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Local Schrödinger flow into Kähler manifolds

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Abstract In this paper we show that there exists a unique local smooth solution for the Cauchy problem of the Schrödinger flow for maps from a compact Riemannian manifold into a complete Kähler manifold, or from a Euclidean space R^m into a compact Kähler manifold. As a consequence, we prove that Heisenberg spin system is locally well-posed in the appropriate Sobolev spaces.

Keywords: **Schrödinger flow, Heisenberg spin system, local existence, Kähler manifold.**

1 Main results

In this paper, we discuss the short time existence of solutions to the Schrödinger flow[1] for maps from a Riemannian manifold (M, g) into a complete Kähler manifold (N, J, h) with complex structure J and Kähler metric h. The Schrödinger flow is defined by the initial value problem

$$\begin{cases}\partial_t u(x,t) = J(u(x,t))\tau(u(x,t)), \\ u(\cdot,0) = u_0 : M \to N;\end{cases} \tag{1.1}$$

where $\tau(u)$ denotes the tension field of u which, in local coordinates, can be written as

$$\tau^\alpha(u) = \Delta_g u^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j},$$

where Δ_g is the Laplace-Beltrami operator on M with respect to the metric g and $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol on the target manifold (N, h). It is well known that u is a harmonic map if and only if $\tau(u) \equiv 0$.

An important example of the Schrödinger flow is the Heisenberg spin chain system (also called ferromagnetic spin chain system[2]) which is given by

$$\frac{\partial u}{\partial t} = u \times \Delta u, \tag{1.2}$$

where u takes values in $S^2 \subset \mathbb{R}^3$ and $\times$ denotes the cross product in $\mathbb{R}^3$[3,4]. This can be written in the form of (1.1) because $u\times : T_uS^2 \to T_uS^2$ is just the standard complex structure on S^2 and $\tau(u)$ is the tangential part of Δu in $\mathbb{R}^3$[1].

In 1991, Zhou, Guo and Tan[5] proved that for any smooth initial data u_0: $S^1 \to S^2$, there exists a unique smooth global solution to the Cauchy problem of the ferromagnetic spin chain system (1.2), i.e. the Schrödinger flow of maps from S^1 into S^2. Ding and Wang[1] generalized the result of Zhou-Guo-Tan to the case of Schrödinger flow from S^1 into a general compact Kähler manifold (N, J, h). They proved that the Cauchy problem admits a unique local smooth solution provided u_0 is smooth. When the target (N, J, h) is a Kähler manifold with constant holomorphic curvature, it is proved in refs. [1,6] that the problem has a unique global smooth solution (see also refs. [7,8] for a generalization to the so-called inhomogeneous Schrödinger flow). Terng and Uhlenbeck[9] considered the Schrödinger flow from $\mathbb{R}^1$ into a Grassmannian manifold. They were able to establish the gauge equivalence of the Schrödinger flow with a nonlinear matrix Schrödinger equation. In the case where $M = \mathbb{R}^1$ and N is a compact Riemann surface, Chang, Shatah and Uhlenbeck[10] proved the global existence and uniqueness in the space $W^{2,2}(\mathbb{R}^1, N)$, using a generalized Hasimoto transformation. For $M = \mathbb{R}^2$ and N a compact Riemann surface with S^1 symmetry, they obtained similar result under the additional conditions that the initial mapping u_0 is radially symmetric or S^1 equivariant with small energy.

In ref. [6], Pang et al. proved the local existence of smooth solutions to the Cauchy problem (1.1) when M is a closed Riemann surface and N is a compact Kähler manifold with non-positive Riemannian curvature. It is our aim in this paper to generalize their result to the cases where M is an arbitrary closed Riemannian manifold or $M = \mathbb{R}^m$, while N is a complete Kähler manifold. As in refs. [1,6], we use parabolic approximation and energy method to obtain *a priori* $W^{k,2}$ estimates for the approximating solutions. It turns out that the geometric structures of the Schrödinger flow are crucial for obtaining such estimates. Instead of directly estimating the $W^{k,2}$-norm of the solutions, one needs to first estimate the L^2-integrals of the covariant differentials $\nabla^k u$ of the solution u, where ∇u is considered as a section on the pull-back tangent bundle $u^* TN$. When N is Kähler, the time derivative of such an L^2-integral can be controlled by a polynomial of similar L^2-integrals of covariant differentials of order $\leqslant k$. In other words, the higher order covariant differentials disappear as a result of good geometric structures in our equations.

Our main results are as follows.

Theorem 1.1. Let (M, g) be a closed Riemannian manifold and (N, J, h) be a complete Kähler manifold. Let $m_0 = \left[\frac{m}{2}\right] + 1$, where $[q]$ denotes the integral part of a positive number q. Then, if N is compact (or noncompact), the Cauchy problem (1.1) with initial map $u_0 \in W^{k,2}(M, N)$, for any integer $k \geqslant m_0 + 1$ (or $k \geqslant m_0 + 2$), admits a local solution $u \in L^\infty([0, T], W^{k,2}(M, N))$, where $T = T(\| u_0 \|_{W^{m_0+1,2}})$ (or $T = T(N, \| u_0 \|_{W^{m_0+2,2}})$); when $k \geqslant m_0 + 3$, the local solution is unique. Moreover, if $u_0 \in C^\infty$, the local solution $u \in C^\infty([0, T] \times M, N)$.

Theorem 1.2. Let $\mathbb{R}^m$ be a Euclidean space and (N, J, h) be a compact Kähler manifold. Let $m_0 = \left[\frac{m}{2}\right] + 1$. Then the Cauchy problem (1.1) with $u_0 \in W^{k,2}(\mathbb{R}^m, N)$, for any integer $k \geqslant m_0 + 1$, admits a local solution $u \in L^\infty([0, T], W^{k,2}(\mathbb{R}^m, N))$, where $T = T(\| \nabla u_0 \|_{W^{m_0,2}})$. If $k \geqslant m_0 + 3$, then the local solution is unique. Moreover, if $u_0 \in \mathscr{H} \equiv \bigcap_{k=1}^{\infty} W^{k,2}(\mathbb{R}^m, N)$, then $u \in C^\infty([0, T], \mathscr{H})$.

In refs. [11,12], it has been shown that the Cauchy problem to the Heisenberg spin chain system defined on a closed Riemannian manifold admits a global, weak solution. Also, it is easy to check that an $L^\infty([0, T], C^2(M, S^2))$ solution to the Cauchy problem of the Heisenberg spin chain system is unique. As a direct consequence of Theorems 1.1 and 1.2 we have

Corollary 1.1. Let (M, g) be a closed Riemannian manifold or a Euclidean space and

S^2 be a sphere with standard metric. Let $m_0 = \left[\frac{m}{2}\right] + 1$ where $m = \dim(M)$, and let $\mathscr{H} \equiv \bigcap_{k=1}^{\infty} W^{k,2}(M,N)$. Then, the Cauchy problem to the Heisenberg spin chain system (1.2) with initial data $u_0 \in W^{k,2}(M,S^2)$, for any integer $k \geqslant m_0 + 1$, admits a local solution $u \in L^{\infty}([0,T], W^{k,2}(M,S^2))$, where $T = T(\|\nabla u_0\|_{W^{m_0,2}})$. The local solution is unique when $k \geqslant m_0 + 2$. In particular, if $u_0 \in \mathscr{H}$, then $u \in C^{\infty}([0,T],\mathscr{H})$.

Sulem et al.[13] also discussed the existence of solutions to Heisenberg spin chain system defined on a Euclidean space $\mathbb{R}^m$ by employing the difference method.

The rest of the paper is organized as follows. In sec. 2 we prove the interpolation inequality of bundle valued Sobolev spaces defined on a compact Riemannian manifold or a Euclidean space $\mathbb{R}^m$, and establish the relations between $W^{k,2}$ norm and $H^{k,2}$ norm. Sec. 3 is devoted to the proof of the above theorems.

