Generalized bent functions and class group of imaginary quadratic fields

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Abstract Several new results on non-existence of generalized bent functions are presented. The results are related to the class number of imaginary quadratic fields.

Keywords: generalized bent functions, imaginary quadratic fields.

1 Preliminaries

Let q and n be positive integers, $q \geqslant 2$, $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$, and $\zeta_q = e^{\frac{2\pi i}{q}}$. A function $f: \mathbb{Z}_q^n \rightarrow \mathbb{Z}_q$ is called a generalized bent function (GBF) if the equality

$$\left| \sum_{x \in \mathbb{Z}_q^n} \zeta_q^{f(x) - x \cdot y} \right| = q^{n/2} \tag{1.1}$$

holds for every $y \in \mathbb{Z}_q^n$ where $x \cdot y$ stands for the dot product. We call $[n, q]$ the type of such GBF f. GBFs have been used in many fields such as code-division multiple-access communication systems and cryptography. For more background information of GBF and its applications, see ref. [1] and references therein.

Bent functions (for $q = 2$) are initiated by Rothaus[2] in 1976 and generalized by Kumar et al.[1] in 1985. For $q = 2$, Rothaus proved that there exists a bent function with type $[n, 2]$ if and only if n is even. The GBFs with type $[n, q]$ have been constructed in ref. [1] for even n or $q \not\equiv 2 \pmod 4$. From now on we assume that

(*) n is odd and $q = 2N$, $2 \nmid N \geqslant 3$.

So far there is no GBF being constructed in the case (*), but several non-existence results of GBF have been presented under the following extra conditions:

(A)[1] there exists an integer $s \geqslant 1$ such that

$$2^s \equiv -1 \pmod N, \tag{1.2}$$

(B)[3] $(n, q) = (1, 14)$,

(C)[4] $n = 1$ and $N = p^l$ where $l \geqslant 1$, p is a prime number such that $p \equiv 7 \pmod 8$ and $p \neq 7$,

(D)[5] $n = 1$, $N = p_1^{e_1} \cdots p_g^{e_g}$ where $g \geqslant 1$, $p_1, \cdots, p_g$ are distinct prime numbers, $e_i \geqslant 1$ $(1 \leqslant i \leqslant g)$ and for each $i (1 \leqslant i \leqslant g)$ there exists $s_i \geqslant 1$ such that

$$p_i^{s_i} \equiv -1 \left(\bmod \frac{N}{p_i^{e_i}} \right).$$

(D) is a generalization of (B) and (C). In this paper we present some new results on the non-existence of GBF with type $[n, q]$ for $2 \nmid n \geqslant 3$.

To show that our results (Theorems 3.1 and 4.1) are new ones, we need a closed form of condition (1.2). Such a closed form was presented in ref. [4] as follows. Let

$$N = \prod_{i=1}^{l} p_i^{e_i}$$

be the decomposition of an odd integer N where $p_i (1 \leqslant i \leqslant l)$ are distinct prime numbers and $e_i \geqslant 1 (1 \leqslant i \leqslant l)$. By the Chinese Remainder Theorem, condition (1.2) means that

$$2^s \equiv -1 \pmod{p_i^{e_i}} \ (1 \leqslant i \leqslant l). \tag{1.3}$$

We denote by $I(p)$ the 2-part of the order of $2 \pmod p$. It is easy to see that condition (1.3) is equivalent to the condition that

$$I(p_i) \ (1 \leqslant i \leqslant l) \text{ are the same even integers.} \tag{1.4}$$

It is easy to see that $I(p^e) = 2, 4, 1$ for $p \equiv 3, 5, 7 \pmod 8$, respectively, so that condition (1.2) contains exactly the following five cases:

(A_1) $N = \prod_{i=1}^{l} p_i^{e_i}$, $p_i \equiv 1 \pmod 8$ $(1 \leqslant i \leqslant l)$ and $I(p_i)$ $(1 \leqslant i \leqslant l)$ are the same even integers.

