\infty}^{\infty} \mathfrak{R} U^{\lambda_0 q}$, where $\mathfrak{R}$ is the bicoset representative $\Gamma'_\infty \setminus A_j\Gamma/\Gamma_\infty$, we can find that

$$\sum_2 = 2 \sum_{\substack{L \in (\mathfrak{R}) \\ c > 0}} c^{-s-r} \bar{\nu}(A_j^{-1}L) e((n+\kappa_j)a/c\lambda_j) \sum_{p=0}^{\infty} \frac{1}{p!} \left[- \frac{2\pi i(n+\kappa_j)}{c^2\lambda_j} \right]^p$$

$$\cdot \sum_{q=-\infty}^{\infty} e(-\kappa_0 q) \left(z + \frac{d}{c} + q\lambda_0 \right)^{-r-p-\frac{s}{2}} \left(\bar{z} + \frac{d}{c} + q\lambda_0 \right)^{-\frac{s}{2}}.$$

Using Poisson's summation formulae and by a long computation we obtain

Theorem 3. For $\mathrm{Re}(s) > 2 - r$, $j = 0, 1, \cdots, t$, $P_{nr}(z, s; \nu, \Gamma, \kappa_j)$ have Fourier expansions as follows: (1) if $\kappa_0 > 0$,

$$P_{nr}(z, s; \nu, \Gamma, \kappa_j) = 2\delta_{0j} e\left(\frac{n+\kappa_0}{\lambda_0} z\right) + \frac{2i^{-r}(2\pi)^{r+s}}{\Gamma(s/2)\lambda_0^{r+s}} \sum_{m=0}^{\infty} (m+\kappa_0)^{r+s-1} e\left(\frac{m+\kappa_0}{\lambda_0} z\right)$$

$$\cdot \sum_{p=0}^{\infty} \frac{[-4\pi^2(n+\kappa_j)(m+\kappa_0)/\lambda_j \lambda_0]^p}{p!\Gamma\left(r+p+\frac{s}{2}\right)} Z_j\left(\frac{r}{2}+\frac{s}{2}+p, m, n\right)$$

$$\cdot \sigma\left(4\pi(m+\kappa_0)y/\lambda_0, r+p+\frac{s}{2}, \frac{s}{2}\right)$$

$$+ \frac{2i^{-r}(2\pi)^{r+s}}{\Gamma(s/2)\lambda_0^{r+s}} \sum_{m=0}^{\infty} (m-\kappa_0)^{r+s-1} e\left(-\frac{m-\kappa_0}{\lambda_0} z\right) \sum_{p=0}^{\infty} \frac{[-4\pi^2(n+\kappa_j)(m-\kappa_0)/\lambda_j \lambda_0]^p}{p!\Gamma\left(r+p+\frac{s}{2}\right)}$$

$$\cdot Z_j\left(\frac{r}{2}+\frac{s}{2}+p, -m, n\right) \sigma\left(4\pi(m-\kappa_0)y/\lambda_0, \frac{s}{2}, r+p+\frac{s}{2}\right);$$

(2) if $\kappa_0 = 0$,

$$P_{nr}(z, s; \nu, \Gamma, \kappa_j) =$$

$$\frac{4\pi i^{-r}}{\lambda_0 (2y)^{r+s-1}\Gamma(s/2)} \sum_{p=0}^{\infty} \frac{1}{p!}\left[-\frac{\pi(n+\kappa_j)}{y\lambda_j}\right]^p \frac{\Gamma(r+p+s-1)}{\Gamma\left(r+p+\frac{s}{2}\right)} Z_j\left(\frac{r}{2}+\frac{s}{2}+p, 0, n\right)$$

$$+ 2\delta_{0j} e\left(\frac{n}{\lambda_0} z\right) + \frac{2i^{-r}(2\pi)^{r+s}}{\Gamma(s/2)\lambda_0^{r+s}} \sum_{m=0}^{\infty} m^{r+s-1} e\left(\frac{m}{\lambda_0} z\right) \sum_{p=0}^{\infty} \frac{[-4\pi^2(n+\kappa_j)m/\lambda_j \lambda_0]^p}{p!\Gamma\left(r+p+\frac{s}{2}\right)}$$

$$\cdot Z_j\left(\frac{r}{2}+\frac{s}{2}+p, m, n\right) \sigma\left(4\pi m y/\lambda_0, r+p+\frac{s}{2}, \frac{s}{2}\right)$$