2 Some inequalities for Sobolev sections on vector bundles

Let $\pi: E \to M$ be a Riemannian vector bundle over a Riemannian manifold M. Then we have the bundle $\Lambda^p T^* M \otimes E \to M$ over M which is the tenser product of the bundle E and the induced p-form bundle over M, where $p = 1,2,\cdots,\dim(M)$. We define $\Gamma(\Lambda^p T^* M \otimes E)$ as the set of all smooth sections of $\Lambda^p T^* M \otimes E \to M$. There exists an induced metric on $\Lambda^p T^* M \otimes E \to M$ from the metric on T^*M and E such that for any $s_1, s_2 \in \Gamma(\Lambda^p T^* M \otimes E)$

$$\langle s_1, s_2\rangle = \sum_{i_1<i_2<\cdots<i_p} \langle s_1(e_{i_1},\cdots,e_{i_p}), s_2(e_{i_1},\cdots,e_{i_p})\rangle,$$

where $\{e_i\}$ is an orthonormal local frame of TM. We define the inner product on $\Gamma(\Lambda^p T^* M \otimes E)$ as follows:

$$(s_1,s_2) = \int_M \langle s_1,s_2\rangle(x)\mathrm{d}M = \int_M \langle s_1,s_2\rangle(x) * 1.$$

The Sobolev space $L^2(M,\Lambda^p T^* M \otimes E)$ is the completion of $\Gamma(\Lambda^p T^* M \otimes E)$ with respect to the above inner product $(\cdot,\cdot)$, we may also define analogously the Sobolev spaces $H^{k,r}(M,\Lambda^p T^* M \otimes E)$ or $H^{k,r}(M,E)$. Let ∇ be the covariant differential induced by the metric on E, then we can take the completion of the smooth sections of E in the norm,

$$\| s \|_{k,r} = \| s \|_{H^{k,r}(M,E)} = \left(\sum_{i=0}^{k}\int_M |\nabla^i s|^r \mathrm{d}M\right)^{\frac{1}{r}},$$

where

$$|\nabla^i s| = \langle \underbrace{\nabla\cdots\nabla}_{i\text{ times}} s, \underbrace{\nabla\cdots\nabla}_{i\text{ times}} s\rangle^{\frac{1}{2}}.$$

We call the above Sobolev spaces the bundle-valued Sobolev spaces.

We first prove the following interpolation inequality for sections on vector bundles, which was proved for functions on $\mathbb{R}^m$ by Gagliardo and Nirenberg, and for functions on Riemannian manifolds by Aubin[14].

Theorem 2.1. Let $s \in C^{\infty}(E)$, where E is a finite-dimensional C^{∞} vector bundle over a closed m-dimensional Riemannian manifold M. Then we have

$$\| \nabla^i s \|_{L^p} \leqslant C \| s \|_{H^{k,q}}^{a} \| s \|_{L^r}^{1-a}, \tag{2.1}$$

where

$$\| s \|_{H^{k,q}} = \sum_{l=0}^{k} \| \nabla^l s \|_{L^q},$$

where $1 \leqslant p, q, r \leqslant \infty$, and $j/k \leqslant a \leqslant 1$ ($j/k \leqslant a < 1$ if $q = m/(k-j) \neq 1$) are numbers such that

$$\frac{1}{p} = \frac{j}{m} + \frac{1}{r} + a\left(\frac{1}{q} - \frac{1}{r} - \frac{k}{m}\right). \tag{2.2}$$

The constant C in (2.1) depends only on M and the numbers j, k, q, r, a.

Proof. We will follow the proof in ref. [14] closely and be sketchy. We first note that, for functions $f \in C^\infty(M)$ (without the assumption that $\int_M f = 0$), one can modify slightly Theorem 3.70 in ref. [14] to get

$$\| \nabla^j f \|_{L^p} \leqslant C \| f \|^a_{W^{k,q}} \| f \|^{1-a}_{L^r}, \tag{2.3}$$

where the constants in (2.3) satisfies the conditions of Theorem 2.1.

Case (i). $j = 0$ and $k = 1$. In this case, letting $f = |s|$ in (2.3) and noticing that Kato's inequality $|\nabla |s|| \leqslant |\nabla s|$ implies $\| |s| \|_{W^{1,q}} \leqslant \| s \|_{H^{1,q}}$, we get inequality (2.1) for $j = 0$, $k = 1$.

In general, one can let $f = |\nabla^j s|$ and use Kato's inequality $|\nabla |\nabla^j s|| \leqslant |\nabla^{j+1} s|$ for $j \geqslant 1$ to derive from (2.3)

$$\| \nabla^j s \|_{L^p} \leqslant C \| s \|^a_{H^{j+1,q}} \| \nabla^j s \|^{1-a}_{L^r}. \tag{2.4}$$

Case (ii). $j = 1$ and $k = 2$. In this case we have $1/2 \leqslant a \leqslant 1$. It is clear that the case $a = 1$ has been treated in (2.4). For the remaining cases, similar to ref. [14], the crucial step is to prove (2.1) for $a = 1/2$, or

$$\| \nabla s \|^2_{L^p} \leqslant C(m,p) \| \nabla^2 s \|_{L^q} \| s \|_{L^r}, \tag{2.5}$$

where $1/q + 1/r = 2/p$.

Proof of (2.5). We assume $p \geqslant 2$, which is what we need for our applications. By direct computation we have

$$\begin{aligned} \mathrm{Div}\langle |\nabla s|^{p-2} \nabla s, s\rangle &= |\nabla s|^p + |\nabla s|^{p-2} \langle \nabla_\alpha \nabla_\alpha s, s\rangle \\ &\quad + (p-2) |\nabla s|^{p-4} \langle \nabla_\beta s, \nabla_\alpha \nabla_\beta s\rangle \langle \nabla_\alpha s, s\rangle. \end{aligned}$$

Integrating the equality over M and noting that

$$\left| \sum \nabla_\alpha \nabla_\alpha s \right|^2 \leqslant m |\nabla^2 s|^2,$$

we get

$$\int_M |\nabla s|^p \leqslant (\sqrt{m} + |p-2|) \int_M |\nabla^2 s| |\nabla s|^{p-2} |s|.$$

Applying the Hölder's inequality (noting $1/q + 1/r + (p-2)/p = 1$), we have

$$\| \nabla s \|^p_{L^p} \leqslant C(m,p) \| \nabla^2 s \|_{L^q} \| s \|_{L^r} \| \nabla s \|^{p-2}_{L^p},$$

which is just (2.5).

For $1/2 < a < 1$, we need to consider two cases.

(a) $q < m$. Using the convexity of $\log(\| f \|^p_{L^p})$ as a function of $p \geqslant 1$, we have

$$\| \nabla s \|_{L^p} \leqslant \| \nabla s \|^a_{L^t} \| \nabla s \|^{1-a}_{L^r},$$

where $1 \leqslant t \leqslant p \leqslant r$ and $a = (p^{-1} - r^{-1})/(t^{-1} - r^{-1})$. Choose t, r so that

$$\frac{1}{r} = \frac{1}{q} - \frac{1}{m}, \text{ and } \frac{2}{t} = \frac{1}{q} + \frac{1}{r} = \frac{2}{q} - \frac{1}{m}.$$

Then by (2.5),

$$\| \nabla s \|_{L^t} \leqslant C \| \nabla^2 s \|_{L^q}^{1/2} \| s \|_{L^r}^{1/2},$$

and by (2.4) with $j = 1$, $a = 1$,

$$\| \nabla s \|_{L^r} \leqslant C \| s \|_{H^{2,q}}.$$

Combining the above three inequalities we get (2.1).

(b) $q \geqslant m$. Under this condition one can choose t such that

$$\frac{2}{t} = \frac{1}{q} + \frac{1}{r},$$

and choose $b \in [0,1]$ such that

$$\frac{1}{p} = \frac{1}{t} + b\left(\frac{1}{q} - \frac{1}{t} - \frac{1}{m}\right).$$

Then by (2.4) with $j = 1$,

$$\| \nabla s \|_{L^p} \leqslant C \| s \|_{H^{2,q}}^{b} \| \nabla s \|_{L^t}^{1-b}.$$

Then by (2.5),

$$\| \nabla s \|_{L^t} \leqslant C \| \nabla^2 s \|_{L^q}^{1/2} \| s \|_{L^r}^{1/2}.$$

Combining the above two inequalities we get (2.1) with $a = (1 + b)/2$.

This completes the proof of Case (ii). All the remaining cases then can be derived by induction.

Now let $u \in C^\infty(M, N)$, where M is a closed Riemannian manifold. Considering ∇u as a section on the bundle $u^*(TN) \otimes T^*M$, then with $s = \nabla u$, we have by Theorem 2.1,

$$\| \nabla^{j+1} u \|_{L^p} \leqslant C \| \nabla u \|_{H^{k,q}}^{a} \| \nabla u \|_{L^r}^{1-a}, \tag{2.6}$$

where the constants in (2.6) satisfy the conditions of Theorem 2.1.

We need to consider the case $M = T_R^m = \mathbb{R}^m/(R \cdot \mathbb{Z})^m$, where $R \geqslant 1$ and the Riemannian metric of T_R^m is just the Euclidean metric.

Proposition 2.1. If $M = T_R^m$, then the constant C in (2.6) does not depend on the diameter $R \geqslant 1$.