(A_2) $N = \prod_{i=1}^{l} p_i^{e_i}$, $p_i \equiv 3 \pmod 8$ $(1 \leqslant i \leqslant l)$,

(A_3) $N = \prod_{i=1}^{l} p_i^{e_i}$, $p_i \equiv 5 \pmod 8$ $(1 \leqslant i \leqslant l)$,

(A_4) $N = \prod_{i=1}^{l} p_i^{e_i} \cdot \prod_{j=1}^{s} p_j'^{f_j}$, $p_i \equiv 3 \pmod 8$ $(1 \leqslant i \leqslant l)$, $p_j' \equiv 1 \pmod 8$ and $I(p_j') = 2$ $(1 \leqslant j \leqslant s)$,

(A_5) $N = \prod_{i=1}^{l} p_i^{e_i} \cdot \prod_{j=1}^{s} p_j'^{f_j}$, $p_i \equiv 5 \pmod 8$ $(1 \leqslant i \leqslant l)$, $p_j' \equiv 1 \pmod 8$ and $I(p_j') = 4$ $(1 \leqslant j \leqslant s)$.

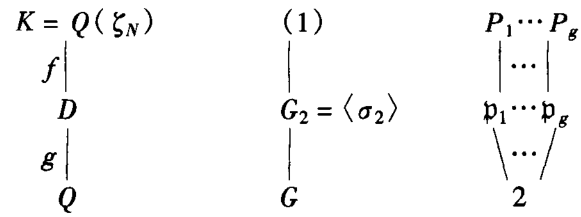
In sec. 3 we will present some new results on the non-existence of GBF in the case of $N = p^e p'^{e'}$.

At the end of this section, we explain the meaning of algebraic number theory of condition (1.3). Let K be the cyclotomic number field $\mathbb{Q}(\zeta_N)$. The Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$ by

$$G \simeq (\mathbb{Z}/N\mathbb{Z})^\times, \ \sigma_a \mapsto a \pmod N, \ (a, N) = 1, \tag{1.5}$$

where σ_a is the isomorphism determined by $\sigma_a(\zeta_N) = \zeta_N^a$. Let D be the decomposition field of 2 in K and $G_2 = \mathrm{Gal}(K/D)$ be the decomposition group of 2 in K. Then G_2 is the cyclic subgroup of G generated by σ_2. Therefore $f = |G_2| = [K:D]$ is the order of σ_2 in G or, by the isomorphism (1.5), f is the order of $2 \pmod N$. And $g = [D:\mathbb{Q}] = [K:\mathbb{Q}]/[K:D] = \frac{\varphi(N)}{f}$ where $\varphi(N)$ stands for the Euler function. For an algebraic number field F, we denote by O_F the ring of integers in F. Then 2 splits completely in $D: 2O_D = \mathfrak{p}_1 \cdots \mathfrak{p}_g$, and each $\mathfrak{p}_i$ is inertia in K.

By isomorphism (1.5), condition (1.3) means that σ_{-1}(the complex conjugation) belongs to $G_2 = \langle \sigma_2 \rangle$ which is also equivalent to the fact that D is a real field. All the results in the next two sections belong to the case $\sigma^{-1} \notin G_2$.



2 Two lemmas

In this and next two sections we fix the following notations.

n is an odd positive integer,

N is an odd positive integer, $N \geqslant 3$,

$q = 2N$,

$K = \mathbb{Q}(\zeta_N)$, $\zeta_N = e^{\frac{2\pi i}{N}}$,

$G = \mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_a : 1 \leqslant a \leqslant N-1,\ (a, N) = 1\}$,

D is the decomposition field of 2 in K,

$G_2 = \mathrm{Gal}(K/D) = \langle \sigma_2 \rangle$ stands for the decomposition group of 2 in K,

$f = [K:D] = |G_2|$ stands for the order of $2 \pmod N$,

O_F the ring of integers in an algebraic number field F,

$g = [D:\mathbb{Q}] = \frac{\varphi(N)}{f}$ stands for the number of prime ideals in O_K above 2.

