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Positive solutions of the semilinear Dirichlet problem with critical growth in the unit disc in $\mathbb{R}^2$

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Abstract. We prove the existence of a positive solution of the following problem

$$-\Delta u = f(r, u) \quad \text{in } D$$
$$u > 0$$
$$u = 0, \quad \text{on } \partial D$$

where D is the unit disc in $\mathbb{R}^2$ and f is a superlinear function with critical growth.

Keywords. Sub-critical growth, critical growth, super critical growth; Laplacian; Palais-Smale condition; Semilinear Dirichlet problem; unit disc.

1. Introduction

Let D be the unit disc in $\mathbb{R}^2$. We are looking for positive radial solutions of the following problem: Find u in $C^2(D) \cap C^0(\bar{D})$ such that

$$\begin{aligned} -\Delta u &= f(r, u) \quad \text{in } D \\ u &> 0 \\ u &= 0, \quad \text{on } \partial D \end{aligned} \tag{1.1}$$

where f is superlinear, $f(r, 0) = 0$, $(\partial f/\partial t)(r, 0) < \lambda_1$ with λ_1 being the first eigenvalue of the Dirichlet problem. For $n \geq 3$ and f of critical growth, Brezis–Nirenberg [4] studied the existence and non-existence of solutions of problem (1·1). For $n = 2$, the critical growth is of exponential type whereas in the case of $n \geq 3$, it is of polynomial type and the method adopted for $n \geq 3$ fails in the case of $n = 2$.

Carleson–Chang [5] obtained a positive solution for $f(u) = \lambda u \exp(\lambda u^2)$ with $0 < \lambda < \lambda_1$ via a variational method. For growths of type $f(u) = u^m \exp(bu^2)$, Atkinson–Peletier [3] used the shooting argument to obtain a solution of (1.1). They assumed that $\log f$ is strictly convex for large u.

In this paper we relax the conditions on f and use a variational method to obtain a solution of (1.1). Since we are interested in radial solutions, (1.1) is equivalent to finding an u in $C^2(D)\cap C^0(\bar{D})$ with u radial and satisfying

$$\begin{aligned} L_1 u \equiv -(ru')' &= f(r,u)r \quad \text{in } [0,1) \\ u &> 0 \quad \text{in } [0,1) \\ u'(0) &= u(1) = 0. \end{aligned} \tag{1.2}$$

where $u' = \mathrm{d}u/\mathrm{d}r$.

The idea of the method is to approximate the energy functional by functionals satisfying Palais–Smale conditions. Then obtain the critical points of these approximate functionals by a constrained minimization problem similar to that of Zeev–Nehari [8] and then pass to the limit. The method of the proof is in the spirit of Brezis–Nirenberg [4]. Here, we also get a constant "a" which is strictly less than the best possible constant and thereby the existence of solutions of (1.2) is guaranteed.

In [1] we also prove the existence of infinitely many solutions of (1.1) when f is odd and of critical growth. Also in [2] we prove the existence of solutions of (1.1) if D is replaced by an arbitrary smooth domain.

2. Statements

Let $E = \{u \in C^1[0,1]; u(1) = 0\}$. For $0 \leqslant \alpha \leqslant 1$ and u in E define

$$|u|_\alpha^2 = \int_0^1 u^2(r) r^\alpha \,\mathrm{d}r$$

$$\|u\|_\alpha^2 = \int_0^1 u'(r)^2 r^\alpha \,\mathrm{d}r.$$

Let H_0^α be the completion of E with respect to $\|\cdot\|_\alpha$. Define the operator L_α by

$$L_\alpha = -\frac{1}{r^\alpha}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^\alpha \frac{\mathrm{d}}{\mathrm{d}r}\right). \tag{2.1}$$

Let $(\lambda_\alpha, \phi_\alpha)$ be the first eigenvalue and the corresponding first eigenvector with $\phi_\alpha(0) = 1$ of the following eigenvalue problem.

$$\begin{aligned} L_\alpha \phi &= \lambda\phi \quad \text{in } [0,1] \\ \phi'(0) &= \phi(1) = 0. \end{aligned} \tag{2.2}$$

DEFINITION 2.1

Let $f:[0,1]\times[0,\infty)\to[0,\infty)$ be a C^1-function. We say f is of class **A** if

(i) $f(r,0) = 0$.
(ii) There exists a $\delta_0 > 0$ and for $(r,t) \in Q_{\delta_0} \equiv [0,\delta_0]\times[0,\infty)(\partial f/\partial r)(r,t) \geqslant 0$.
(iii) There exists a $t_0 > 0$ such that $f(r,t) < \lambda_1 t$ for all $(r,t) \in [0,1]\times[0,t_0]$.
(iv) There exist constants $t_1 > 0$, $\beta > 2$ such that $\beta F(r,t) \leqslant f(r,t)t$ for all $(r,t) \in [0,1]\times[t_1,\infty)$ where $F(r,t) = \int_0^t f(r,s)\,\mathrm{d}s$.

Let

$$A' = \left\{ f \in A; \frac{\partial f}{\partial t} > \frac{f}{t} \text{ in } [0,1] \times (0,\infty) \right\}.$$

We consider the following three types of functions in our discussions.