Proof. For each $u \in C^\infty(T_R^m, N)$ let $u_R \in C^\infty(T_1{}^m, N)$ be defined by

$$u_R(x) = u(Rx), \quad \forall x \in T_1^m.$$

Then it is easy to find that for any integer $l \geqslant 1$,

$$\| \nabla^l u_R \|_{L^p(T_1{}^m)} = R^{l - m/p} \| \nabla^l u \|_{L^p(T_R^m)}. \tag{2.7}$$

Since $R \geqslant 1$, one deduces from (2.7)

$$\| \nabla u_R \|_{H^{k,q}(T_1{}^m)} \leqslant (k + 1) R^{k+1-m/q} \| \nabla u \|_{H^{k,q}(T_R^m)}. \tag{2.8}$$

Note u_R satisfies (2.6) with the constant $C = C(T_1^m, j, k, p, q, r)$. Combining (2.6) for $u = u_R$ with (2.7) and (2.8) we get

$$\| \nabla^{j+1} u \|_{L^p} \leqslant C R^h \| \nabla u \|_{H^{k,q}}^{a} \| \nabla u \|_{L^r}^{1-a},$$

for some constant h. However, using (2.2) one checks that $h = 0$. This proves the proposition.

In the following we consider the problem of comparing the $W^{k,2}$ norm with $H^{k,2}$ norm of maps $u \in C^\infty(M, N)$. We assume that M is a closed Riemannian manifold and N is a compact Riemannian manifold with or without boundary. It will be convenient to imbed N isometrically into some Euclidean space $\mathbb{R}^K$, and consider N as a compact submanifold of $\mathbb{R}^K$. Then the map u can be represented as $u = (u^1, \cdots, u^K)$ with u^i being globally defined functions on M. Then we have

$$\| u \|^2_{W^{k,2}} = \sum_{i=0}^{k} \| D^i u \|^2_{L^2},$$

where

$$\| D^i u \|^2_{L^2} = \sum_{|\boldsymbol{a}|=i} \| D_{\boldsymbol{a}} u \|^2_{L^2},$$

and D denotes the covariant derivative for functions on M. The $H^{k,2}$ norm of u is defined similarly, we only need to replace D by ∇, where ∇ is the covariant derivative for sections of the bundle $u^*(TN)$ over M (for simplicity we also write $\nabla u = Du$.).

In general, V is a section of $u^*(TN)$ if and only if $V \in C^\infty(M, \mathbb{R}^K)$ such that $V(x) \in T_{u(x)}N$ for all $x \in M$. For each $y \in N \subset \mathbb{R}^K$, let $P(y)$ be the orthogonal projection from $\mathbb{R}^K$ onto T_yN, then we have

$$V(x) = P(u(x))V(x), \quad \forall x \in M.$$

Applying the operator D_α to the identity we get

$$D_\alpha V = P(u)D_\alpha V + A(u)(D_\alpha u, V),$$

or equivalently,

$$D_\alpha V = \nabla_\alpha V + A(u)(\nabla_\alpha u, V),$$

where A is the second fundamental form of N in $\mathbb{R}^K$. Using this, it is easy to derive by induction the following identity (with $u_\alpha = V$)

$$D_{\boldsymbol{a}} u = \nabla_{\boldsymbol{a}} u + \sum_\sigma B_{\sigma(\boldsymbol{a})}(u)(\nabla_{\boldsymbol{a}_1} u, \cdots, \nabla_{\boldsymbol{a}_s} u), \tag{2.9}$$

where $|\boldsymbol{a}| \geqslant 2$ and the sum is over all multi-indices $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_s$ such that $|\boldsymbol{a}_i| \geqslant 1$ for all i and

$$(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_s) = \sigma(\boldsymbol{a})$$

is a permutation of $\boldsymbol{a}$. The $B_{\sigma(\boldsymbol{a})}$ in (2.9) is the multi-linear form on TN, whose norm as an operator depends only on the geometry of N.

It follows that

$$|D_{\boldsymbol{a}} u| \leqslant |\nabla_{\boldsymbol{a}} u| + C(N) \sum |\nabla_{\boldsymbol{a}_1} u| \cdots |\nabla_{\boldsymbol{a}_s} u|,$$

and

$$\| D^k u \|_{L^2} \leqslant \| \nabla^k u \|_{L^2} + C(N) \sum_{j_1 + \cdots + j_l = k,\ j_i \geqslant 1} \| |\nabla^{j_1} u| \cdots |\nabla^{j_l} u| \|_{L^2}. \tag{2.10}$$

Inversely, by the definition of covariant differential we can also deduce that there are $\widetilde{B}_i$ which are multi-linear vector valued functions on $\mathbb{R}^K$ such that

$$\nabla_{\boldsymbol{a}} u = D_{\boldsymbol{a}} u + \sum_\sigma \widetilde{B}_{\sigma(\boldsymbol{a})}(u)(D_{\boldsymbol{a}_1} u, \cdots, D_{\boldsymbol{a}_s} u), \tag{2.11}$$

where $|\boldsymbol{a}| \geqslant 2$ and the sum is over all multi-indices $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_s$ such that $|\boldsymbol{a}_i| \geqslant 1$ for all i and

$$(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_s) = \sigma(\boldsymbol{a})$$

is a permutation of $\boldsymbol{a}$. There also holds

$$\| \nabla^k u \|_{L^2} \leqslant \| D^k u \|_{L^2} + C \sum_{j_1 + \cdots + j_l = k,\ j_i \geqslant 1} \| |D^{j_1} u| \cdots |D^{j_l} u| \|_{L^2}. \tag{2.12}$$

Proposition 2.2. Assume that $k > m/2$. Then there exists a constant $C = C(N,k)$ such that for all $u \in C^\infty(M, N)$,

$$\| Du \|_{W^{k-1,2}} \leqslant C \sum_{t=1}^{k} \| \nabla u \|_{H^{k-1,2}}^{t}, \tag{2.13}$$

and

$$\| \nabla u \|_{H^{k-1,2}} \leqslant C \sum_{t=1}^{k} \| Du \|_{W^{k-1,2}}^{t}. \tag{2.14}$$

Proof. Consider the terms in the sum in (2.10). Let $2 \leqslant p_i \leqslant \infty$ $(i = 1, \cdots, l)$ be such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_l} = \frac{1}{2}.$$

Then by Hölder's inequality,

$$\| \, | \nabla^{j_1} u | \cdots | \nabla^{j_l} u | \, \|_{L^2} \leqslant \| \nabla^{j_1} u \|_{L^{p_1}} \cdots \| \nabla^{j_l} u \|_{L^{p_l}}.$$

We assume $j_1 + \cdots + j_l = j \leqslant k$. We claim that under the condition $k > m/2$ one can check that there exist p_i and $(j_i - 1)/(k-1) \leqslant a_i \leqslant 1$ with

$$\frac{1}{p_i} = \frac{j_i - 1}{m} + \frac{1}{2} - \frac{a_i(k-1)}{m}.$$

By Theorem 2.1, with $s = \nabla u$, we have

$$\| \nabla^{j_i} u \|_{L^{p_i}} \leqslant C \| \nabla u \|_{H^{k-1,2}}^{a_i} \| \nabla u \|_{L^2}^{1-a_i} \leqslant C \| \nabla u \|_{H^{k-1,2}}. \tag{2.15}$$

Using this in (2.10) we get

$$\| D^j u \|_{L^2} \leqslant C \sum_{l=1}^{j} \| \nabla u \|_{H^{k-1,2}}^{l}$$

for $1 \leqslant j \leqslant k$. It follows easily that (2.13) holds.

Now we turn to proving the above claim. From Theorem 2.1 we see that the condition $a_i \geqslant \frac{j_i - 1}{k-1}$ is equivalent to $p_i \geqslant 2$, and the condition $a_i \leqslant 1$ is equivalent to

$$\frac{1}{p_i} \geqslant \frac{1}{2} + \frac{j_i}{m} - \frac{k}{m} \equiv \gamma_i. \tag{2.16}$$

Also, we see

$$\gamma_i > 0 \quad \text{if and only if} \quad j_i > k - \frac{m}{2}.$$

We may assume that

$$j_1 \geqslant j_2 \geqslant \cdots j_t > k - \frac{m}{2} \geqslant j_{t+1} \geqslant \cdots \geqslant j_l.$$

That is, for $i \leqslant t$ we have $\gamma_i > 0$, while for $i > t$ we have $\gamma_i \leqslant 0$.