Suppose that there exists a GBF with type $[n, q]$, $q = 2N$. Since N is odd, condition (1.1) implies that there exists $\xi \in O_K = \mathbb{Z}[\zeta_N]$ such that $\xi \bar{\xi} = q^n = 2^n N^n$ where $\bar{\xi} = \sigma_{-1}(\xi)$ is the complex conjugation of ξ. The first result we use in this paper is

Lemma 2.1[5]. Let $N = \prod_{i=1}^{l} p_i^{e_i}$ where $l \geqslant 1$, $p_1, \cdots, p_l$ are distinct prime numbers, $e_i \geqslant 1$ $(1 \leqslant i \leqslant l)$, and for each i $(1 \leqslant i \leqslant l)$ there exists a positive integer s_i such that

$$p_i^{s_i} \equiv -1 \left(\bmod \frac{N}{p_i^{e_i}}\right).$$

If we have $\xi \in O_K$ such that $\xi \bar{\xi} = 2^n N^n$, then we have $\alpha \in O_K$ such that $\alpha \bar{\alpha} = 2^n$.

Remark. The condition of Lemma 2.1 is trivial for $l = 1$ (so N is a power of an odd prime number). For $l \geqslant 2$, we denote by I_{ij} the 2-part of multiplicative order of $p_i \pmod{p_j}$. It is easy to see that the condition of Lemma 2.1 is equivalent to the saying that for each i $(1 \leqslant i \leqslant l)$, I_{ij} $(1 \leqslant j \leqslant l,\ j \neq i)$ are the same even integers (depending on i only).

The second result we need in this paper says that number α in Lemma 2.1 can be found in a smaller field.

Lemma 2.2. If $\alpha \bar{\alpha} = 2^n$ for some $\alpha \in O_K$, then there exists $\beta \in O_K$ such that $\beta^2 \in O_D$ and $\beta \bar{\beta} = 2^n$. Moreover, $\beta \in O_D$ if f is odd.

Proof. We follow the idea in the proof of the Lemma 2 in ref. [4], but make some simplifications. Since σ_2 fixes all prime ideals of O_K above 2, from $\alpha \bar{\alpha} = 2$ we know that

$$\alpha O_K = \sigma_2(\alpha O_K) = \sigma_2(\alpha) O_K.$$

Therefore $\sigma_2(\alpha) = \alpha\varepsilon$ where $\varepsilon \in U_k$ (the unit group of O_K). For each $\sigma \in G$,

$$\sigma(\alpha)\overline{\sigma(\alpha)} = \sigma(\alpha\bar{\alpha}) = 2^n, \quad \sigma\sigma_2(\alpha) = \sigma(\alpha\varepsilon) = \sigma(\alpha)\sigma(\varepsilon).$$

Thus

$$2^n = \sigma\sigma_2(\alpha)\overline{\sigma\sigma_2(\alpha)} = \sigma(\alpha)\sigma(\varepsilon)\overline{\sigma(\alpha)\sigma(\varepsilon)} = 2^n\sigma(\varepsilon)\overline{\sigma(\varepsilon)},$$

which means that $|\sigma(\varepsilon)| = 1$ for all $\sigma \in G$. Thus ε is a root of 1 in K, namely $\varepsilon = \pm\delta$ and $\delta = \zeta_N^i$ for some integer i. Let $\beta = \alpha\delta^{-1}$. Then $\beta\bar{\beta} = \alpha\bar{\alpha} = 2^n$ and

$$\sigma_2(\beta) = \sigma_2(\alpha)\sigma_2(\delta)^{-1} = \alpha\varepsilon\delta^{-2} = \pm\alpha\delta^{-1} = \pm\beta.$$

Therefore $\sigma_2(\beta^2) = \beta^2$ which means that $\beta^2 \in O_D$. Moreover, we have $D \subseteq D(\beta) \subseteq K$ and $[D(\beta):D] \leqslant 2$. If $f = [K:D]$ is odd, then $D(\beta) = D$ so that $\beta \in O_D$. This completes the proof of Lemma 2.2.

