Existence of the atmosphere attractor *

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Abstract The global asymptotic behavior of solutions for the equations of large-scale atmospheric motion with the non-stationary external forcing is studied in the infinite-dimensional Hilbert space. Based on the properties of operators of the equations, some energy inequalities and the uniqueness theorem of solutions are obtained. On the assumption that external forces are bounded, the exsitence of the global absorbing set and the atmosphere attractor is proved, and the characteristics of the decay of effect of initial field and the adjustment to the external forcing are revealed. The physical sense of the results is discussed and some ideas about climatic numerical forecast are elucidated.

Keywords: operator equations, atmosphere attractor, global absorbing set, non-stationary external forcing, decay, adjustment.

The atmosphere is a forced dissipative nonlinear system. The essential characteristics of the atmospheric motion are formed by the basic actions such as frictional dissipation, thermal forcing, nonlinear advection, rotational force field and gravitational field. The motion obeys some physical laws and can be written as partial differential equations in the mathematical language. However, the nonlinear partial differential equations are too complicated to be solved analytically. Although we can use computers to carry out numerical experiment on it, the properties of final state of all possible initial values can by no means be made clear as time tends to infinity. But if the properties of its solutions are known directly by the equations themselves without solving the differential equations, we may understand a lot of the macroscopic properties of the atmosphere.

Under the stationary external forcing, Chou Jifan[1—4] studied the global asymptotic behavior of solutions for the system of nonlinear atmosphere, and proved that the system is bound to evolve into a state of an absorbing set in R^n, whatever the initial values might be. In the physical sense, that is the adjustment of the system to the external forcing. Chou's results were extended to the infinite-dimensional Hilbert space[5]. For the real atmospheric system, the external forces are non-stationary. A study of large-scale motion of weather and climate under the non-stationary external forcing is also of basic significance. Therefore, we extended the above results to R^n under the non-stationary external forcing[6]. Do the results hold true in infinite-dimensional Hilbert space in that case? This paper gives a discussion on the problem.

1 Basic equations

The present study addresses the equations of large-scale atmospheric motion in the spherical coordinate system (λ, θ, p, t). In order to save space, for their expressions see references [1, 3—5, 7] (the corresponding notations are all the same).

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The domain of solutions of the equations $\Omega = S^2 \times (p_0, P_s)$, with $0 < p_0 < P_s < \infty$. Here $p_0 > 0$ is a certain small number, and P_s the surface pressure. The boundary value conditions are given below.

On the surface of the earth $p = P_s$

$$V_\lambda = V_\theta = \omega = 0, \tag{1}$$

$$\frac{\partial T}{\partial p} = a_s(T_s - T), \tag{2}$$

where $T_s = T_s(\lambda, \theta, t)$ is the temperature on the surface of the earth (sea surface or land surface), and a_s is a positive constant related to turbulent thermal conductivity.

On the upper surface of the atmosphere $p = p_0$

$$\frac{\partial V_\lambda}{\partial p} = \frac{\partial V_\theta}{\partial p} = 0. \qquad \omega = 0, \qquad \frac{\partial T}{\partial p} = 0. \tag{3}$$

The initial value conditions are

$$(V_\lambda, V_\theta, T)\Big|_{t=0} = \left(V^{(0)}_\lambda, V^{(0)}_\theta, T^{(0)}\right). \tag{4}$$

2 Fundamental function spaces, operator equation, properties of operator and assumption

By introducing the vector function $\varphi = (V_\lambda, V_\theta, \omega, \Phi, T)'$ (here the sign $'$ denotes transposition) and the operators B, N and L, the partial differential equations of large-scale atmospheric motion can be written as

$$B\frac{\partial \varphi}{\partial t} + (N(\varphi) + L)\varphi = \xi(t), \tag{5}$$

where

$$B\varphi\Big|_{t=0} = B\varphi_0, \tag{6}$$

$$B = \mathrm{diag}(1, 1, 0, 0, R^2/C^2), \tag{7}$$

$$N(\varphi) = \begin{bmatrix} \Lambda & 2\Omega\cos\theta + \frac{\mathrm{ctg}\theta}{a}V_\lambda & 0 & \frac{1}{a\sin\theta}\frac{\partial}{\partial} & 0 \\ \left(-2\Omega\cos\theta + \frac{\mathrm{ctg}\theta}{a}V_\lambda\right) & \Lambda & 0 & \frac{1}{a}\frac{\partial}{\partial\theta} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial p} & \frac{R}{p} \\ \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda} & \frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}\sin\theta & \frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & -\frac{R}{p} & 0 & \frac{R^2}{C^2}\Lambda \end{bmatrix}, \tag{8}$$