When $t \geqslant 1$, let $p_i (i = 1, \cdots, t)$ be a set of positive numbers such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_t} = \frac{1}{2}. \tag{2.17}$$

We need each $p_i (1 \leqslant i \leqslant t)$ to satisfy (2.16), i.e.

$$\frac{1}{p_i} \geqslant \gamma_i \equiv \frac{1}{2} + \frac{j_i}{m} - \frac{k}{m} > 0.$$

We can choose p_i to satisfy both (2.16) and (2.17) if only

$$\sum_{i=1}^{t} \gamma_i \leqslant \frac{1}{2}.$$

This can be verified by the following computation

$$\sum_{i=1}^{t} \gamma_i = \left(\frac{1}{2} - \frac{k}{m}\right)t + \frac{1}{m}\sum_{i=1}^{t} j_i \leqslant \left(\frac{1}{2} - \frac{k}{m}\right)t + \frac{k}{m}$$
$$= \frac{1}{2} + (t-1)\left(\frac{1}{2} - \frac{k}{m}\right) \leqslant \frac{1}{2}.$$

For $i > t$, to apply (2.15) we need only to select $p_i = \infty$.

When $t = 0$, it is obvious that to apply (2.15) we need only to choose $p_1 = 2$ and $p_i = \infty$ for $i = 2, \cdots, l$.

By the same spirit as above, we can use (2.11) and (2.12) to derive (2.14). Thus, we complete the proof of the proposition.

3 Local existence of Schrödinger flow

In this section we prove the local existence of smooth solutions for the initial value problem of the Schrödinger flow

$$\begin{cases} u_t = J(u)\tau(u), \\ u(\cdot,0) = u_0 \in C^\infty(M,N). \end{cases} \tag{3.1}$$

We need to employ an approximate procedure and solve first the following perturbed problem

$$\begin{cases} u_t = \varepsilon\tau(u) + J(u)\tau(u), \\ u(\cdot,0) = u_0 \in C^\infty(M,N), \end{cases} \tag{3.2}$$

where $\varepsilon > 0$ is a small number.

The advantage of (3.2) is that the equation with $\varepsilon > 0$ is uniformly parabolic. Hence the initial value problem has a unique smooth solution $u_\varepsilon \in C^\infty(M \times [0, T_\varepsilon), N)$ for some $T_\varepsilon > 0$. The problem is then to obtain a uniform positive lower bound T of T_ε, and uniform bounds for various norms of $u_\varepsilon(t)$ in suitable spaces for t in the time interval $[0, T)$ (Since we shall use L^2 estimates, the norms are $W^{k,2}(M, N)$-norms for all positive integer k.). Once we get these bounds it is clear that the u_ε subconverge to a smooth solution of (3.1) as $\varepsilon \to 0$.

Now let $u = u_\varepsilon$ be a solution of (3.2). Then it is easy to see that the energy $E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_{L^2}$ is uniformly bounded for $t \in [0, T_\varepsilon)$. Actually,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(u(t)) = -\int_M \langle \tau(u), u_t \rangle$$
$$= -\varepsilon\int_M |\tau(u)|^2 - \int_M \langle \tau(u), J(u)\tau(u) \rangle.$$

The last integral vanishes since the complex structure J is anti-symmetric. It follows that the time derivative for the energy is non-positive, and

$$E(u(t)) \leqslant E(u_0). \tag{3.3}$$

In the following we will make estimations on L^2-norms of all covariant derivatives $\nabla^k u$ ($k = 2, 3, \cdots$). We will assume M is flat, i.e. the Riemannian curvature of M vanishes identically, to simplify the computations. For the general case, the additional terms involving the curvatures of M actually do not provide additional difficulties, since the derivatives of u appearing in these terms are of lower orders. We formulate our estimates into a lemma.

Lemma 3.1. Let $m_0 = [m/2] + 1$, where $[q]$ denotes the integral part of a positive number q, and let $u_0 \in C^\infty(M, N)$. There exists a constant $T = T(\| u_0 \|_{H^{m_0+1,2}}) > 0$, independent of $\varepsilon \in [0,1]$, such that if $u \in C^\infty(M \times [0, T_\varepsilon])$ is a solution of (3.1) with $\varepsilon \in (0, 1]$ then

$$T_\varepsilon \geqslant T(\| u_0 \|_{H^{m_0+1,2}})$$

and

$$\| u(t) \|_{H^{k+1,2}} \leqslant C(k, \| u_0 \|_{H^{k+1,2}}) \quad t \in [0, T]$$

for all $k \geqslant m_0$.

Proof. As N may not be compact we let $\Omega \stackrel{\text{def}}{=} \{p \in N: \text{dist}_N(p, u_0(M)) < 1\}$, which is an open subset of N with compact closure $\bar{\Omega}$. Let

$$T' = \sup\{t > 0: u(M, t) \subset \Omega\}.$$

Fix a $k \geqslant m_0$, and let l be any integer with $1 \leqslant l \leqslant k$. Suppose that $\boldsymbol{a}$ be a multi-index of length l, i.e. $\boldsymbol{a} = (a_1, \cdots, a_l)$. Then we have for $t \leqslant T'$

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \nabla_{\boldsymbol{a}} \nabla_i u \|_{L^2}^2 = \int_M \langle \nabla_{\boldsymbol{a}} \nabla_i u, \nabla_t \nabla_{\boldsymbol{a}} \nabla_i u \rangle. \tag{3.4}$$

Exchanging the order of covariant differentiation we have (cf. ref. [15])

$$\nabla_t \nabla_{\boldsymbol{a}} \nabla_i u = \nabla_{\boldsymbol{a}} \nabla_i \nabla_t u + \sum \nabla_{\boldsymbol{b}} R(u)(\nabla_{\boldsymbol{c}} u, \nabla_{\boldsymbol{d}} \nabla_t u) \nabla_{\boldsymbol{e}} \nabla_i u,$$

where the sum is over all multi-indexes $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ with possible zero lengths, except that $|\boldsymbol{c}| > 0$ always holds, such that

$$(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}) = \sigma(\boldsymbol{a})$$

is a permutation of $\boldsymbol{a}$. Noting that we may replace $\nabla_t u$ in the terms of the summation by the right hand side of eq. (3.2), the above identity can be rewritten as

$$\nabla_t \nabla_{\boldsymbol{a}} \nabla_i u = \nabla_{\boldsymbol{a}} \nabla_i \nabla_t u + Q \tag{3.5}$$

with

$$| Q | \leqslant C(l, \Omega) \sum |\nabla^{j_1} u| \cdots |\nabla^{j_s} u| \tag{3.6}$$

where the summation is over all $(j_1, \cdots, j_s)$ satisfying

$$j_1 \geqslant j_2 \geqslant \cdots \geqslant j_s, \quad l + 1 \geqslant j_i \geqslant 1, \quad j_1 + \cdots + j_s = l + 3, \quad s \geqslant 3. \tag{3.7}$$

For the first term in the right hand side of (3.5), we may use eq. (3.2) to get

$$\begin{aligned} \nabla_{\boldsymbol{a}} \nabla_i \nabla_t u &= \nabla_{\boldsymbol{a}} \nabla_i (\varepsilon \tau(u) + J(u) \tau(u)) \\ &= \varepsilon \nabla_{\boldsymbol{a}} \nabla_i \nabla_k \nabla_k u + J(u) \nabla_{\boldsymbol{a}} \nabla_i \nabla_k \nabla_k u, \end{aligned} \tag{3.8}$$

where we have used the integrability of the complex structure J of the Kähler manifold N. By exchanging the orders of covariant differentiation as the above, we get from (3.5) and (3.8)

$$\nabla_t \nabla_{\boldsymbol{a}} \nabla_i u = \varepsilon \nabla_k \nabla_k \nabla_{\boldsymbol{a}} \nabla_i u + J(u) \nabla_k \nabla_k \nabla_{\boldsymbol{a}} \nabla_i u + Q$$

where Q satisfies (3.6) and (3.7). Substituting this into (3.4) and integrating by part we then have

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \nabla_{\boldsymbol{a}} \nabla_i u \|_{L^2}^2 \\ &= \int_M (-\varepsilon |\nabla \nabla_{\boldsymbol{a}} \nabla_i u|^2 - \langle \nabla_k \nabla_{\boldsymbol{a}} \nabla_i u, J(u) \nabla_k \nabla_{\boldsymbol{a}} \nabla_i u \rangle + \langle \nabla_{\boldsymbol{a}} \nabla_i u, Q \rangle). \end{aligned}$$

Note that the first integrand is non-positive and the second vanishes, so we have by (3.6)

$$\frac{\mathrm{d}}{\mathrm{d}t} \| \nabla_{\boldsymbol{a}} \nabla_i u \|_{L^2}^2 \leqslant C(l, \Omega) \sum \int_M |\nabla^{l+1} u| |\nabla^{j_1} u| \cdots |\nabla^{j_s} u|,$$

and consequently

$$\frac{d}{dt}\| \nabla^{l+1}u \|_{L^2}^2 \leqslant C(l,\Omega)\sum\int_M |\nabla^{l+1}u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_s}u|, \tag{3.9}$$

where the summation is over all $(j_1,\cdots,j_s)$ satisfying (3.7).