3 Non-existence result for case $N = p^l$, $p \equiv 7 \pmod 8$

Now we present some new result on non-existence of GBF for $n > 1$.

Theorem 3.1. Let $N = p^l$ where $l \geqslant 1$ $p \equiv 7 \pmod 8$, let f be the order of $2 \pmod{p^l}$, $s = \frac{g}{2}\left(= \frac{\varphi(p^l)}{2f} \text{ is odd}\right)$, let m be the smallest odd integer such that $x^2 + py^2 = 2^{m+2}$ has integral solution (x, y), and let n be an odd positive integer. If $n < m/s$, then there is no GBF with type $[n, p]$ $(q = 2N)$.

Proof. Suppose that there exists a GBF with type $[n, p]$. From (1.1) we know that $\beta\bar{\beta} = q^n = 2^nN^n = 2^np^{ln}$ for some $\beta \in O_K$. By Lemmas 2.1 and 2.2 we know that $\beta\bar{\beta} = 2^n$ for some $\beta \in O_D$. Let $E = \mathbb{Q}(\sqrt{-p})$ be the unique quadratic subfield of D. Then $[D:E] = \frac{g}{2} = s$ is odd. Let $\gamma = N_{D/E}(\beta)$. Then $\gamma\bar{\gamma} = N_{D/E}(\beta\bar{\beta}) = 2^{sn}$ and $\gamma \in O_E$ so that $\gamma = \frac{1}{2}(A + B\sqrt{-p})$ where $A, B \in \mathbb{Z}$. Therefore we have $A^2 + pB^2 = 4\gamma\bar{\gamma} = 2^{sn+2}$. By the definition of m we know $m \leqslant sn$. Therefore there is no GBF with type $[n, p]$ if $m > sn$. This completes the proof of Theorem 1.

Remark 1. Let p be a fixed odd prime number. For all $l \geqslant 1$, we denote by f_l the order of $2 \pmod{p^l}$ and $g_l = \varphi(p^l)/f_l$. It is easy to see that if $2^{p-1} \not\equiv 1 \pmod{p^2}$, then $f_l = p^{l-1}f_1$ and $g_l = \frac{\varphi(p^l)}{f_l} = \frac{p^{l-1}(p-1)}{p^{l-1}f_1} = g_1$ for all $l \geqslant 1$. It is a well-known fact that the formula $2^{p-1} \not\equiv 1 \pmod{p^2}$ holds for all odd prime numbers $p < 6 \times 10^9$ except $p = 1093$ and 3511 (see Ribenboim's book[6] for instance). Therefore we have $g_l = g_1$ for all $l \geqslant 1$ so that it is enough to compute $g = g_1$ for such a prime number p.

Remark 2. The definition of m is elementary; it has a clear algebraic number theory meaning. Since m is the smallest odd integer such that the equation $x^2 + py^2 = 2^{m+2}$ has integral solution $(x, y) = (A, B)$, we know that both of A and B should be odd, so that $\delta = \frac{1}{2}(A + B\sqrt{-p}) \in O_E$ $(E = \mathbb{Q}(\sqrt{-p}))$ and $\delta\bar{\delta} = 2^m$. We know that 2 splits in O_E as $2O_E = \mathfrak{p}\bar{\mathfrak{p}}$. The minimum property of m implies that $\delta O_E = \mathfrak{p}^m$ or of $\bar{\mathfrak{p}}^m$. Therefore $\mathfrak{p}^m$ is a principal ideal, so that m is a factor of the class number $h(-p)$ of $E = \mathbb{Q}(\sqrt{-p})$. By Gauss' genus theory we know that $h(-p)$ is odd for $p \equiv 7 \pmod 8$. On the other hand, we have $2^{m+2} = A^2 + pB^2 > p$ which gives a lower bound of m, $m > \frac{\log p}{\log 2} - 2$. Particularly we have $m \geqslant 3$ if $p \equiv 7 \pmod 8$ and $p \neq 7$. Therefore if $h(-p)$ is a prime number, then $m = h(-p)$.