Sub-critical: f in A is said to be sub-critical if there exists a $\delta > 0$ and for every $\varepsilon > 0$

$$\sup_{(r,t)\in[0,\delta]\times[0,\infty)} f(r,t)\exp(-\varepsilon t^2) < \infty \tag{2.3}$$

Critical: f in A' is said to be critical if there exists $\delta_1 > 0$ such that

(i) $f(r,t) = h(r,t)\exp(b(r)t^2) \quad \forall (r,t) \in Q_{\delta_1} \equiv [0,\delta_1] \times [0,\infty)$
(ii) $\forall \varepsilon > 0$,

$$\sup_{(r,t)\in Q_{\delta_1}} h(r,t)\exp(-\varepsilon t^2) < \infty \tag{2.4}$$

(iii) For every $\varepsilon > 0$, $h(0,t)\exp(\varepsilon t^2) \to \infty$ as $t \to \infty$.

Super critical: $f \in A'$ is said to be super critical if for every $c > 0$

$$\sup_{\|w\|_1 = 1} \int_0^1 f(r, cw) wr \, \mathrm{d}r = \infty. \tag{2.5}$$

For $f \in A$, $0 \leqslant \alpha \leqslant 1$, let $\sum_\alpha$ be the set of C^2-solutions of the following problem

$$\begin{aligned} L_\alpha u &= f(r,u) \quad \text{in } [0,1] \\ u &> 0 \\ u'(0) &= u(1) = 0. \end{aligned} \tag{2.6}$$

DEFINITION 2.2

u in $H_0^1(D)$ is said to be a weak solution of (1.2) if

(i) $u > 0 \quad \text{in } [0,1)$

(ii) $\displaystyle\int_0^1 f(r,u) ur \, \mathrm{d}r < \infty$ (2.7)

(iii) $\forall \phi \in C^2[0,1]$ with $\phi(1) = 0$

$$\int_0^1 u(L_1 \phi) r \, \mathrm{d}r = \int_0^1 f(r,u) \phi r \, \mathrm{d}r.$$

Since we are interested in only positive solutions of (1.2) and hence extending f for $t \leqslant 0$ is irrelevent. Therefore we make the following conventions.

1) Whenever we say f is in A, then we extend f by $f(r,t) = 0$ for $t \leqslant 0$ and $r \in [0,1]$.
2) Whenever we say f is in A', then we extend f by $f(r,t) = -f(r,-t)$ for $t \leqslant 0$. (2.8)

For u in H_0^α, define

$$\bar{I}_\alpha(u) = \tfrac{1}{2}\|u\|_\alpha^2 - \int_0^1 F(r,u)r^\alpha \,\mathrm{d}r. \tag{2.9}$$

$$l_\alpha = \inf_{\Sigma_\alpha} \bar{I}_\alpha.$$

Then we have

Theorem 2.1. *Let f be in A. Then there exists an $\alpha_0 < 1$ such that for every $\alpha_0 \leqslant \alpha < 1$, $\sum_\alpha$ is non-empty and $\{l_\alpha\}$ is bounded. Let $l = \underline{\lim}_{\alpha\to 1} l_\alpha$. Suppose there exists $b > 0$, $M > 0$ such that*

(i) $f(r,t) \leqslant M\exp(bt^2)$ for all $(r,t)\in[0,\delta]\times[0,\infty)$
(ii) $bl < 1$. (2.10)

Then there exists a solution u of (1.2).

COROLLARY 2.1

If f is sub-critical, then there exists a solution.

Proof. If f is sub-critical, we can take b as small as we want and satisfying (i) and (ii) of Theorem (2.1). Hence the solution exists.

Criterion to satisfy (2.10). Let f be in A satisfying (i) of Theorem (2.1). Suppose there exists an $m > 0$ such that

$$\int_0^{1/2} F\left(r, \frac{m}{2}\right) r\,\mathrm{d}r \geqslant 2m^2.$$
$$2m^2 b < 1 \tag{2.11}$$

Then f satisfies (ii) of Theorem (2.1).

For f in A^1 and for $0 \leqslant \alpha < 1$, define

$$B_\alpha = \left\{u\in H_0^\alpha\backslash\{0\};\ \|u\|_\alpha^2 \leqslant \int_0^1 f(r,u)ur^\alpha\,\mathrm{d}r\right\}$$

$$\partial B_\alpha = \left\{u\in B_\alpha;\ u\geqslant 0;\ \|u\|_\alpha^2 = \int_0^1 f(r,u)ur^\alpha\,\mathrm{d}r\right\}$$

$$B_1 = \left\{u\in H_0^1\cap L^\infty\backslash\{0\};\ \|u\|_1^2 \leqslant \int_0^1 f(r,u)ur\,\mathrm{d}r\right\}$$

$$B_1^* = \left\{u\in H_0^1\backslash\{0\}; u \text{ is non-increasing, } \|u\|_1^2 \leqslant \int_0^1 f(r,u)ur\,\mathrm{d}r\right\}$$

$$\partial(B_1\cup B_1^*) = \left\{u\in B_1\cup B_1^*;\ u\geqslant 0;\ \|u\|_1^2 = \int_0^1 f(r,u)ur\,\mathrm{d}r\right\}$$

$$B_{01} = \{u\in B_1;\ u \text{ is constant in a nhd of zero}\}.$$

$$\partial B_{01} = \left\{u\in B_{01};\ u\geqslant 0,\ \|u\|_1^2 = \int_0^1 f(r,u)ur\,\mathrm{d}r\right\}$$