$$+ \frac{2i^{-r}(2\pi)^{r+s}}{\Gamma(s/2)\lambda_0^{r+s}} \sum_{m=0}^{\infty} m^{r+s-1} e\left(-\frac{m}{\lambda_0} z\right) \sum_{p=0}^{\infty} \frac{[-4\pi^2(n+\kappa_j)m/\lambda_j \lambda_0]^p}{p!\Gamma\left(r+p+\frac{s}{2}\right)}$$

$$\cdot Z_j\left(\frac{r}{2}+\frac{s}{2}+p, -m, n\right) \sigma\left(4\pi m y/\lambda_0, \frac{s}{2}, r+p+\frac{s}{2}\right),$$

where $\delta_{0j} = 1$ or 0 if $j = 0$ or $j \neq 0$, and

$$S_j(c, m, n) = \sum_{d \in D_c} \bar{\nu}(A_j^{-1}L) e\left(\frac{n+\kappa_j}{\lambda_j} \frac{a}{c} + \frac{m+\kappa_0}{\lambda_0} \frac{d}{c}\right);$$

$$Z_j(w, m, n) = \sum_{c \in C'_{j0}} S_j(c, m, n) c^{-2w}, \quad \mathrm{Re}(w) > 1;$$

$$D_c = D_c(j, 0, 0, \lambda_0), \quad C'_{j0} = \{c \in C_{j0} \mid c > 0\};$$

$$\sigma(y, \alpha, \beta) = \int_0^{\infty} (u+1)^{\alpha-1} u^{\beta-1} e^{-yu} du.$$

From above we get the Fourier expansions of $P_{nr}(z, s; \nu, \Gamma, \kappa_j)$ for $\mathrm{Re}(s) > 2 - r$. Now we deal with the analytic continuation of $P_{nr}(z, s; \nu, \Gamma, \kappa_j)$. Denoting two infinite series in Theorem 1 by F_1 and F_2 respectively, we hope that F_1 and F_2 can be analytically continued to an area containing $s = 0$.

Lemma 5[3,4]**.** Let $L^2(\Gamma \setminus H, \nu, r)$ be the Hilbert space consisting of the following functions:

(1) $g: H \to \mathbb{C}$;

(2) $g(\gamma z) = \nu(\gamma)\left(\frac{cz+d}{|cz+d|}\right)^r g(z)$, $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$;

(3) $\int_{D_\Gamma} |g(z)|^2 \frac{dxdy}{y^2} < +\infty$,

where D_Γ is a fundamental domain of Γ. Then the spectrum of Laplacian operator

$$\Delta_r = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iry\frac{\partial}{\partial x}$$

is contained in the half-line $\left[\frac{|r|}{2}\left(1 - \frac{|r|}{2}\right), +\infty\right)$.

Lemma 6[5]. Selberg's zeta functions are meromorphic on $\mathrm{Re}(w) > 1/2$ which has only finite simple poles on $\mathrm{Re}(w) > 1/2$ all contained in real interval $\left(\frac{1}{2}, 1\right)$. And we have

$$|Z_j(w, m, n)| = O\left(\frac{t^{1/2}}{\sigma - \frac{1}{2}}|n + \kappa_j||m + \kappa_0|\right),$$

where $w = \sigma + it$, $\sigma > 1/2$, $|t| \geqslant 1$.

Using the method in ref. [6] and by Lemmas 5 and 6, we can analytically continue $Z_j\left(\frac{r}{2} + \frac{s}{2} + p, m, n\right)$ to a half-plane $\mathrm{Re}(s) > -\delta$ if there are not any holomorphic modular forms with weight $2 - r$ and multiplier system $\bar{\nu}$ for the group Γ, where $0 < \delta < r - 1$ and δ is independent of m.

By the definition of $\sigma(y, \alpha, \beta)$ and integration by parts we can continue analytically the factor $\sigma\left(4\pi(m - \kappa_0)y/\lambda_0, \frac{s}{2}, r + p + \frac{s}{2}\right)$ and $\sigma\left(4\pi(m + \kappa_0)y/\lambda_0, r + p + \frac{s}{2}, \frac{s}{2}\right)$ to the half-plane $\mathrm{Re}(s) > -\delta$. In this half-plane $\sigma\left(4\pi(m - \kappa_0)y/\lambda_0, \frac{s}{2}, r + p + \frac{s}{2}\right)$ is holomorphic and $\sigma\left(4\pi(m + \kappa_0)y/\lambda_0, r + p + \frac{s}{2}, \frac{s}{2}\right)$ has unique simple pole $s = 0$.

By the above continuations and some estimations we can prove that F_2 is convergent absolutely and uniformly in any compact subset of the half-plane $\mathrm{Re}(s) > -\delta$ which implies that F_2 can be continued analytically to the half-plane $\mathrm{Re}(s) > -\delta$.