Example 1. There are 11 pirme numbers $p \equiv 7 \pmod 8$ within 200. The following table shows the values of g, $h(-p)$ and m for these primes.

p	7	23	31	47	71	79	103	127	151	191	199
$g=2s$	2	2	6	2	2	2	2	18	10	2	2
$h(-p)$	1	3	3	5	7	5	5	5	7	13	9
m	1	3	3	5	7	5	5	5	7	13	0

For all $23 \leqslant p \leqslant 191$, $p \equiv 7 \pmod 8$ and $h(-p)$ are prime number so that $m = h(-p)$. For $p = 199$, $m \mid 9 = h(-199)$ and $m > \frac{\log 199}{\log 2} - 2 > 3$; thus $m = 9$.

For $p = 23, 47, 71, 79, 103, 191$, we have $s = \frac{g}{2} = 1$, so that there is no GBF with type $[n, 2p^l]$ for all $l \geqslant 1$ if n is odd and less than m.

From the above observation Theorem 1 has following corollaries.

Corollary 1. Suppose that $p \equiv 7 \pmod 8$, $p \geqslant 7$, $2^{p-1} \not\equiv 1 \pmod{p^2}$ and the order f of $2 \pmod p$ is $\frac{p-1}{2}$. Then there is no GBF with type $[n, 2p^l]$ for all $l \geqslant 1$ if n is odd and less than m where m is defined in Theorem 1.

Corollary 2. Suppose that $p \equiv 7 \pmod 8$, $p > 7$, $2^{p-1} \not\equiv 1 \pmod{p^2}$ and the class number $h(-p)$ of $\mathbb{Q}(\sqrt{-p})$ is a prime number. Then there is no GBF with type $[n, 2p^l]$ for all $l \geqslant 1$ if n is odd and less than $h(-p)/s$ where $s = \frac{p-1}{2f}$ and f is the order of $2 \pmod p$.

4 Non-existence result: $N = p^l p'^{l'}$

In this section we consider the case $N = p^l p'^{l'}$, where $l, l' \geqslant 1$, p and p' are distinct prime numbers, and do not belong to cases (A_1)—(A_5) in sec. 1. But we assume that N satisfies the condition of Lemma 2.1; that is, there exist positive integers s and s' such that

$$p^s \equiv -1 \pmod{p'^{l'}}, \quad p'^{s'} \equiv -1 \pmod{p^l}.$$

It is easy to see that this condition is equivalent to the following condition:

(*) both of the order of $p \pmod{p'}$ and the order of $p' \pmod p$ are even.

Theorem 4.1. Suppose that $N = p^l p'^{l'}$ where $l, l' \geqslant 1$, $p \equiv 3 \pmod 4$, $p' \equiv 5 \pmod 8$, and primes p and p' satisfy condition (*) (which is equivalent to $\left(\frac{p}{p'}\right) = -1$). Let f be the order of $2 \pmod N$, and $g = \frac{\varphi(N)}{f}$. Then g is even and $s = g/2$ is odd.

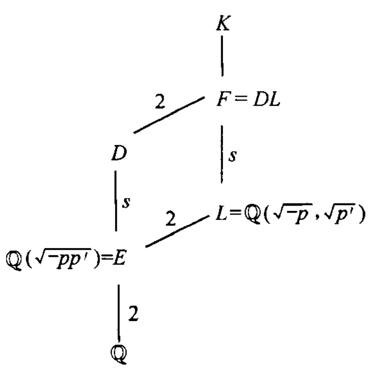
(1) In the case $p \equiv 3 \pmod 8$. Let $E = \mathbb{Q}(\sqrt{-pp'})$. Then $2O_E = P\overline{P}$ where P and $\overline{P}$ are distinct prime ideals in O_E. Let m be the smallest positive integer such that the equation $p'y^2 + pz^2 = 2^{m+2}$ has an integral solution (y, z). Then m is odd and the order of ideal class $[P]$ in the class group of E is $2m$. Moreover, there is no GBF with type $[n, 2N]$ if $2 \nmid n < m/s$.