$$L = \mathrm{diag}(L_1, L_1, 0, 0, L_2 + l_2\alpha_s T_s^2/T^2), \tag{9}$$

$$\xi(t) = (0,0,0,0, R^2\varepsilon(t)/C^2C_p + l_2\alpha_s T_s^2(t)/T. \tag{10}$$

where $L_i = -\partial_p l_i \partial_p - \mu_i \nabla^2$, $l_i = \nu_i(gp/R\overline{T})$, $i = 1, 2$. Operator $N(\varphi)$ embodies the actions of the nonlinear advection, the Coriolis force, the spherical action and the gravity, etc.; L shows the dissipation terms.

On the set formed by the whole vector function $\varphi = (V_\lambda, V_\theta, \omega, \Phi, T)'$, we define inner product and norm as follows:

$$(\varphi_1, \varphi_2) = \int_\Omega \varphi'_1 \varphi_2 \mathrm{d}\Omega = \int_{P_0}^{Ps}\int_0^\pi\int_0^{2\pi} \varphi'_1\varphi_2 a^2 \sin\theta \mathrm{d}\lambda \mathrm{d}\theta \mathrm{d}p, \tag{11}$$

$$\| \varphi \|_0 = (\varphi, \varphi)^{1/2}. \tag{12}$$

So we get a Hilbert space H_0.

Let B^*, L^* and $N^*(\varphi)$ be the adjoint operators of B, L and $N(\varphi)$, respectively. Then we have

Property 1. $B = B^*$, $L = L^*$, $N(\varphi) = -N^*(\varphi)$. (13)

We call B and L the self-adjoint operators, and $N(\varphi)$ the anti-adjoint operator.

Property 2. B and L are the positively definite operators,

$$(\varphi, B\varphi) \geqslant 0, \tag{14}$$

$$(\varphi, L\varphi) \geqslant 0, \tag{15}$$

$$(\varphi, N(\varphi_1)\varphi) = 0, \tag{16}$$

$\forall\, \varphi, \varphi_1 \in H_0(\Omega)$, the equalities in (14) and (15) are true if and only if $\| \varphi \|_0 = 0$.

(14) shows that $(\varphi, B\varphi)$ represents energy. (15) shows that the self-conjugate and positively definite properties of operator L embody the characterization that the dissipative action always dissipates energy. (16) shows that the anti-adjoint property of $N(\varphi)$ embodies the important essence that the actions of the nonlinear advection, the Coriolis force, the spherical action and the gravity do not change the total energy of the system.

Under the adiabatic condition without friction, according to (14)—(16), we know (5) has the conservation of total energy, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi, B\varphi) = 0, \tag{17}$$

namly

$$\| B_1\varphi \|_0^2 = \int_\Omega (V_\lambda^2 + V_\theta^2 + \frac{R^2}{C^2}T^2)\mathrm{d}\Omega = \text{const}, \tag{18}$$

where $B_1 = \mathrm{diag}(1, 1, 0, 0, R/C)$.

Let $H_1(\Omega)$ be the complete space with the norm as follows:

$$\| \varphi \|_1 = (\| V_\lambda \|^2 + \| V_\theta \|^2 + \| \omega \|^2 + \| \Phi \|^2 + \| T \|^2)^{1/2}, \tag{19}$$

$\forall\, \varphi = (V_\lambda, V_\theta, \omega, \Phi, T)'$, where $\| V_\lambda \|$, $\| V_\theta \|$ and $\| T \|$ take $H^1(\Omega)$-norm, $\| \omega \|$ and $\| \Phi \|$ take $Q(\Omega)$-norm. Here $H^1(\Omega)$ is the standard Sobolev space. $Q(\Omega)$ is the complete space with the norm as follows:

$$\| q \| = \left(\int_\Omega (q^2 + (\partial q/\partial p)^2) \mathrm{d}\Omega \right)^{1/2}, \quad (q = \omega \text{ or } \Phi). \tag{20}$$

In $Q(\Omega)$, we can use the following equivalent norm:

$$\| q \| = \left(\int_\Omega (\partial q/\partial p)^2 \mathrm{d}\Omega \right)^{1/2}, \quad (q = \omega \text{ or } \Phi), \tag{21}$$

Lemma 1. *There exist constants* $K_1, K_2 > 0$ *such that*

$$K_1(\| V_\lambda \|^2 + \| V_\theta \|^2 + \| T \|^2) \leqslant \| \varphi \|_1 \leqslant K_2(\| V_\lambda \|^2 + \| V_\theta \|^2 + \| T \|^2), \tag{22}$$

$\forall \varphi = (V_\lambda, V_\theta, \omega, \Phi, T)' \in H_1(\Omega)$.