We can continue analytically F_1 to the half-plane $\mathrm{Re}(s) > -\delta$ by a similar way.

These show that $P_{nr}(z, s; \nu, \Gamma, \kappa_j)$ can be continued analytically to the half-plane $\mathrm{Re}(s) > -\delta$. In particular, $P_{nr}(z, s; \nu, \Gamma, \kappa_j)$ is holomorphic at points $s = 0$.

Definition 6. The functions

$$P_{nr}(z, \nu, \Gamma, \kappa_j) \overset{\text{def}}{=} P_{nr}(z, 0; \nu, \Gamma, \kappa_j)$$

are called generalized Poincaré series of group Γ with weight r and multiplier system v.

A direct computation shows that

$$P_{nr}(z, \nu, \Gamma, \kappa_j) = 2\delta_{0j}\, e\left(\frac{n + \kappa_0}{\lambda_0}z\right) + \frac{2i^{-r}(2\pi)^r}{\lambda_0^r}\sum_{m=0}^{\infty}(m + \kappa_0)^{r-1} e\left(\frac{m + \kappa_0}{\lambda_0}z\right) \cdot \sum_{p=0}^{\infty}\frac{[-4\pi^2(n + \kappa_j)(m + \kappa_0)/\lambda_j\lambda_0]^p}{p!\,\Gamma(r + p)} Z_j\left(\frac{r}{2} + p, m, n\right).$$

2 Properties of Poincaré series and the proof of Theorem 2

In this section we want to show that $P_{nr}(z,\nu,\Gamma,\kappa_j)$ is holomorphic in H and deal with some properties of it. Let $\hat{P}_n(m)$ be the m-th Fourier coefficient of $P_{nr}(z,\nu,\Gamma,\kappa_j)$, i.e.

$$\hat{P}_n(m)=\frac{2\mathrm{i}^{-r}(2\pi)^r}{\lambda_0^r}(m+\kappa_0)^{r-1}\sum_{p=0}^{\infty}\frac{[-4\pi^2(n+\kappa_j)(m+\kappa_0)/\lambda_j\lambda_0]^p}{p!\Gamma(r+p)}Z_j\left(\frac{r}{2}+p,m,n\right)$$

$$=\frac{2\mathrm{i}^{-r}(2\pi)^r}{\lambda_0^r\Gamma(r)}(m+\kappa_0)^{r-1}Z_m(r/2)+\frac{2\mathrm{i}^{-r}(2\pi)^r}{\lambda_0^r}(m+\kappa_0)^{r-1}$$

$$\cdot\sum_{p=1}^{\infty}\frac{[-4\pi^2(n+\kappa_j)(m+\kappa_0)/\lambda_j\lambda_0]^p}{p!\Gamma(r+p)}\sum_{c>0}\frac{S_j(c,m,n)}{c^{r+2p}}.$$

Since by Lemma 2 $S_j(c,m,n)=O(c)$, where the constant in O is only dependent on j and Γ, we see that

$$\sum_{c>0}\left|\frac{S_j(c,m,n)}{c^{r+2p}}\right|\leqslant C_1\sum_{c>0}\frac{1}{c^{r+2p-1}}\leqslant C_2<+\infty,$$

where $C_2=C_1\sum_{c>0}\frac{1}{c^{1+r}}$ is a constant (only dependent on j, Γ). Observing that $Z_m(r/2)=O(m)$, the first term in $\hat{P}_n(m)$ is $O(m^r)$. And for the second term we see that it

$$\ll m^{r-1}\sum_{p=1}^{\infty}\frac{[4\pi^2|n+\kappa_j|(m+\kappa_0)/\lambda_0\lambda_j]^p}{(p!)^2}$$

$$\leqslant m^{r-1}\left[\sum_{p=1}^{\infty}\frac{\left(\sqrt{4\pi^2|n+\kappa_j|(m+\kappa_0)/\lambda_0\lambda_j}\right)^p}{p!}\right]^2$$

$$\leqslant m^{r-1}\mathrm{e}^{4\pi\sqrt{|n+\kappa_j|(m+\kappa_0)/\lambda_0\lambda_j}}\leqslant m^{r-1}\mathrm{e}^{\gamma\sqrt{m}},$$

where $\gamma=4\pi\sqrt{|n+\kappa_j|/\lambda_j\lambda_0}$. So

$$\hat{P}_n(m)=O(m^{r-1}\mathrm{e}^{\gamma\sqrt{m}}).$$

Lemma 7. Let $\{a_m\}_{m=0}^{\infty}$ be an infinite series satisfying the conditions $a_m=O(m^{r-1}\cdot\mathrm{e}^{\gamma\sqrt{m}})$, where r,γ are any positive reals. Then the function defined by

$$f(z)=\sum_{m=0}^{\infty}a_m e\left(\frac{m+\kappa_0}{\lambda_0}z\right)$$

is holomorphic in H, where κ_0, λ_0 are positive reals.