For $0 \leqslant \alpha \leqslant 1$, $f \in A'$, u in H_0^α, define

$$I_\alpha(u) = \frac{1}{2}\int_0^1 f(r,u)ur^\alpha \, dr - \int_0^1 F(r,u)r^\alpha \, dr \tag{2.13}$$

since $f \in A'$; $f(r,t)t - 2F(r,t) \geqslant 0$ for all $(r,t) \in [0,1] \times \mathbb{R}$, hence $I_\alpha(u) \geqslant 0$. Define a_α by

$$\frac{a_\alpha^2}{2} = \inf_{\Sigma_\alpha} I_\alpha. \tag{2.14}$$

Theorem 2.2. *Let f be in A'. Then there exists an $\alpha_0 < 1$ such that for $\alpha_0 \leqslant \alpha < 1$, $\sum_\alpha$ is non-empty and $\{a_\alpha\}$ is bounded and satisfying*

$$\frac{a_\alpha^2}{2} = \inf_{B_\alpha} I_\alpha(u) = \inf_{\partial B_\alpha} I_\alpha(u). \tag{2.15}$$

Case 1. If f is super critical then $\lim_{\alpha \to 1} a_\alpha = 0$.

Case 2. If f is critical and suppose there exists a $t_2 > 0$ such that

$$\begin{aligned} &t_2 h\left(0, \left(\frac{2}{b(0)}\right)^{1/2} t_2\right) > 2\left(\frac{2}{b(0)}\right)^{1/2} \\ &\exp(-t_2) < \delta_1 \quad [\text{see } (2.4)] \end{aligned} \tag{2.16}$$

then $\lim_{\alpha \to 1} a_\alpha = a$ exists and is non-zero. Moreover there exists u satisfying (1.2) such that

$$I_1(u) = \frac{a^2}{2} = \inf_{B_1 \cup B_1^*} I_1 = \inf_{B_{01}} I_1 = \inf_{\partial B_{01}} I_1 \tag{2.17}$$

Remark 2.1. Suppose there exists a sequence $t_n \to \infty$ such that $h(0, t_n)t_n \to \infty$, then (2.16) is satisfied.

Examples

1. *Carleson–Chang.* Let $f_\lambda(t) = \lambda t \exp(\lambda t^2)$ for $0 < \lambda < \lambda_1$. Then f_λ is in A' and satisfies (2.16). Hence (1.1) has a solution.

2. *Atkinson–Peletier.* $f(t) = t^m \exp(bt^2)$, $m > 1$, $b > 0$. Then f is in A' satisfying (2.16). Hence (1.1) has a solution.

3. $f(t) = \lambda t^m \exp(bt^2 + \sin t^2)$, $\quad b \geqslant 1$

$$m = 1, \quad 0 < \lambda < \lambda_1,$$

$$m > 1, \quad \lambda > 0.$$

Then f is in A' and satisfying (2.16). Hence (1.1) has a solution. Here $\log f$ is not convex for large t.

4. Let $b(r)$ be a C^1-function on $[0,1]$ such that $0 \leqslant b(r) \leqslant 1$, $b(r) \equiv 1$ in a neighbourhood of zero. Let $f(r,t) = t^m \exp(b(r)t^2 + (1 - b(r))\exp(t))$. Then f is in A' satisfying (2.16). Hence (1.1) has a solution.

3. Proofs of theorems (2.1) and (2.2)

Lemma 3.1. For $0 \leqslant \alpha < 1$, we have

(i) *H_0^α is compactly embedded in $C[0,1]$.*
(ii) *$\lambda_\alpha < \lambda_1$ and $\lambda_\alpha \to \lambda_1$ as $\alpha \to 1$*
(iii) *u in H_0^1, $r_1 < r_2$,*

$$|u(r_1) - u(r_2)|^2 \leqslant \|u\|_1^2 \log\frac{r_2}{r_1}.$$

Proof. Let $r_1 \leqslant r_2$ and u is in H_0^α. Then by integration by parts

$$\begin{aligned}|u(r_2) - u(r_1)|^2 &= \left(\int_{r_1}^{r_2} u'(r)\,\mathrm{d}r\right)^2 \\ &\leqslant \|u\|_\alpha^2 \int_{r_1}^{r_2} r^{-\alpha}\,\mathrm{d}r \\ &= \|u\|_\alpha^2 \frac{r_2^{1-\alpha} - r_1^{1-\alpha}}{1-\alpha}. \qquad (3.1)\end{aligned}$$

Hence (i) follows from (3.1) and Arzela–Ascoli's theorem. Let u is in H_0^1, then

$$\begin{aligned}|u(r_2) - u(r_1)|^2 &= \left(\int_{r_1}^{r_2} u'(r)\,\mathrm{d}r\right)^2 \\ &\leqslant \|u\|_1^2\left(\int_{r_1}^{r_2} r^{-1}\,\mathrm{d}r\right) \\ &= \|u\|_1^2 \log\frac{r_2}{r_1}. \qquad (3.2)\end{aligned}$$

This proves (iii).