So we can use the following equivalent norm in $H_1(\Omega)$:

$$\| \varphi \|_1 = (\| V_\lambda \|^2 + \| V_\theta \|^2 + \| T \|^2)^{1/2}. \tag{23}$$

In the following discussion, operator $N(\varphi)$ should be decomposed into $N^{(1)}(\varphi)$ and $N^{(2)}$:

$$N(\varphi) = N^{(1)}(\varphi) + N^{(2)} \tag{24}$$

$$N^{(1)}(\varphi) = \begin{bmatrix} \Lambda & \frac{\mathrm{ctg}\theta}{a} V_\lambda & 0 & 0 & 0 \\ -\frac{\mathrm{ctg}\theta}{a} V_\lambda & \Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{R^2}{C^2}\Lambda \end{bmatrix},$$

$$N^{(2)} = \begin{bmatrix} 0 & 2\Omega\cos\theta & 0 & \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda} & 0 \\ -2\Omega\cos\theta & 0 & 0 & \frac{1}{a}\frac{\partial}{\partial\theta} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial p} & \frac{R}{p} \\ \frac{1}{a\sin\theta}\frac{\partial}{\partial\lambda} & \frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}\sin\theta & \frac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & -\frac{R}{p} & 0 & 0 \end{bmatrix}.$$

Both $N^{(1)}(\varphi)$ and $N^{(2)}$ are anti-adjoint operators.

Lemma 2. $N(\varphi_1 + \varphi_2) = N^{(1)}(\varphi_1 + \varphi_2) + N^{(2)}$ (25)

$$N^{(1)}(\alpha\varphi_1 + \beta\varphi_2) = \alpha N^{(1)}(\varphi_1) + \beta N^{(1)}(\varphi_2), \ \forall \varphi_1, \varphi_2 \in H_0(\Omega). \tag{26}$$

Lemma 3. *There exist constants* $C1, C2 > 0$ *such that*

$$C_1 \| \varphi \|_1^2 \leqslant (\varphi, L\varphi), \ \forall \varphi \in H_1(\Omega). \tag{27}$$

Lemma 4. *There exists a constant C such that*

$$| (N^{(1)}(\varphi)\varphi_1, \varphi) | = | (N^{(1)}(\varphi)\varphi, \varphi_1) | \leqslant C \begin{cases} \| \varphi \|_1 \ \| B_0\varphi \|_0 \ \| \varphi_1 \|_3, \\ \| \varphi \|_1 \ \| B_0\varphi \|_0 \ \| B_1\varphi_1 \|_3. \end{cases} \tag{28}$$

In this paper, the exteranl forcing is non-stationary. So $\xi = \xi(t)$ as a function of time. In reality, the external forces are always bounded. Therefore, we assume that the external forces are bounded in subsequent sections, namely

$$0 < \| \zeta(t) \|^2 \leqslant M < \infty, \tag{29}$$

where $\| \xi(t) \|^2 = \| R^2\varepsilon(t)/C^2C_p \|^2 + l_2\alpha_s \| T_s(t) \|^2$. $\varepsilon(t)$, $T_s(t)$ may be the quasi-periodic or the asymptotically almost periodic or the functions that can be expanded by Fourier series.

3 Energy inequalities and uniqueness of solutions

Theorem 1. *Any solution φ of the operator equations* (5) *and* (6) *satisfies*

$$\begin{aligned} &\| B_1\varphi \|_0^2 + 2C_1\int_0^t \| \varphi(t) \|_1^2 \mathrm{d}t \\ &\leqslant \| B_1\varphi_0 \|_0^2 + 2\int_0^t (\xi(t), \varphi(t))\mathrm{d}t, \qquad t \in [0, T], a.e. \end{aligned} \tag{30}$$

where C_1 is given by (27).