To treat the integrals in the summation of (3.9), i.e.

$$I = \int_M |\nabla^{l+1}u|\,|\nabla^{j_1}u|\cdots|\nabla^{j_s}u|, \tag{3.10}$$

we need the following lemmas.

Lemma 3.2. Let I be the integral (3.10), where $(j_1,\cdots,j_s)$ satisfy (3.7). If $1\leqslant l\leqslant m_0$, then there exists a constant $C = C(M,l)$ such that

$$I \leqslant C\| \nabla u \|_{H^{m_0,2}}^{A} \| \nabla u \|_{L^2}^{B} \| \nabla^{l+1}u \|_{L^2},$$

where $A = [\,l+3+(m/2-1)s-m/2]/m_0$ and $B = s-A$.

Proof. Let $2\leqslant p_i\leqslant\infty$ $(i=1,\cdots,s)$ be real numbers (to be chosen later) such that

$$\sum_{i=1}^{s}\frac{1}{p_i} = \frac{1}{2}.$$

Then by Hölder's inequality

$$I \leqslant \| \nabla^{l+1}u \|_{L^2} \| \nabla^{j_1}u \|_{L^{p_1}}\cdots \| \nabla^{j_s}u \|_{L^{p_s}}. \tag{3.11}$$

Now by Theorem 2.1, we have

$$\| \nabla^{j_i}u \|_{L^{p_i}} \leqslant C\| \nabla u \|_{H^{m_0,2}}^{a_i} \| \nabla u \|_{L^2}^{1-a_i}, \tag{3.12}$$

where $(j_i-1)/m_0\leqslant a_i<1$, and

$$\frac{1}{p_i} = \frac{j_i-1}{m} + \frac{1}{2} - \frac{a_i m_0}{m}. \tag{3.13}$$

Note that when (3.13) holds, $p_i\geqslant 2$ implies $a_i\geqslant (j_i-1)/m_0$, while the condition $a_i<1$ is equivalent to

$$\frac{1}{p_i} > \frac{j_i-1}{m} + \frac{1}{2} - \frac{m_0}{m} \equiv \gamma_i. \tag{3.14}$$

Also note that

$$\gamma_i \geqslant 0 \quad \text{if and only if} \quad j_i - 1 \geqslant m_0 - \frac{m}{2} \quad \text{if and only if} \quad j_i \geqslant 2.$$

Assume that $j_i\geqslant 2$ for $i\leqslant t$, and $j_i = 1$ for $t\geqslant t+1$. We choose $p_i=\infty$ for $i\geqslant t+1$ so that

$$\sum_{i=1}^{t}\frac{1}{p_i} = \frac{1}{2}. \tag{3.15}$$

Then we may choose p_i for $i=1,\cdots,t$ to satisfy both (3.14) and (3.15) if and only if

$$\frac{1}{2} > \sum_{i=1}^{t}\gamma_i = \frac{1}{m}\Big(\sum_{i=1}^{t}j_i - t\Big) + \frac{t}{2} - \frac{m_0 t}{m}.$$

But $\sum_{i=1}^{s} j_i = l+3$ and $s\geqslant 3$, we have

$$\sum_{i=1}^{t} j_i - t = l+3-(s-t)-t \leqslant l \leqslant m_0,$$

hence

$$\sum_{i=1}^{t}\gamma_i \leqslant \frac{m_0}{m} + \frac{t}{2} - \frac{m_0 t}{m} = \frac{1}{2} + \left(\frac{1}{2} - \frac{m_0}{m}\right)(t-1) \leqslant \frac{1}{2}.$$

The last equality can hold only if $t = 1$, which implies $p_1 = 2$ and $j_1 = l + 1$. If $l < m_0$ we have $\gamma_1 < \frac{1}{2}$ and (3.14) holds true. If $l = m_0$, we see inequality (3.12) is trivially true for $j_1 = m_0 + 1$ and $a_1 = 1$.

We have shown that in any case we can choose p_i and a_i such that (3.12) holds for all i. It follows from (3.11) and (3.12) that

$$I \leqslant C \| \nabla u \|_{H^{m_0,2}}^{A} \| \nabla u \|_{L^2}^{B} \| \nabla^{l+1} u \|_{L^2},$$

with $A = \sum_{i=1}^{s} a_i = A = [l + 3 + (m/2 - 1)s - m/2]/m_0$ and $B = \sum_{i=1}^{s}(1 - a_i) = s - A$. This finishes the proof of Lemma 3.2.

Lemma 3.3. Assume $l > m_0$. Then there exists a constant $C = C(M, l)$ such that

(i) if $j_1 = l + 1$,

$$I \leqslant C \| \nabla^{l+1} u \|_{L^2}^{2} \| \nabla u \|_{H^{m_0,2}}^{m/m_0} \| \nabla u \|_{L^2}^{2-m/m_0}.$$

(ii) if $j_1 \leqslant l$,

$$I \leqslant C(1 + \| \nabla u \|_{H^{l,2}}^{2})(1 + \| \nabla u \|_{H^{l-1,2}}^{A})$$

where $A = A(m, l)$.

Proof. (i) If $j_1 = l + 1$, we see from (3.7) that $s = 3$ and $j_2 = j_3 = 1$, i.e.

$$I = \int_M |\nabla^{l+1} u|^2 |\nabla u|^2.$$

Then it is clear that

$$I \leqslant \| \nabla^{l+1} u \|_{L^2}^{2} \| \nabla u \|_{L^\infty}^{2}.$$

The claimed inequality follows immediately

$$\| \nabla u \|_{L^\infty} \leqslant C \| \nabla u \|_{H^{m_0,2}}^{a} \| \nabla u \|_{L^2}^{1-a},$$

where $a = m/(2m_0)$.

(ii) Assume $j_1 \leqslant l$. We need to use a special case of Theorem 2.1, where $s = \nabla^j u$ $(j \geqslant 1)$, $r = q = 2$ and k is replaced by $k - j > 0$. We have

$$\| \nabla^j u \|_{L^p} \leqslant C \| \nabla^j u \|_{H^{k-j,2}}^{a} \| \nabla^j u \|_{L^2}^{1-a}, \tag{3.16}$$

where

$$\frac{1}{p} = \frac{1}{2} - \frac{(k - j)a}{m}, \tag{3.17}$$

with $0 \leqslant a \leqslant 1$, unless $j = k - \frac{m}{2}$ in which case $0 \leqslant a < 1$.

From (3.17) we see that the condition $a \geqslant 0$ is equivalent to $p \geqslant 2$, and the condition $a < 1$ is equivalent to

$$\frac{1}{p} > \frac{1}{2} - \frac{k - j}{m} \equiv \gamma_{k,j}. \tag{3.18}$$

Also, we see

$$\gamma_{k,j} \geqslant 0 \quad \text{if and only if } j \geqslant k - \frac{m}{2}.$$

Now, we turn to the integral I. We may assume that for some integer $t \geqslant 0$,

$$l - \frac{m}{2} > j_{t+1} \geqslant \cdots \geqslant j_s \geqslant 1,$$

while if $t > 0$,

$$l \geqslant j_1 \geqslant \cdots \geqslant j_t \geqslant l - \frac{m}{2}.$$

That means, for $i \leqslant t$ we have $\gamma_{l,j_i} \geqslant 0$, while for $i \geqslant t+1$ we have $\gamma_{l,j_i} < 0$. Then from (3.16) we have for $i \geqslant t+1$ (with $k = l$, $p_i = \infty$)

$$\| \nabla^{j_i} u \|_{L^\infty} \leqslant C \| \nabla^{j_i} u \|^{a_i}_{H^{l-j_i,2}} \| \nabla^{j_i} u \|^{1-a_i}_{L^2},$$

where $a_i = \dfrac{m}{2(l - j_i)} < 1$.