We have

$$-(r\phi_\alpha')' = \lambda_\alpha \phi_\alpha r - (1-\alpha)\phi_\alpha'$$
$$-(r\phi_1')' = \lambda_1 \phi_1 r.$$

Hence

$$\begin{aligned}\lambda_1 \int_0^1 \phi_1 \phi_\alpha r\,\mathrm{d}r &= -\int_0^1 (r\phi_1')'\phi_\alpha\,\mathrm{d}r \\ &= -\int_0^1 (r\phi_1')'\phi_\alpha\,\mathrm{d}r \\ &= \lambda_\alpha \int_0^1 \phi_\alpha \phi_1 r\,\mathrm{d}r - (1-\alpha)\int_0^1 \phi_\alpha' \phi_1\,\mathrm{d}r.\end{aligned}$$

i.e.

$$(\lambda_1 - \lambda_\alpha)\int_0^1 \phi_1 \phi_\alpha r\,\mathrm{d}r = -(1-\alpha)\int_0^1 \phi_\alpha' \phi_1\,\mathrm{d}r.$$

Since $\phi'_\alpha \leqslant 0$ and hence $\lambda_\alpha \leqslant \lambda_1$ and $\lambda_\alpha \to \lambda_1$ as $\alpha \to 1$. This proves (ii).

Lemma 3.2. Let f be in A, then there exists an $\alpha_0 < 1$ such that for $\alpha_0 \leqslant \alpha < 1$,

i) *$\bar{I}_\alpha$ satisfies the Palais–Smale condition.*
ii) *Let $m > 0$ be such that*

$$\int_0^{1/2} F\left(r, \frac{m}{2}\right) r \, \mathrm{d}r \geqslant 2m^2 \tag{3.3}$$

[*Such a m exists because of the condition* (iv) *of definition* (2.1)].
Then there exists a u_α in $C^2[0,1]$ satisfying

$$\begin{aligned} L_\alpha u_\alpha &= f(r, u_\alpha) \quad \text{in } [0,1) \\ u_\alpha &> 0 \\ u'_\alpha(0) &= u_\alpha(1) = 0. \end{aligned} \tag{3.4}$$

and

$$\bar{I}_\alpha(u_\alpha) \leqslant 2m^2.$$

Proof. Proof of this lemma is standard (see [7]). For the sake of completeness we will prove it.

Step 1. Let u_n in H_0^α be a sequence such that

$$\begin{aligned} &|\bar{I}_\alpha(u_n)| \leqslant M \\ &\bar{I}'_\alpha(u_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\beta \bar{I}_\alpha(u_n) - \langle \bar{I}'_\alpha(u_n), u_n \rangle \\ &= \left(\frac{\beta}{2} - 1\right) \int_0^1 u'_n(r)^2 r^\alpha \, \mathrm{d}r - \int_0^1 [\beta F(r, u_n) - f(r, u_n) u_n] r^\alpha \, \mathrm{d}r \\ &\geqslant \left(\frac{\beta}{2} - 1\right) \int_0^1 u'_n(r)^2 r^\alpha \, \mathrm{d}r - \int_{|u_n| \leqslant t_1} [\beta F(r, u_n) - f(r, u_n) u_n] r^\alpha \, \mathrm{d}r \\ &\geqslant \left(\frac{\beta}{2} - 1\right) \int_0^1 u'_n(r)^2 r^\alpha \, \mathrm{d}r + C, \end{aligned} \tag{3.6}$$

where C is a constant depending only on F. Since $\beta > 2$, (3.5) and (3.6) imply $\{\|u_n\|_\alpha\}$ is bounded. Let u_n converge to u weakly in H_0^α and strongly in $C[0,1]$.

$$\begin{aligned} \langle \bar{I}'(u_n), u_n - u \rangle &= \int_0^1 u'_n(r)^2 r^\alpha \, \mathrm{d}r - \int_0^1 u'_n(r) u'(r) r^\alpha \, \mathrm{d}r \\ &\quad - \int_0^1 f(r, u_n)(u_n - u) r^\alpha \, \mathrm{d}r \end{aligned} \tag{3.7}$$

(3.5) and (3.7) imply

$$\int_0^1 u'_n(r)^2 r^\alpha \, \mathrm{d}r \to \int_0^1 u'(r)^2 r^\alpha \, \mathrm{d}r.$$

Hence u_n converges strongly to u and this proves (i).

Step 2. From (ii) of Lemma (3.1) and (iii) of Definition (2.1) there exists an $\alpha_0 < 1$ and a $\lambda > 0$ such that

$$F(r,t) \leqslant \frac{\lambda t^2}{2} < \frac{\lambda_\alpha t^2}{2} \quad \text{for all } r \in [0,1], \quad 0 < |t| < t_0. \tag{3.8}$$

Let u in H_0^α be such that

$$\|u\|_\alpha^2 \leqslant \frac{(1-\alpha)}{2} t_0^2. \tag{3.9}$$

From (3.1) and (3.9) we have

$$|u(r)|^2 \leqslant t_0^2. \tag{3.10}$$

Hence (3.8) and (3.10) give

$$F(r,u(r)) \leqslant \frac{\lambda u(r)^2}{2} \tag{3.11}$$

$$\begin{aligned} \bar{I}_\alpha(u) &= \tfrac{1}{2}\|u\|_\alpha^2 - \int_0^1 F(r,u) r^\alpha \, \mathrm{d}r \\ &\geqslant \tfrac{1}{2}\|u\|_\alpha^2 - \frac{\lambda}{2}\int_0^1 u(r)^2 r^\alpha \, \mathrm{d}r \\ &\geqslant \tfrac{1}{2}\left[\|u\|_\alpha^2 - \frac{\lambda}{\lambda_\alpha}\|u\|_\alpha^2 \right] \\ &= \tfrac{1}{2}\left(1 - \frac{\lambda}{\lambda_\alpha}\right)\|u\|_\alpha^2. \end{aligned} \tag{3.12}$$

Hence zero is a local minima.