(2) In the case $p \equiv 7 \pmod 8$. Let $E = \mathbb{Q}(\sqrt{-p})$. Then $2O_E = P\overline{P}$. Let m be the smallest odd integer such that the equation $x^2 + pz^2 = 2^{m+2}$ has an integral solution (x, z). Then m is the order of $[P]$ in the class group of E. Moreover, there is no GBF with type $[n, 2N]$ if $2 \nmid n < m/s$.

Proof. Let t and t_1 be the orders of $2 \pmod{p^l}$ and $\pmod{p'^{l'}}$, respectively. From $p \equiv 3 \pmod 4$ and $p' \equiv 5 \pmod 8$ we know that $t_1 = 4t'$ where t' is odd. Therefore $f = [t, 4t'] = 4a$ and a is odd, so that $s = \frac{g}{2} = \frac{\varphi(N)}{2f}$ is odd. Let D be the decomposition field of 2 in $K = \mathbb{Q}(\zeta_N)$. Then $[D:\mathbb{Q}] = g = 2s$.

Suppose that there exists GBF with type $[n, 2N]$ where $2 \nmid n \geqslant 1$. Then we have $\xi \in O_K$ such that $\xi\bar{\xi} = (2N)^n$. Since N satisties condition (*), by Lemma 2.1 we have $\alpha \in O_K$ such that $\alpha\bar{\alpha} = 2^n$. Then by Lemma 2.2 we have $\beta \in O_K$ such that $\beta^2 \in O_D$ and $\beta\bar{\beta} = 2^n$.

(1) Consider the case $p \equiv 3 \pmod 8$ first. Since $E = \mathbb{Q}(\sqrt{-pp'})$ is a subfield of K and 2 splits in E, we know that $E \subset D$ and $[D:E] = s$. From $\beta^2 \in O_D$ we know that β belongs to the unique quadratic extension F of D in K. By the Galois correspondence, $\mathrm{Gal}(K/F)$ is the cyclic subgroup of $G = \mathrm{Gal}(K/\mathbb{Q})$ generated by $\sigma_2^2 = \sigma_4$ from which we know that $L = \mathbb{Q}(\sqrt{-p}, \sqrt{p'})$ is a subfield of F and $F = DL$, $[F:L] = s$. Let $\gamma = N_{F/L}(\beta) \in O_L$. Then $\gamma\bar{\gamma} = N_{F/L}(\beta\bar{\beta}) = N_{F/L}(2^n) = 2^{ns}$ and $\gamma^2 = N_{F/L}(\beta^2) = N_{D/E}(\beta^2) \in O_E$.



It is well known that $\left\{1,\ \alpha = \frac{1+\sqrt{p'}}{2},\ \beta = \frac{1+\sqrt{-p}}{2},\ \alpha\beta\right\}$ is an integral basis of O_L (see exercise 42(d) of ref. [3], page 52 for instance). Therefore

$$\gamma = A + B\alpha + C\beta + D\alpha\beta \quad (A, B, C, D \in \mathbb{Z})$$
$$= \frac{1}{4}(X + Y\sqrt{p'} + Z\sqrt{-p} + W\sqrt{-pp'}),$$

where

$$X = 4A + 2B + 2C + D,\ Y = 2B + D,\ Z = 2C + D,\ W = D,$$

implying that $D = W$, $C = \frac{1}{2}(Z - W)$, $B = \frac{1}{2}(Y - W)$, $A = \frac{1}{4}(X - Y - Z + W)$ and

$$X \equiv Y \equiv Z \equiv W \pmod 2,\ X + W \equiv Y + Z \pmod 4. \tag{4.1}$$

The equality $\overline{\gamma}\gamma = 2^{ns}$ becomes

$$2^{ns+4} = X^2 + p'Y^2 + pZ^2 + pp'W^2 + 2(XY + pZW)\sqrt{p'};$$

that is, (X, Y, Z, W) satisfies congruences (4.1) and equations

$$\begin{cases} X^2 + p'Y^2 + pZ^2 + pp'W^2 = 2^{ns+4}, \\ XY = -pZW. \end{cases} \tag{4.2}$$