First, consider the case $t = 0$. For this case we have

$$I \leqslant C \| \nabla u \|^{A_1}_{H^{l-1,2}} \Big(\sum_{i=2}^{s} \| \nabla^{j_i} u \|^{1-a_i}_{L^2} \Big) \int_M |\nabla^{l+1} u| \, |\nabla^{j_1} u|,$$

where $A_1 = a_2 + \cdots + a_s$. Then we have

$$I \leqslant C(1 + \| \nabla u \|^{A_1+1}_{H^{l-1,2}}) \| \nabla^{l+1} u \|_{L^2} \| \nabla^{j_1} u \|_{L^2} \leqslant C(1 + \| \nabla u \|^{A_1+1}_{H^{l-1,2}}) \| \nabla u \|^2_{H^{l,2}}.$$

Assume now that $t \geqslant 1$ then we have

$$I \leqslant C \| \nabla u \|^{A_2}_{H^{l-1,2}} \Big(\sum_{i=t+1}^{s} \| \nabla^{j_i} u \|^{1-a_i}_{L^2} \Big) \int_M |\nabla^{l+1} u| \, |\nabla^{j_1} u| \cdots |\nabla^{j_t} u|,$$

where $A_2 = a_{t+1} + \cdots + a_s$. So we can also write

$$I \leqslant C(1 + \| \nabla u \|^{A_2+1}_{H^{l-1,2}}) \int_M |\nabla^{l+1} u| \, |\nabla^{j_1} u| \cdots |\nabla^{j_t} u|. \tag{3.19}$$

Let $p_i (i = 1, \cdots, t)$ be a set of positive numbers such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_t} = \frac{1}{2}.$$

Then by Hölder inequality, the integral on the right hand side of (3.19) is no greater than

$$\| \nabla^{l+1} u \|_{L^2} \| \nabla^{j_1} u \|_{L^{p_1}} \cdots \| \nabla^{j_t} u \|_{L^{p_t}}. \tag{3.20}$$

If $t = 1$, we have $p_1 = 2$ and obviously

$$\| \nabla^{j_1} u \|_{L^2} \leqslant C \| \nabla u \|_{H^{l-1,2}}.$$

In view of (3.19) and (3.20), the lemma is proved. So in the following we assume $t \geqslant 2$.

For the second term in (3.20) we need to treat two cases differently.

Case 1. $j_1 \geqslant l + 1 - \dfrac{m}{2}$. In this case we may apply (3.16) with $k = l+1$, $a = 1 - \varepsilon$, where $\varepsilon = 0$ if $j_1 > l + 1 - m/2$, and $\varepsilon > 0$ arbitrarily small when $j_1 = l + 1 - m/2$, to get

$$\| \nabla^{j_1} u \|_{L^{p_1}} \leqslant C \| \nabla^{j_1} u \|^{1-\varepsilon}_{H^{l+1-j_1,2}} \| \nabla^{j_1} u \|^{\varepsilon}_{L^2} \leqslant C \| \nabla u \|_{H^{l,2}},$$

where

$$\frac{1}{p_1} = \frac{1}{2} - \frac{(l + 1 - j_1)(1 - \varepsilon)}{m} > 0.$$

Note that we now have

$$\frac{1}{p_2} + \cdots + \frac{1}{p_t} = \frac{1}{2} - \frac{1}{p_1} = \frac{(l + 1 - j_1)(1 - \varepsilon)}{m}. \tag{3.21}$$

To apply inequality (3.16) with $k = l$ and $j = j_i$ for $2 \leqslant i \leqslant t$, we need each p_i to satisfy

(3.18), i.e.

$$\frac{1}{p_i} > \gamma_{l,j_i} \equiv \frac{1}{2} - \frac{l - j_i}{m} \geqslant 0.$$

We can choose p_i to satisfy both (3.18) and (3.23) if only

$$\sum_{i=2}^{t} \gamma_{l,j_i} < \frac{(l+1-j_1)(1-\varepsilon)}{m}.$$

This can be verified by the following computation.

$$\begin{aligned}\sum_{i=2}^{t} \gamma_{l,j_i} &= (t-1)\left(\frac{1}{2} - \frac{l}{m}\right) + \frac{1}{m}\sum_{i=2}^{t} j_i \\ &\leqslant (t-1)\left(\frac{1}{2} - \frac{m_0}{m}\right) - (t-1)\frac{l - m_0}{m} + \frac{(l+3-j_1)-(s-t)}{m} \\ &\leqslant (t-1)\left(\frac{1}{2} - \frac{m_0}{m}\right) - \frac{t-1}{m} + \frac{(l+3-j_1)-(s-t)}{m} < \frac{(l+1-j_1)}{m},\end{aligned}$$

since $s \geqslant 3$, $t \geqslant 2$ and $l > m_0 > \frac{m}{2}$. Choosing ε sufficiently small we get what we need. Now, applying (3.16) to the terms in (3.20) with $i \geqslant 2$ we get

$$\| \nabla^{j_i} u \|_{L^{p_i}} \leqslant C \| \nabla u \|_{H^{l-1,2}}^{a_i} \| \nabla^{j_i} u \|_{L^2}^{1-a_i} \leqslant C \| \nabla u \|_{H^{l-1,2}},$$

where a_i is determined by (3.17) with $p = p_i$, $k = l$, $j = j_i$. In view of (3.19) and (3.20), the lemma is proved.

Case 2. $j_1 < l + 1 - \frac{m}{2}$. We may apply (3.16) with $k = l + 1$ and $p = \infty$ to get

$$\| \nabla^{j_1} u \|_{L^\infty} \leqslant C \| \nabla^{j_1} u \|_{H^{l+1-j_1,2}}^{a_1} \| \nabla^{j_1} u \|_{L^2}^{1-a_1} \leqslant c \| \nabla u \|_{H^{l,2}},$$

where

$$a_1 = \frac{m}{2(l+1-j_1)} < 1.$$

Similar to the proof in Case 1, what we need to show now is

$$\Gamma \equiv \sum_{i=2}^{t} \gamma_{l,j_i} < \frac{1}{2},$$

since $p_1 = \infty$. We still have

$$\Gamma = (t-1)\left(\frac{1}{2} - \frac{l}{m}\right) + \frac{1}{m}\sum_{i=2}^{t} j_i.$$

But by (3.7)

$$\sum_{i=2}^{t} j_i = l + 3 - j_1 - \sum_{i=t+1}^{s} j_i < l + 3 - \left(l - \frac{m}{2}\right) - s + t \leqslant \frac{m}{2} + t.$$

Since the left hand side is an integer, we must have

$$\sum_{i=2}^{t} j_i \leqslant \left[\frac{m}{2}\right] + t = m_0 - 1 + t,$$

with equality holds if and only if $s = 3$, $j_{t+1} = \cdots = j_s = 1$ and $j_1 = l + 1 - m_0$. So we have

$$\Gamma \leqslant (t-1)\left(\frac{1}{2} - \frac{l}{m}\right) + \frac{m_0}{m} + \frac{t-1}{m} = \frac{m_0}{m} + (t-1)\left(\frac{1}{2} - \frac{l-1}{m}\right).$$

Noting that $l > m_0$ implies $l \geqslant m_0 + 1$, hence $1/2 < (l-1)/m$, we see (by $t \geqslant 2$)

$$\Gamma \leqslant \frac{1}{2} - \frac{1}{m}(l - m_0 - 1) \leqslant \frac{1}{2}.$$

Note that equality can hold if and only if $l = m_0 + 1$, $j_1 = l + 1 - m_0$, $j_{t+1} = \cdots = j_s = 1$, $s = 3$ and $t = 2$. The first two equalities imply that $j_1 = 2$. It follows that we must have $j_1 = j_2 = 2$, $j_3 = 1$, hence by (3.7) $l = 2$, and by $l = m_0 + 1$ it holds $m_0 = 1$. This then shows that $m = 1$. Except for this special case, our proof for Case 2 can go through just as we did for Case 1. Finally, we remark that the remaining special case can also be treated, and the proof is omitted. This finishes our proof of Lemma 3.3.

Now, return to the proof of Lemma 3.1. We first consider the case $1 \leqslant l \leqslant m_0$ in (3.9). Then Lemma 3.2 together with (3.3) leads to

$$\frac{d}{dt} \| \nabla u \|_{H^{m_0,2}} \leqslant C \sum_{l=1}^{m_0} \sum_{s=3}^{l+3} \| \nabla u \|_{H^{m_0,2}}^{A(s,l)},$$

where

$$A(s,l) = [l + 3 + (m/2 - 1)s - m/2]/m_0.$$

If we let $f(t) = \| \nabla u(t) \|_{H^{m_0,2}} + 1$, then we have

$$f' \leqslant C f^{A_0}, \quad f(0) = \| \nabla u_0 \|_{H^{m_0,2}} + 1, \tag{3.22}$$

where $A_0 = \max\{A(s,l): 3 \leqslant s \leqslant l + 3,\ 1 \leqslant l \leqslant m_0\}$. The constant C in (3.22) depends only on m_0, M and N. It follows from (3.22) that there exists $T_0 > 0$ and $K_0 > 0$ such that

$$\| \nabla u(t) \|_{H^{m_0,2}} \leqslant K_0, \quad t \in [0, \min(T_0, T')]. \tag{3.23}$$

For any $k > m_0$, we need to consider the case $m_0 < l \leqslant k$ in (3.9). Lemma 3.3, (3.3) and (3.23) then imply

$$\frac{d}{dt} \| \nabla u \|_{H^{k,2}}^2 \leqslant C(1 + \| \nabla u \|_{H^{k,2}}^2)(1 + \| \nabla u \|_{H^{k-1,2}}^A). \tag{3.24}$$

For $k = m_0 + 1$, we see from (3.23) that the summation in (3.24) is bounded since $k - 1 = m_0$. Then, since (3.24) is a linear differential inequality for $\| \nabla u \|_{H^{k,2}}^2$, there exists a constant $K_1 > 0$ such that

$$\| \nabla u(t) \|_{H^{m_0+1,2}} \leqslant K_1, \quad t \in [0, \min(T_0, T')]. \tag{3.25}$$

It is now clear that inductively using (3.24) one can show the existence of $K_i > 0$ for any $i \geqslant 1$ such that

$$\| \nabla u(t) \|_{H^{m_0+i,2}} \leqslant K_i, \quad t \in [0, \min(T_0, T')]. \tag{3.26}$$

Since we assume Ω is compact, consequently $\| u(t) \|_{L^\infty}$ is uniformly bounded for $t \in [0, \min(T_0, T')]$.