Furthermore, if $\xi(t) \in H_1^*(\Omega)$, where $H_1^*(\Omega)$ is the dual space of $H_1(\Omega)$, then

$$(\xi, \varphi) \leqslant \frac{1}{2C_1} \| \zeta \|_{H_1^*}^2 + \frac{C_1}{2} \| \varphi \|_1^2. \tag{31}$$

Therefore,

$$\begin{aligned} &\| B_1\varphi \|_0^2 + C_1\int_0^t \| \varphi(t) \|_1^2 \mathrm{d}t \\ &\leqslant \| B_1\varphi_0 \|_0^2 + \frac{1}{C_2}\int_0^t \| \zeta(t) \|_{H_1^*}^2 \mathrm{d}t, \qquad t \in [0, T], \text{a.e.} \end{aligned} \tag{32}$$

On the other hand, using $\| B_1\varphi \|_0^2 \leqslant C_1^* \| \varphi \|_1^2$ and

$$| (\xi(t), \varphi(t)) | \leqslant \frac{1}{C_2} \| \zeta(t) \|^2 + \frac{C_1}{2} \| \varphi(t) \|_0^2,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \| B_1\varphi \|_0^2 + \tilde{C}_1 \| B_1\varphi(t) \|_0^2 \leqslant \frac{1}{\tilde{C}_2} \| \zeta(t) \|^2. \tag{33}$$

Applying the classical Gronwall inequality, we get

Theorem 2. *Any solution φ of* (5) *and* (6) *satisfies*

$$\| B_1\varphi \|_0^2 \leqslant \left\{ \| B_1\varphi_0 \|_0^2 + \frac{1}{C_2}\int_0^t e^{\tilde{C}_1 t} \| \zeta(t) \|^2 dt \right\} e^{-\tilde{C}_1 t}, \tag{34}$$

$t \in [0, T]$, $a.e.$, $C_2 \tilde{C}_1 > 0$.

(34) has obvious physical sense. On its right-hand side, the first term shows the effect of initial value, and the second term shows the effect of the external forcing. As the time $t \to \infty$, we obtain

$$\| B_1\varphi_0 \|_0^2 e^{-\tilde{C}_1 t} \longrightarrow 0, \tag{35}$$

$$\lim_{t\to\infty} \| B_1\varphi \|_0^2 \leqslant \lim_{t\to\infty} \left\{ \| B_1\varphi_0 \|_0^2 + \frac{1}{C_2}\int_0^t e^{\tilde{C}_1 t} \| \zeta(t) \|^2 dt \right\} e^{-\tilde{C}_1 t}$$
$$= \lim_{t\to\infty}\frac{1}{C_2}\left\{ \int_0^t e^{\tilde{C}_1 t} \| \zeta(t) \|^2 dt \right\} e^{-\tilde{C}_1 t}. \tag{36}$$

They show in the general sense that the system described by (5) has the characteristic of the decay of effect of initial field[7], and that the long-range evolution of the system will be dependent on the change of external forcing.

By use of assumption (29), we have

$$\| B_1\varphi \|_0^2 \leqslant \left\{ \| B_1\varphi_0 \|_0^2 + \frac{M}{C_2}\int_0^t e^{\tilde{C}_1 t} dt \right\} e^{-\tilde{C}_1 t}$$
$$= \| B_1\varphi_0 \|_0^2 e^{-\tilde{C}_1 t} + \frac{M}{C_2\tilde{C}_1}(1 - e^{-\tilde{C}_1 t}), \qquad t \in [0, T], \text{a.e.} \tag{37}$$

Theorem 3. *There is a unique smoothing solution of the initial-boundary value problem of* (5), (6) *and* (1)—(3).

Proof. Let $\varphi_1 = (V_{1\lambda}, V_{1\theta}, \omega_1, \Phi_1, T_1)'$, $\varphi_2 = (V_{2\lambda}, V_{2\theta}, \omega_2, \Phi_2, T_2)'$ be solutions of the initial-boundary value problem of (5) and (6). Besides, let

$$\varphi_1 - \varphi_2 = \varphi = (V_\lambda, V_\theta, \omega, \Phi, T)'.$$

So by the results mentioned before, we get

$$\frac{\partial}{\partial t}B\varphi + N^{(1)}(\varphi_1)\varphi_1 - N^{(1)}(\varphi_2)\varphi_2 + N^{(2)}\varphi + L^*\varphi = 0, \tag{38}$$

$$\varphi(\lambda, \theta, p; 0) = 0, \tag{39}$$

$$(V_\lambda, V_\theta, \omega) = 0, \partial T/\partial p = -\alpha_s T, \qquad \text{on } p = P_s, \tag{40}$$

$$(\partial V_\lambda/\partial p, \partial V_\theta/\partial p, \omega, \partial T/\partial p) = 0, \qquad \text{on } p = p_0 \tag{41}$$

where $L^* = \text{diag}(L_1, L_1, 0, 0, L_2)$.

By Lemma 2, we have

$$\frac{\partial}{\partial t}B\varphi + N^{(1)}(\varphi + \varphi_2)\varphi + N^{(1)}(\varphi)\varphi_2 + N^{(2)}\varphi + L^*\varphi = 0. \tag{42}$$

Therefore,