Note that $\gamma^2 \in O_E = \mathbb{Z} + \left[\frac{1 + \sqrt{-pp'}}{2}\right]\mathbb{Z}$. If $\gamma \in O_E$, then $Y = Z = 0$ and $X^2 + pp'W^2 = 2^{4+ns}$ which implies that $\left(\frac{2}{p}\right) = \left(\frac{2}{p}\right)^{4+ns} = 1$ since $2 \nmid ns$. But $\left(\frac{2}{p}\right) = -1$ for $p \equiv 3 \pmod 8$. Therefore $\gamma \notin O_E$ and $\gamma \in O_L$, so that $L = E(\gamma)$ ($\gamma^2 \in E$). Let σ be the non-trivial automorphism in Gal(L/E). Then $\sigma(\gamma) = -\gamma$ which means that $X - Y\sqrt{p'} - Z\sqrt{-p} + W\sqrt{-pp'} = -X - Y\sqrt{p'} - Z\sqrt{-p} - W\sqrt{-pp'}$. So we have $X = W = 0$. And congruence (4.1) becomes $Y \equiv Z \equiv 0 \pmod 2$, and (4.2) becomes $p'Y^2 + pZ^2 = 2^{4+ns}$. Let $y = \frac{Y}{2} \in \mathbb{Z}$, $z = \frac{Z}{2} \in \mathbb{Z}$. Then

$$p'y^2 + pz^2 = 2^{ns+2},\ y \equiv z \pmod 2. \tag{4.3}$$

Let m be the smallest positive integer such that the equation $p'y^2 + pz^2 = 2^{m+2}$ has integral solution $(y, z) = (A, B)$. From $-1 = \left(\frac{p'}{p}\right) = \left(\frac{2}{p}\right)^m = (-1)^m$ we know that m is odd. The minimality of m implies that $2 \nmid AB$, and

$$2^m p = \left(\frac{Bp + A\sqrt{-pp'}}{2}\right)\left(\frac{Bp - A\sqrt{-pp'}}{2}\right).$$

We have $2O_E = P\overline{P}$ and $pO_E = P'^2$. From minimality of m we know that $\left(\frac{Bp + A\sqrt{-pp'}}{2}\right)O_E = P^m P'$ or $\overline{P}^m P'$. Therefore $[P]^m[P'] = 1$. It is well known that $[P']$ is an ideal class of order 2. Therefore the order of $[P]$ is $2m$. From (3.3) and the minimality of m we have $sn \geqslant m$. Therefore there is no GBF with type $[n, 2N]$ if $2 \nmid n < \frac{m}{s}$.

(2) Next we consider the case $p \equiv 7 \pmod 8$. In this case we take $E = \mathbb{Q}(\sqrt{-p})$. By similar argument as in (1), we know that there exists

$$\gamma = \frac{1}{4}(X + Y\sqrt{p'} + Z\sqrt{-p} + W\sqrt{-pp'}) \in O_L,$$

$$\gamma^2 \in O_E = \mathbb{Z} + \left[\frac{1 + \sqrt{-p}}{2}\right]\mathbb{Z},\ \gamma\overline{\gamma} = 2^{ns}.$$