Note that a positive lower bound of T' can be derived from (3.26). In deed, it is easy to see that

$$\sup_{t \in [0,T]} \| u_t \|_{H^{m_0,2}} \leqslant K_1.$$

However, by Theorem 2.1, for some $0 < a < 1$ there holds

$$\| u_t \|_{C^0} \leqslant C(M) \| u_t \|_{H^{m_0,2}}^a \| u_t \|_{L^2}^{1-a}.$$

This implies

$$\| u_t \|_{C^0} \leqslant \mathscr{M}$$

for some $\mathscr{M} > 0$, assuming that $t < \min\{T_0, T'\}$. Thus we have

$$\sup_{x \in M} d_N(u(x,t), u_0(x)) \leqslant \mathscr{M} t \quad \text{for} \quad t < \min\{T_0, T'\}.$$

If $T' > T_0$ we get the lower bound, so we may assume that $T' \leqslant T_0$. Then letting $t \to T'$ in the above inequality we get $\mathscr{M}\, T' \geqslant 1$. Therefore, if we set $T = \min\left\{\frac{1}{\mathscr{M}}, T_0\right\}$, then the desired estimates hold for $t \in [0, T]$. It is worthwhile pointing out that, if N is compact (noncompact), $T = T(N, \|\nabla u_0\|_{W^{m_0,2}})$ ($T = T(N, \|\nabla u_0\|_{W^{m_0+1,2}})$) depends only on N, u_0, not on $0 < \varepsilon < 1$.

It is easy to find that the solution to (3.2) with $\varepsilon \in (0,1)$ must exist on the time interval $[0, T]$. Otherwise, we always extend the time interval of existence to cover $[0, T]$, i.e. we always have $T_\varepsilon \geqslant T$. Thus, Lemma 3.1 has been proved.

Remark 3.1. In the proofs of Lemmas 3.1—3.3, we only use the interpolation inequalities in Theorem 2.1 and the Hölder inequality. All estimates do not depend on the volume of M, but on the Sobolev constant of M.

Proof of Theorem 1.1. First, we would like to mention that N is always regarded as an embedded submanifold of $\mathbb{R}^K$. If $u_0: M \to N$ is C^∞, then, Lemma 3.1 claims that the Cauchy problem (3.2) admits a unique smooth solution u_ε which satisfies the estimates in Lemma 3.1. It follows from Proposition 2.2 that, for any $k > 0$ and $\varepsilon \in (0,1]$, there holds

$$\max_{t \in [0,T]} \| u_\varepsilon \|_{W^{k,2}(M)} \leqslant C_k(\Omega, u_0),$$

where $C_k(\Omega, u_0)$ does not depend on ε. Hence, by sending $\varepsilon \to 0$ and applying the embedding theorem of Sobolev spaces to u, we have $u_\varepsilon \to u \in C^k(M \times [0, T], N)$ for any k. It is very easy to check that u is a solution to the Cauchy problem (3.1). The uniqueness was addressed in Proposition 2.1 in ref. [1].

Finally, if u_0: $M \to N$ is not C^∞, but $u_0 \in W^{k,2}(M, N)$, we may always select a sequence of C^∞ maps from M into N, denoted by u_{i0}, such that

$$u_{i0} \to u_0 \quad \text{in} \quad W^{k,2}, \text{ as } i \to \infty.$$

This together with (2.11) leads to

$$\| \nabla u_{i0} \|_{H^{k-1,2}} \to \| \nabla u_0 \|_{H^{k-1,2}}, \text{ as } i \to \infty.$$

Thus, there exists a unique, smooth solution u_i, defined on time interval $[0, T_i]$, of the Cauchy problem (3.1) with u_0 replaced by u_{i0}. Furthermore, it is not difficult to see from the arguments in Lemma 3.1 that if i is large enough, then there exists a uniform positive lower bound of T_i, denoted by T, such that the following holds uniformly with respect to large enough i:

$$\sup_{t \in [0,T]} \| \nabla u_i(t) \|_{H^{k-1,2}} \leqslant C(T, \| \nabla u_0 \|_{H^{k-1,2}}).$$

Here, we would like to point out that $T = T(\|\nabla u_0\|_{W^{m_0,2}})$ ($T = T(\|\nabla u_0\|_{W^{m_0+1,2}})$) when N is compact (noncompact). It follows from Proposition 2.2 and the last inequality that

$$\sup_{t \in [0,T]} \| Du_i(t) \|_{W^{k-1,2}} \leqslant C'(T, \| Du_0 \|_{W^{k-1,2}}),$$

where D denotes the covariant derivative for functions on M. Therefore, there exists a $u \in L^\infty([0, T], W^{k,2}(M, N))$ such that

$$u_i \to u \text{ [weakly}^* \text{] in } L^\infty([0, T], W^{k,2}(M, N))$$

upon extracting a subsequence and re-indexing if necessary. It remains to verify that u is a strong solution to (3.1), i.e. we need to check that for any $v \in C^\infty([0, T] \times M, \mathbb{R}^K)$ there holds

$$\int_0^T \int_M \langle \partial_t u, v \rangle = \int_0^T \int_M \langle J(u)\tau(u), v \rangle.$$

However, for each u_i, the following is always true

$$\int_0^T\int_M\langle\partial_t u_i, v\rangle = \int_0^T\int_M\langle J(u_i)\tau(u_i), v\rangle.$$

As $u\in L^\infty([0,T], W^{k,2}(M,N))$ where $k\geq m_0+1$ (or $k\geq m_0+2$ when N is noncompact), we can see easily (see also (2.14))

$$\Delta_g u, A(u)(Du,Du)\in L^\infty([0,T], L^2(M,\mathbb{R}^K)).$$

Therefore, in the sense of distribution $\tau(u)$ can be written as

$$\tau(u) = \Delta_g u + A(u)(Du,Du) = P(u)\Delta_g u.$$

Indeed, for any $\eta\in C_0^1(M,\mathbb{R}^K)$, we have

$$\begin{aligned}\int_M\langle P(u)\Delta_g u,\eta\rangle &= \int_M\langle\Delta_g u, P(u)\eta\rangle\\ &= -\int_M\langle Du, P(u)D\eta\rangle - \int_M\langle Du, D(P(u))\eta\rangle\\ &= -\int_M(\langle Du, D\eta\rangle - \langle A(u)(Du,Du),\eta\rangle)\\ &= \int_M\langle\Delta_g u + A(u)(Du,Du),\eta\rangle.\end{aligned}$$

Now, we consider

$$\begin{aligned}&\int_{M\times[0,T]}|\langle J(u_i)P(u_i)\Delta_g u_i - J(u)P(u)\Delta_g u, v\rangle|\\ \leq &\int_{M\times[0,T]}|\langle(J(u_i)P(u_i) - J(u)P(u))\Delta_g u_i, v\rangle|\\ &+\int_{M\times[0,T]}|\langle J(u)P(u)(\Delta_g u_i - \Delta_g u), v\rangle|.\end{aligned}$$

When N is compact, obviously

$$\|D(J(\cdot)P(\cdot))\|_{L^\infty(N)} < \infty.$$

When N is noncompact, by the Sobolev embedding theorem, we can infer that there is a compact subset of N, denoted by $\mathscr{S}$, such that $u_i(M\times[0,T])\subset\mathscr{S}$ for i large enough and $u(M\times[0,T])\subset\mathscr{S}$. Hence, we also have

$$\|D(J(\cdot)P(\cdot))\|_{L^\infty(\mathscr{S})} < \infty.$$

Therefore, it is not difficult to see that every term on the right hand side of the last inequality converges to zero as i goes to infinity, no matter whether N is compact or not. Hence,

$$\lim_{i\to\infty}\int_0^T\int_M\langle J(u_i)P(u_i)\Delta_g u_i, v\rangle = \int_0^T\int_M\langle J(u)P(u)\Delta_g u, v\rangle.$$

On the other hand, we also have

$$\lim_{i\to\infty}\int_0^T\int_M\langle\partial_t u_i, v\rangle = -\int_0^T\int_M\langle u,\partial_t v\rangle + \int_M(\langle u(T), v(T)\rangle - \langle u_0, v(0)\rangle).$$

The last two equalities lead to

$$\int_0^T\int_M\langle J(u)P(u)\Delta_g u, v\rangle = -\int_0^T\int_M\langle u,\partial_t v\rangle + \int_M(\langle u(T), v(T)\rangle - \langle u_0, v(0)\rangle). \tag{3.27}$$

Noting $J(u)\tau(u)\in L^2(M\times[0,T], \mathbb{R}^K)$, (3.27) also implies that $\partial_t u\in L^2(M\times[0,T], \mathbb{R}^K)$. So, for any smooth function v we always have

$$\int_0^T\int_M\langle J(u)\tau(u), v\rangle = \int_0^T\int_M\langle\partial_t u, v\rangle,$$

this means that u is a strong solution.