Step 3. Define u_0 in H_0^0 by

$$u_0(r) = \begin{cases} \dfrac{m}{2} & 0 \leqslant r < \tfrac{1}{2} \\ m(1-r) & \tfrac{1}{2} \leqslant r \leqslant 1 \end{cases} \tag{3.13}$$

Then

$$\begin{aligned} \bar{I}_\alpha(u_0) &= \frac{1}{2}\int_{1/2}^1 m^2 r^\alpha \, \mathrm{d}r - \int_0^1 F(r,u_0) r^\alpha \, \mathrm{d}r \\ &\leqslant \frac{m^2}{2(1+\alpha)}\left(1 - \frac{1}{2^{1+\alpha}}\right) - \int_0^{1/2} F(r,u_0) r^\alpha \, \mathrm{d}r \\ &\leqslant \frac{m^2}{2(1+\alpha)}\left(1 - \frac{1}{2^{1+\alpha}}\right) - \int_0^{1/2} F\left(r, \frac{m}{2}\right) r^\alpha \, \mathrm{d}r \\ &\leqslant \frac{m^2}{2(1+\alpha)}\left(1 - \frac{1}{2^{1+\alpha}}\right) - 2m^2 < 0 \end{aligned} \tag{3.14}$$

and for $0 \leqslant t \leqslant 1$,

$$\begin{aligned} \bar{I}_\alpha(tu_0) &\leqslant \frac{t^2}{2} \| u_0 \|_\alpha^2 \\ &\leqslant \frac{m^2}{2(1+\alpha)}\left(1 - \frac{1}{2^{1+\alpha}}\right) \leqslant 2m^2 \end{aligned} \tag{3.15}$$

Hence $\bar{I}_\alpha$ satisfies all the hypotheses of Mountain pass theorem and hence there exists a critical point u_α of $\bar{I}_\alpha$ such that

$$\bar{I}_\alpha(u_\alpha) \leqslant \sup_{t \in [0,1]} \bar{I}_\alpha(tu_0).$$

Now from (3.15) it follows that

$$\bar{I}_\alpha(u_\alpha) \leqslant 2m^2$$

and u_α satisfies (3.4).

Lemma 3.3. Let f be in A', then there exists $\alpha_0 < 1$ such that for all $\alpha_0 \leqslant \alpha < 1$, $\sum_\alpha$ is non-empty and an $u_\alpha \in \sum_\alpha$ satisfying

$$\frac{a_\alpha^2}{2} = I_\alpha(u_\alpha) = \inf_{u \in \partial B_\alpha} I_\alpha(u) = \inf_{u \in B_\alpha} I_\alpha(u) \tag{3.16}$$

and for all w in H_0^α, $\| w \|_\alpha = 1$,

$$\int_0^1 f(r, a_\alpha w) w r^\alpha \, \mathrm{d}r \leqslant a_\alpha. \tag{3.17}$$

Proof. Let u be in B_α. Define $\gamma \leqslant 1$ such that

$$\| u \|_\alpha^2 = \frac{1}{\gamma} \int_0^1 f(r, \gamma u) u r^\alpha \, \mathrm{d}r. \tag{3.18}$$

Such a γ exists because $f(r,t)/t$ is an increasing function and u is in B_α and $|f(r,t)| < \lambda_\alpha |t|$ for $|t| < t_0$; $\alpha_0 \leqslant \alpha < 1$.

Define $v = \gamma u$, then

$$\begin{aligned} \| v \|_\alpha^2 = \gamma^2 \| u \|_\alpha^2 &= \int_0^1 f(r, \gamma u)(\gamma u) r^\alpha \, \mathrm{d}r \\ &= \int_0^1 f(r, v) v r^\alpha \, \mathrm{d}r. \end{aligned} \tag{3.19}$$

Hence v is in ∂B_α and since $\gamma \leqslant 1$, and $f \in A'$, we have

$$I_\alpha(v) = I_\alpha(\gamma u) \leqslant I_\alpha(u).$$

this together with $\partial B_\alpha \subset B_\alpha$ imply that

$$d_\alpha = \inf_{\partial B_\alpha} I_\alpha = \inf_{B_\alpha} I_\alpha. \tag{3.20}$$

Let u_n in ∂B_α be a sequence such that $u_n \geqslant 0$ and $I_\alpha(u_n) \to d_\alpha$. Such a sequence exists because for u in ∂B_α implies $|u|$ is in ∂B_α and $I_\alpha(u) = I_\alpha(|u|)$.

We claim that $\{\|u_n\|_\alpha\}$ is bounded. Let N be such that for all $n \geqslant N$,

$$d_\alpha \leqslant I_\alpha(u_n) \leqslant d_\alpha + 1 \tag{3.21}$$

$$\begin{aligned} d_\alpha + 1 \geqslant I_\alpha(u_n) &= \frac{1}{2}\int_0^1 [f(r,u_n)u_n - 2F(r,u_n)]r^\alpha \, dr \\ &= \frac{1}{2}\int_0^1 [f(r,u_n)u_n - \beta F(r,u_n)]r^\alpha \, dr \\ &\quad + \left(\frac{\beta}{2} - 1\right)\int_0^1 F(r,u_n)r^\alpha \, dr. \end{aligned} \tag{3.22}$$

From (iv) of Definition (2.1), there exists a constant C depending only on f such that for all v in H_0^α,

$$\int_0^1 [f(r,v)v - \beta F(r,v)]r^\alpha \, dr \geqslant C. \tag{3.23}$$

From (3.22) and (3.23) there exists a constant C_1 independent of n such that

$$\int_0^1 F(r,u_n)r^\alpha \, dr \leqslant C_1. \tag{3.24}$$

From (3.21) and (3.24) we have

$$\begin{aligned} \|u\|_\alpha^2 &= 2I_\alpha(u_n) + 2\int_0^1 F(r,u_n)r^\alpha \, dr \\ &\leqslant 2(d_\alpha + 1) + 2C_1 \end{aligned}$$

and this proves the claim.