If $\gamma \notin O_E$, then $L = E(\gamma)$ ($\gamma^2 \in O_E$) and $X = Z = 0$. Therefore $\overline{\gamma\gamma} = 2^{ns}$ means that $2^{ns+4} = p'(Y^2 + pW^2)$ which is impossible. Thus $\gamma \in O_E$ which implies $Y = W = 0$, $X^2 + pZ^2 = 2^{ns+4}$ and $X \equiv Z \equiv 0 \pmod 2$. Let $x = \frac{X}{2} \in \mathbb{Z}$, $z = \frac{Z}{2} \in \mathbb{Z}$. Then

$$x^2 + pz^2 = 2^{ns+2}. \tag{4.4}$$

Let m be the smallest odd integer such that the equation $x^2 + pz^2 = 2^{m+2}$ has integral solution $(x, z) = (A, C)$. Then $2 \nmid AC$ and $\left(\frac{A + C\sqrt{-p}}{2}\right)\left(\frac{A - C\sqrt{-p}}{2}\right) = 2^m$. From $2O_E = P\overline{P}$ and the minimality of m we have $\left(\frac{A + C\sqrt{-p}}{2}\right)O_E = P^m$ or $\overline{P}^m$ so that m is the order of $[P]$. From (4.4) we know that $ns \geqslant m$. Therefore there is no GBF with type $[n, 2N]$ if $2 \nmid n < \frac{m}{s}$. This completes the proof of Theorem 2.

Remark. We denote the class number of $Q(\sqrt{-d})$ by $h(-d)$ (there is a big table in the ref. [2] for class number of imaginary quadratic fields). For $p \equiv 3 \pmod 8$ we have $h(-pp') = 2t$, $2 \nmid t$ and $m \mid t$. Particularly, if t is a prime number, then $m = t = h(-pp')/2$. For $p \equiv 7 \pmod 8$ and $p > 7$, $m \mid h(-p)$, $m > 1$ and $2 \nmid h(-p)$. If $h(-p)$ is prime number, then $m = h(-p)$.

Example 2. Computation shows that all (p, p') in the following table satisfies conditions (*) (which is equivalent to $\left(\frac{p'}{p}\right) = -1$) and $p \equiv 3 \pmod 8$, $p' \equiv 5 \pmod 8$. The values of s and $h(-pp')$ are listed in the table. If $h(-pp')/2$ is prime, then $m = h(-pp')/2$. Otherwise m can be determined by definition and the fact that $m \mid h(-pp')/2$.

(p,p')	(67,5)	(83,5)	(11,13)	(59,13)	(67,13)	(83,13)	(11,29)	(19,29)	(43,29)	(19,37)
s	1	1	1	1	1	1	1	1	3	1
$h(-pp')$	18	10	10	22	22	34	10	26	26	14
m	9	5	5	11	11	17	5	13	13	7

(p,p')	(59,37)	(3,53)	(19,53)	(67,53)	(83,53)	(11,61)	(43,61)	(59,61)	(67,61)
s	1	1	1	1	1	1	3	1	1
$h(-pp')$	42	10	30	58	50	30	22	66	30
m	21	5	15	29	25	15	11	33	5

By Theorem 4.1 we know that there is no GBF with types $[3, 2\cdot 43^l \cdot 29^{l'}]$ and $[3, 2\cdot 43^l \cdot 61^{l'}]$ for all $l, l' \geqslant 1$. For remaining cases in the table we have $s = 1$, so that there is no GBF with type $[n, 2p^l p'^{l'}]$ if $2 \nmid n < m$ (for $n = 1$ the result is known[5]).

Example 3. Computation shows that all (p, p') ($p \equiv 7 \pmod 8$, $p' \equiv 5 \pmod 8$) in the following table satisfy condition (*) (namely $\left(\frac{p'}{p}\right) = -1$). For all cases $s = 1$ and $h(-p)$ is prime so that $m = h(-p)$.

(p, p')	(47, 5) (47, 13) (47,29)	(71, 13) (71,53) (71,61)	(79,29) (79,37) (79,53) (79,61)
$m = h(-p)$	5	7	5

By Theorem 4.1 we know that there is no GBF with type $[n, 2p^l p'^{l'}]$ if $2 \nmid n < h(-p)$ and (p, p') belongs to the table (for $n = 1$ the result is known[5]).

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