The uniqueness of u follows Proposition 2.1 in ref. [1], when $k \geqslant m_0 + 3$. Obviously, u is smooth as k is large enough. Thus, the proof of Theorem 1.1 is complete.

Let N be a closed submanifold of the Euclidean space $\mathbb{R}^K$. We say $u \in W^{k,p}(\mathbb{R}^m, N)$ if and only if there is a point $O \in N$ such that if we take the origin of $\mathbb{R}^K$ at O then $u \in W^{k,p}(\mathbb{R}^m, \mathbb{R}^K)$ and $u(x) \in N$ for a.e. $x \in \mathbb{R}^m$. In order to prove Theorem 1.2, we need to use the following:

Lemma 3.4. Let $k > \frac{m}{2}$ and $u \in W^{k,2}(\mathbb{R}^m, N)$. Then there exists a sequence of maps $u_i \in W^{k,2}(\mathbb{R}^m, N) \cap C_0^\infty(\mathbb{R}^m, \mathbb{R}^K)$ such that $u_i \to u$ in $W^{k,2}(\mathbb{R}^m, N)$.

Proof. By the density theorem of Sobolev spaces, there exists a sequence $\{v_i\}$, $v_i \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^K)$ such that $v_i \to u$ in $W^{k,2}(\mathbb{R}^m, \mathbb{R}^K)$.

It is well-known that there exists a tubular neighbourhood $\mathscr{T}(N)$ of N such that the projection map $\pi: \mathscr{T}(N) \to N$, defined by

$$\operatorname{dist}(p, \pi(p)) = \operatorname{dist}(p, N) = \inf\{|p - q| : q \in N\}$$

for $p \in \mathscr{T}(N)$, is a smooth map. Define

$$u_i = \pi(v_i).$$

We need to show

$$\| D_a u_i - D_a u \|_{L^2} \to 0$$

for $|a| \leqslant k$.

It is well known that we have

$$Du_i = d\pi(v_i) Dv_i,$$

$$D^2 u_i = d\pi(v_i) D^2 v_i + d^2 \pi(v_i)(Dv_i, Dv_i).$$

In general, we have

$$D_a u_i = D_a \pi(v_i) = \sum B_{\sigma(a)}(v_i)(D_{a_1} v_i, \cdots, D_{a_s} v_i),$$

where the sum is over all multi-indices $a_1, \cdots, a_s$ such that $|a_j| \geqslant 1$ for all $j = 1, \cdots, s$ and

$$(a_1, \cdots, a_s) = \sigma(a)$$

is a permutation of a, and $B_{\sigma(a)}$ is a multi-linear form with uniformly bounded norm.

Since we have

$$\pi(u) = u, \quad D_a u = D_a \pi(u),$$

so

$$\begin{aligned} D_a u_i - D_a u &= D_a \pi(v_i) - D_a \pi(u) \\ &= \sum [B_{\sigma(a)}(v_i)(D_{a_1} v_i, \cdots, D_{a_s} v_i) - B_{\sigma(a)}(u)(D_{a_1} u, \cdots, D_{a_s} u)]. \end{aligned}$$

Set $v_i^t = u + t(v_i - u)$ for $t \in [0,1]$. Then, we get

$$\begin{aligned} |D_a u_i - D_a u| &= \left| \sum \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} B_{\sigma(a)}(v_i^t)(D_{a_1} v_i^t, \cdots, D_{a_s} v_i^t) \mathrm{d}t \right| \\ &\leqslant C \sum \int_0^1 |D^{j_1} v_i^t| \cdots |D^{j_l} v_i^t| \, |D^{j_{l+1}}(v_i - u)| \mathrm{d}t, \end{aligned}$$

where the sum is over all $j_1, \cdots, j_{l+1}$ such that $j_1 + \cdots + j_{l+1} \leqslant |a|$, $j_1, \cdots, j_l \geqslant 1$, $j_{l+1} \geqslant 0$.

It follows that

$$\| D_a u_i - D_a u \|_{L^2} \leqslant \sum \max_{t \in [0,1]} \| |D^{j_1} v_i^t| \cdots |D^{j_l} v_i^t| \, |D^{j_{l+1}}(v_i - u)| \|_{L^2}.$$

The proof that the right hand side of the above goes to 0 follows essentially from the proof of Proposition 2.2. The detailed estimates are hence omitted.

Proof of Theorem 1.2. Since u_0: $\mathbb{R}^m \to N$ belongs to $W^{k,2}(\mathbb{R}^m, N)$, by Lemma 3.4 we may always choose a sequence of C_0^∞-maps in $W^{k,2}(\mathbb{R}^m, N)$, denoted by $\{u_{i0}\}$, such that the supports of $\{u_{i0}\}$ are compact, and

$$u_{i0} \to u_0 \quad \text{in} \quad W^{k,2}, \text{ as } \quad i \to \infty.$$

Moreover, since $k > \frac{m}{2}$ we can apply (2.11) to prove easily

$$\| \nabla u_{i0} \|_{H^{k-1,2}} \to \| \nabla u_0 \|_{H^{k-1,2}}, \text{ as } i \to \infty.$$

If the support of u_{i0} is denoted by Ω_i, we may pick a large enough R_i such that $\Omega_i \subset\subset \widetilde{\Omega}_i \equiv \overbrace{[-R_i, R_i] \times \cdots \times [-R_i, R_i]}^{m}$. Thus, u_{i0} can be regarded as a function defined on a flat torus $T_i^m = \mathbb{R}^m/(2R_i \cdot \mathbb{Z})^m$.

We consider the following Cauchy problem

$$\begin{cases} \partial_t u(x,t) = J(u(x,t))\tau(u(x,t)) & \text{on } T_i^m \times (0,T], \\ u(x,0) = u_{i0} & u_{i0}: T_i^m \to N. \end{cases}$$

Since the constant in Proposition 2.1 does not depend on the diameter of a flat torus, by checking the proof of Lemma 3.1, we can see easily that there exists $T > 0$, which does not depend on i, such that the above Cauchy problem admits a unique, smooth solution u_i on $T_i^m \times [0, T]$. Furthermore, the following holds uniformly with respect to i:

$$\sup_{t \in [0,T]} \| \nabla u_i(t) \|_{H^{k-1,2}} \leqslant \widetilde{C}(T, \| \nabla u_0 \|_{H^{k-1,2}}).$$

It follows from Proposition 2.2 and the last inequality that

$$\sup_{t \in [0,T]} \| Du_i(t) \|_{W^{k-1,2}} \leqslant \widetilde{C}'(T, \| Du_0 \|_{W^{k-1,2}}).$$

We regard each u_i as a map from $\widetilde{\Omega}_i \times [0, T]$ into N. Therefore, there exists a $u \in L^\infty([0, T], W^{k,2}(\mathbb{R}^m, N))$ such that for any compact domain $\mathscr{S} \subset \mathbb{R}^m$

$$u_i \to u \text{ [weakly}^*\text{] in } L^\infty([0,T], W^{k,2}(\mathscr{S}, N))$$

upon extracting a subsequence and re-indexing if necessary. It is easy to see that u is a strong solution to the following Cauchy problem:

$$\begin{cases} \partial_t u(x,t) = J(u(x,t))\tau(u(x,t)) & \text{on } \mathbb{R}^m \times (0,T], \\ u(x,0) = u_0 & u_0: \mathbb{R}^m \to N. \end{cases}$$

From the process of the proof of Proposition 2.1 in ref. [1], we can see easily that u is unique when $k \geqslant m_0 + 3$. When k is large enough, u is smooth. Thus, the proof of Theorem 1.2 is complete.

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