Let $u_\alpha =$ weak limit of u_n and α_0 be as in Lemma (3.2). We claim that for $\alpha_0 \leqslant \alpha < 1$, $u_\alpha \in \sum_\alpha$ satisfying (3.16).

First we will show that u_α is non-zero. Suppose $u_\alpha \equiv 0$, then from Lemma (3.1). u_n converges to 0 in $C[0,1]$. Let N be an integer such that

$$u_n(r) < t_0 \quad \text{for all } n \geqslant N, r \in [0,1]. \tag{3.25}$$

Then from (iii) of Definition (2.1) and the choice of α_0,

$$f(r,u_n(r)) < \lambda_\alpha u_n(r). \tag{3.26}$$

Since $u_n \in \partial B_\alpha$, we have from (3.26)

$$\begin{aligned} \|u_n\|_\alpha^2 &= \int_0^1 f(r,u_n)u_n r^\alpha \, dr \\ &< \lambda_\alpha \int_0^1 u_n(r)^2 r^\alpha \, dr \leqslant \|u_n\|_\alpha^2 \end{aligned}$$

which is a contradiction and hence $u_\alpha \not\equiv 0$ and

$$I_\alpha(u_\alpha) = \lim_{n\to\infty} I_\alpha(u_n) = d_\alpha$$

$$\|u_\alpha\|_\alpha^2 \leq \varliminf_{n\to\infty} \|u_n\|_\alpha^2 = \int_0^1 f(r, u_\alpha)u_\alpha r^\alpha \, dr, \tag{3.27}$$

u_α is in ∂B_α. If not, then by (3.27) we can choose a $\gamma < 1$ such that

$$\|u_\alpha\|^2 = \frac{1}{\gamma}\int_0^1 f(r, \gamma u_\alpha)u_\alpha r^\alpha \, dr.$$

Then γu_α is in ∂B_α and

$$d_\alpha \leq I(\gamma u_\alpha) < I(u_\alpha) = d_\alpha.$$

This proves that u_α is in ∂B_α. Since u_α is a minimizer and hence there exists a real number ρ such that for all ϕ in H_0^α,

$$\int_0^1 u_\alpha'(r)\phi'(r)r^\alpha \, dr - \int_0^1 f(r, u_\alpha)\phi r^\alpha \, dr$$
$$= \rho\left\{2\int_0^1 u_\alpha'(r)\phi'(r)r^\alpha \, dr - \int_0^1 f(r, u_\alpha)\phi r^\alpha \, dr - \int_0^1 \frac{\partial f}{\partial t}(r, u_\alpha)u_\alpha \phi r^\alpha \, dr\right\}. \tag{3.28}$$

Putting $\phi = u_\alpha$ in (3.28) and using the fact that $u_\alpha \in \partial B_\alpha$, we have

$$\rho\left\{2\int_0^1 u_\alpha'(r)^2 r^\alpha \, dr - \int_0^1 f(r, u_\alpha)u_\alpha r^\alpha \, dr - \int_0^1 \frac{\partial f}{\partial t}(r, u_\alpha)u_\alpha(r)^2 r^\alpha \, dr\right\} = 0.$$

Since u_α is in ∂B_α, we have

$$\rho \int_0^1 \left[\frac{f(r, u_\alpha)}{u_\alpha} - \frac{\partial f}{\partial t}(r, u_\alpha)\right] u_\alpha(r)^2 r^\alpha \, dr = 0.$$

Since f is in A', and u is not zero, it implies that $\rho = 0$. Hence from (3.28) and by regularity of elliptic operator, it follows that u_α is in $\sum_\alpha$ and $I_\alpha(u_\alpha) = d_\alpha$. Since $\sum_\alpha \subset \partial B_\alpha$, we have $a_\alpha^2/2 = \inf_{\sum_\alpha} I_\alpha = I_\alpha(u_\alpha) = d_\alpha$ and this proves (3.16). Let $\|w\|_\alpha = 1$. Choose $\gamma > 0$ such that

$$1 = \frac{1}{\gamma}\int_0^1 f(r, \gamma w)w r^\alpha \, dr. \tag{3.29}$$

Then γw is in ∂B_α. Hence

$$\frac{a_\alpha^2}{2} \leq I_\alpha(\gamma w) \leq \frac{\gamma^2}{2}\|w\|_\alpha^2 = \frac{\gamma^2}{2}$$

implies $a_\alpha \leqslant \gamma$. Since f is in A', we have

$$\frac{1}{a_\alpha}\int_0^1 f(r, a_\alpha w)wr^\alpha \,dr \leqslant \frac{1}{\gamma}\int_0^1 f(r, \gamma w)wr^\alpha \,dr = 1$$

i.e.

$$\int_0^1 f(r, a_\alpha w)wr^\alpha \,dr \leqslant a_\alpha$$

proving (3.17).