Proof. Let $z=x+\mathrm{i}y$. Then

$$\left|a_m e\left(\frac{m+\kappa_0}{\lambda_0}z\right)\right|=|a_m|\,\mathrm{e}^{-2\pi(m+\kappa_0)y/\lambda_0}\ll m^{r-1}\mathrm{e}^{\gamma\sqrt{m}}\mathrm{e}^{-2\pi(m+\kappa_0)y/\lambda_0}.$$

For any $\varepsilon>0$, if $y\geqslant\varepsilon$, then

$$\left|\sum_{m=0}^{\infty}a_m e\left(\frac{m+\kappa_0}{\lambda_0}z\right)\right|\ll\sum_{m=0}^{\infty}m^{r-1}\mathrm{e}^{\gamma\sqrt{m}}\mathrm{e}^{-2\pi(m+\kappa_0)\varepsilon/\lambda_0}<+\infty,$$

which shows that $\sum a_m e\left(\frac{m+\kappa_0}{\lambda_0}z\right)$ is absolutely and uniformly convergent in the domain $\overline{D}_\varepsilon=\{z\in\mathrm{H}\,|\,\mathrm{Im}(z)\geqslant\varepsilon\}\subset\mathrm{H}$ and therefore $f(z)$ is holomorphic in H.

Lemma 7 implies that $P_{nr}(z,\nu,\Gamma,\kappa_j)$ are holomorphic on H for $j=1,\cdots,t$. And $P_{nr}(z,\nu,\Gamma):=P_{nr}(z,\nu,\Gamma,\kappa_0)$ is meromorphic on H with pole at ∞.

$P_{nr}(z, s; \nu, \Gamma, \kappa_j)$ satisfies the following transformation formulae:

$$P_{nr}(Lz, s, \nu, \Gamma, \kappa_j) = \nu(L)(cz + d)^r \mid cz + d \mid^s P_{nr}(z, s; \nu, \Gamma, \kappa_j),$$

where $\mathrm{Re}(s) > 2 - r$, $L \in \Gamma$. But $P_{nr}(z, s, \nu, \Gamma, \kappa_j)$ have an analytic continuation to the half-plane $\mathrm{Re}(s) > -\delta$. Hence two ends of the above equation are holomorphic for variable s on the half-plane $\mathrm{Re}(s) > -\delta$. By identical principle of analytic functions the above equality holds for all s on $\mathrm{Re}(s) > -\delta$. Especially these imply that

$$P_{nr}(Lz, \nu, \Gamma, \kappa_j) = \nu(L)(cz + d)^r P_{nr}(z, \nu, \Gamma, \kappa_j)$$

for all $L = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$, $z \in \mathrm{H}$.

We now compute the values of $P_{nr}(z, \nu, \Gamma)$ at cusp points. Using the similar method as above we have

$$P_{nr}(z, \nu', \Gamma', \kappa'_j) = (-z)^{-r} P_{nr}(A_j^{-1} z, \nu, \Gamma).$$

Hence

$$P_{nr}(p_j, \nu, \Gamma) = \lim_{z \to i\infty} P_{nr}(A_j^{-1} z, \nu, \Gamma) = \lim_{z \to i\infty} (-z)^r P_{nr}(z, \nu', \Gamma', \kappa'_j)$$

$$= \lim_{z \to i\infty} (-z)^r \sum_{m + \kappa_0 > 0} \hat{p}'_n(m) e\left(\frac{m + \kappa_0}{\lambda_0} z\right) = 0$$

for $j = 1, \cdots, t$.

We therefore obtain the following

Theorem 4. Suppose $r > 1$ and there are not non-trivial cusp forms with weight $2 - r$ and multiplier system $\bar{\nu}$ of group Γ. Then $P_{nr}(z, \nu, \Gamma)$ $(n < 0)$ are meromorphically automorphic forms of group Γ with weight r and multiplier system ν with only pole at ∞ and the values of it at cusp points $p_k (\neq \infty)$ are zero.

Corollary 1. Suppose $r > 1$ and there are not non-trivial cusp forms with weight $2 - r$ and multiplier system $\bar{\nu}$ of group Γ. Then there exists an automorphic form with weight r and multiplier system ν of group Γ which has arbitrarily prescribed principal part at any given cusp point and holomorphic in H.

Proof. As in Theorem 4, $P_{nr}(z, \nu, \Gamma)$ has only pole term $2e\left(\frac{n + \kappa_0}{\lambda_0} z\right)$ at $z = \infty$ if $n < 0$. Similarly we can prove that there exists an automorphic form which has only pole term $2e\left(\frac{n + \kappa_j}{\lambda_j} A_j z\right)$ at $z = p_j$ if $n < 0$. On the other hand, we can obtain an automorphic form which is regular at all cusp points from each automorphic form by minus a linear combination of such Poincaré series. These complete the proof.

Proof of Theorem 2. The proof is similar to that of Theorem 1 in ref. [1]. For the sake of convenience of readers, we give a sketch as follows. For the detail, please see ref. [1].

Now suppose $-2 < r < -1$ and $\{\varphi_V\}$ is a parabolic cocycle in P. We shall prove that there exists $\Phi^* \in P$ such that

$$\Phi^* \Big|_{\nu}^{r} M - \Phi^* = \varphi_M, \ M \in \Gamma,$$

which implies Theorem 2. Without loss of generality, assume that $\varphi_S = 0$ for $S = \begin{pmatrix} 1 & \lambda_0 \\ 0 & 1 \end{pmatrix}$ since $\varphi_0 \in P$.

Let Φ be the function just as in Theorem 3 of ref. [1]. Since $-2 < r < -1$ and $C^0(\Gamma,$

$-r-2,\nu)=0$, we know that there exists $G\in\{\Gamma, r, \nu\}$ such that G is holomorphic in H and has the same principal part at each p_j as the principal part at p_j of the expansion of Φ by Corollary 1. Let $\Phi^* = \Phi - G$. Then

(I) $\Phi^* \big|_\nu^r M - \Phi^* = \varphi_M$, $M\in\Gamma$;

(II) Φ^* is holomorphic in H;

(III) Φ^* has an expansion at each $p_j\,(0\leqslant j\leqslant t)$ in which no negative power of the local parameter appears.

Condition (III) implies that there exist positive constants K, ρ, σ such that

$$|\Phi^*(z)| < K(|y|^\rho + y^{-\sigma}) \text{ for all } z\in\overline{\mathfrak{R}}\cap \mathrm{H}, \tag{1}$$

where $\mathfrak{R} = \left\{z\in \mathrm{H}\,\middle|\, |\mathrm{Re}(z)| < \lambda_0/2 \text{ and } |cz+d|>1 \text{ for all } V=\begin{pmatrix} * & * \\ c & d\end{pmatrix}\in\Gamma-\Gamma_\infty\right\}$ is the Ford fundamental domain of Γ. Put $f(z) = y^{-r/2}|\Phi^*(z)|$, $y=\mathrm{Im}z>0$. Then by (I)

$$f(Vz) = y^{-r/2}|cz+d|^r|\Phi^*(Vz)|\leqslant f(z) + y^{-r/2}|\varphi_V(z)|. \tag{2}$$

Let $\mathfrak{F} = \bigcup_{V\in\Gamma/\Gamma_\infty} V(\overline{\mathfrak{R}})\cap \mathrm{H}$. Then

$$|\Phi^*(z)| < K(|y|^\alpha + y^{-\beta}) \quad \text{for all } z\in\mathfrak{F}, \tag{3}$$

where α,β,K are positive constants independent of z. In fact, if $z\in\mathfrak{F}$, then $z=V\tau$ with $V\in\Gamma/\Gamma_\infty$ and $\tau\in\overline{\mathfrak{R}}\cap\mathrm{H}$,

$$|\Phi^*(z)| = y^{r/2}f(z) = y^{r/2}f(V\tau)\leqslant y^{r/2}\{f(\tau)+t^{-r/2}|\varphi_V(z)|\} = y^{r/2}t^{-r/2}\{|\Phi^*(\tau)|+|\varphi_V(z)|\},$$

where $t=\mathrm{Im}(\tau)$. By (1) and Lemma 8 in ref. [2] we deduce inequality (3).

By the definition of fundamental domain it follows that for any $z\in\mathrm{H}$, there exists an integer m such that $S^m z\in\mathfrak{F}$. Then

$$|\Phi^*(S^m z)| = |(\Phi^* \mid S^m)(z)| = |\Phi^*(z)|$$

since $\varphi_S = 0$. Therefore (3) implies that

$$|\Phi^*(z)| = |\Phi^*(S^m z)| < K(|y|^\alpha + y^{-\beta})$$

for $z\in\mathrm{H}$ since $\mathrm{Im}(S^m z) = \mathrm{Im}(z) = y$ which shows that $\Phi^*\in P$ and hence Theorem 2.

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