Lemma 3.4. Let f be in A' and α_0 is as in Lemma (3.3). *Then $\{a_\alpha\}$ is bounded on* $[\alpha_0, 1)$. *Let $a = \overline{\lim}_{\alpha\to 1} a_\alpha$. Then for all $w \in H_0^1$ with $\|w\|_1 = 1$. we have*

$$\int_0^1 f(r, aw)wr \,dr \leqslant a. \tag{3.30}$$

Proof. From Lemma (3.2) and (3.3) we have $l_\alpha = a_\alpha^2/2$ and $l_\alpha \leqslant 2m^2$. Hence $\{a_\alpha\}$ is bounded on $[\alpha_0, 1)$. Let α_n be a sequence such that $a_{\alpha_n} \to a$ as $\alpha_n \to 1$ and w be in E with $\|w\|_1 = 1$. Let $v_n = w/\|w\|_{\alpha_n}$. Then from (3.17) we have

$$\int_0^1 f(r, a_{\alpha_n} v_n)v_n r^\alpha \,dr \leqslant a_{\alpha_n}.$$

Letting $\alpha_n \to 1$, $v_n \to w$, $a_{\alpha_n} \to a$, we get

$$\int_0^1 f(r, aw)wr \,dr \leqslant a. \tag{3.31}$$

Since f is odd, and hence by Fatou's (3.31) holds for all w in H_0^1.

Lemma 3.5. Let f be in A, $0 \leqslant \alpha < 1$, $0 \leqslant \varepsilon \leqslant 1$, and u in $\sum_\alpha$. Then we have

$$u(r) = \frac{1 - r^{1-\alpha}}{1 - \alpha}\int_0^r f(t, u(t))t^\alpha \,dt + \int_r^1 t^\alpha\left(\frac{1 - t^{1-\alpha}}{1 - \alpha}\right) f(t, u(t))\,dt \tag{3.32}$$

$$\tfrac{1}{2}\varepsilon^{1+\alpha}u'(\varepsilon)^2 = (1 + \alpha)\int_0^\varepsilon F(r, u)r^\alpha \,dr + \int_0^\varepsilon \frac{\partial F}{\partial r}(r, u)r^{1+\alpha}\,dr$$
$$+ \frac{1 - \alpha}{2}\int_0^\varepsilon u'(r)^2 r^\alpha \,dr - \varepsilon^{1+\alpha}F(\varepsilon, u(\varepsilon)). \tag{3.33}$$

Proof. If $v(r)$ is the right hand side of (3.32), then by differentiating twice, v satisfies

$$\begin{aligned} L_\alpha v &= f(r, u) \\ v'(0) &= v(1) = 0. \end{aligned} \tag{3.34}$$

Hence by uniqueness, $v = u$. This proves (3.32). u is in $\sum_\alpha$, hence

$$(r^\alpha u')' = -f(r, u(r))r^\alpha. \tag{3.35}$$

multiply (3.35) by $ru'(r)$ and integrate from 0 to ε we get

$$\int_0^\varepsilon (r^\alpha u'(r))' u'(r) r \, \mathrm{d}r = -\int_0^\varepsilon f(r,u) u' r^{1+\alpha} \, \mathrm{d}r. \tag{3.36}$$

Since $(\mathrm{d}F/\mathrm{d}r)(r,u(r)) = (\partial F/\partial r)(r,u(r)) + f(r,u(r))u'(r)$, we have

$$\begin{aligned}\tfrac{1}{2}\varepsilon^{1+\alpha}u'(\varepsilon)^2 - \frac{(1-\alpha)}{2}\int_0^\varepsilon u'(r)^2 r^\alpha \, \mathrm{d}r &= -\int_0^\varepsilon \frac{\mathrm{d}F}{\mathrm{d}r} r^{1+\alpha} \, \mathrm{d}r + \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+\alpha} \, \mathrm{d}r \\ &= -F(\varepsilon, u(\varepsilon))\varepsilon^{1+\alpha} + (1+\alpha)\int_0^\varepsilon F(r,u) r^\alpha \, \mathrm{d}r \\ &\quad + \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+\alpha} \, \mathrm{d}r.\end{aligned}$$

Hence

$$\begin{aligned}\tfrac{1}{2}\varepsilon^{1+\alpha}u'(\varepsilon)^2 = (1+\alpha)\int_0^\varepsilon F(r,u) r^\alpha \, \mathrm{d}r + \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+\alpha} \, \mathrm{d}r + \frac{1-\alpha}{2}\int_0^\varepsilon u'(r)^2 r^\alpha \, \mathrm{d}r \\ - F(\varepsilon, u(\varepsilon))\varepsilon^{1+\alpha}.\end{aligned}$$

This proves (3.33).

Lemma 3.6. Let f be in A, $\alpha_n \to 1$, u_n is in $\sum_{\alpha_n}$ and a constant M independent of n such that

$$\begin{aligned}&\text{(i)}\ \ \|u_n\|_{\alpha_n} \leqslant M \\ &\text{(ii)}\ \lim_{n\to\infty} u_n'(1) = \eta \neq 0.\end{aligned} \tag{3.37}$$

Then there exists a subsequence (still denoted by α_n) such that the weak limit u of u_n in H_0^1 is a weak solution of (1.2). *Furthermore*

$$\lim_{n\to\infty} \int_0^1 F(r,u_n) r^{\alpha_n} \, \mathrm{d}r = \int_0^1 F(r,u) r \, \mathrm{d}r. \tag{3.38}$$

Proof. $\|u_n\|_1 \leqslant \|u_n\|_{\alpha_n} \leqslant M$, hence by going to a subsequence the weak limit u of u_n in H_0^1 exists. From (iii) of Lemma (3.1), u_n converges to u uniformly on compact subsets of $(0,1]$. We claim u is not identically zero. For, if $u \equiv 0$, then, since u_n in $\sum_{\alpha_n}$, we have for $0 < r \leqslant 1$,

$$r^{\alpha_n} u_n'(r) = u_n'(1) + \int_r^1 f(r,u_n) r^{\alpha_n} \, \mathrm{d}r. \tag{3.39}$$

From (ii) of (3.37) and (3.39) and using $u_n \to 0$ on $[r,1]$ uniformly

$$r \lim_{n\to\infty} u_n'(r) = \eta. \tag{3.40}$$

Hence by Fatou's lemma, and (3.40)

$$\infty = \eta^2 \int_0^1 \frac{r\,\mathrm{d}r}{r^2} < \int_0^1 \varliminf_{n\to\infty} u_n'(r)^2 r^{\alpha_n} \, \mathrm{d}r \leqslant \varliminf \|u_n\|_{\alpha_n}^2 \leqslant M$$

which is a contradiction. Hence $u \not\equiv 0$ and u satisfies

$$\begin{aligned} -(ru')' &= f(r,u)r \quad \text{in } (0,1] \\ u(1) &= 0. \end{aligned} \tag{3.41}$$

Now by Fatous, we have

$$\int_0^1 f(r,u)ur\,\mathrm{d}r \leqslant \underline{\lim} \int_0^1 f(r,u_n)u_n r^\alpha\,\mathrm{d}r \leqslant M. \tag{3.42}$$

Hence

$$\int_0^1 f(r,u)r\,\mathrm{d}r \leqslant \int_{u\leqslant 1} f(r,u)r\,\mathrm{d}r + \int_{u>1} f(r,u)ur\,\mathrm{d}r < \infty. \tag{3.43}$$

For any $0 < r \leqslant 1$, integrating (3.41) from r to 1, we get

$$ru'(r) = u'(1) + \int_r^1 f(t,u)t\,\mathrm{d}t. \tag{3.44}$$

(3.44) gives $ru'(r)$ is monotone and hence limit $r \to 0$ exists. We claim that

$$\lim_{r\to 0} ru'(r) = 0. \tag{3.45}$$

For, if $\lim_{r\to 0} ru'(r) = C < 0$, then there exists $\varepsilon > 0$ such that $-u'(r) \geqslant C/r$ for $0 < r \leqslant \varepsilon$. Hence

$$\infty = C^2 \int_0^\varepsilon \frac{r\,\mathrm{d}r}{r^2} \leqslant \int_0^\varepsilon ru'(r)^2\,\mathrm{d}r < \infty.$$

Hence (3.45) is true. Using (3.44) and (3.45) we get

$$u'(1) = -\int_0^1 f(t,u)t\,\mathrm{d}t. \tag{3.46}$$

Let ϕ be in $C^2[0,1]$ with $\phi(1) = 0$. Multiply ϕ' to (3.44) and integrate from 0 to 1, and using (3.46) we have

$$\begin{aligned} \int_0^1 u'(r)\phi'(r)r\,\mathrm{d}r &= u'(1)(\phi(1) - \phi(0)) + \int_0^1 \phi'(r) \int_r^1 f(t,u)t\,\mathrm{d}t\,\mathrm{d}r \\ &= u'(1)(\phi(1) - \phi(0)) + \int_0^1 f(t,u)\phi(t)t\,\mathrm{d}t \\ &\quad - \phi(0) \int_0^1 f(t,u)t\,\mathrm{d}t \\ &= \int_0^1 f(t,u)\phi(t)t\,\mathrm{d}t \end{aligned}$$

and hence u is a weak solution of (1.2).

From (3.33) and (3.37) we have

$$\lim_{n\to\infty}\left\{(1+\alpha_m)\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r+\int_0^1\frac{\partial F}{\partial r}r^{1+\alpha_n}\,\mathrm{d}r\right\}=\tfrac{1}{2}\eta^2 \tag{3.47}$$

Now multiply $ru'(r)$ to (3.41) and integrate from r to 1, we have

$$-\tfrac{1}{2}r^2u'(r)^2+\tfrac{1}{2}u'(1)^2=-\int_r^1\frac{\mathrm{d}F}{\mathrm{d}t}t^2\,\mathrm{d}t+\int_{r_1}^1\frac{\partial F}{\partial t}t^2\,\mathrm{d}t$$
$$=F(r,u(r))r^2+2\int_r^1 F(t,u)t+\int_r^1\frac{\partial F}{\partial t}t^2\,\mathrm{d}t. \tag{3.48}$$

Since $ru'(r)\to 0$, $\int_0^1 F(t,u)t\,\mathrm{d}t<\infty$, $\partial F/\partial r>0$ in $[0,\delta_0]$ and $\int_{\delta_0}^1(\partial F/\partial t)t^2\,\mathrm{d}t<\infty$, we conclude that $\lim_{r\to 0}F(r,u(r))r^2$ exists and claim that

$$\lim_{r\to 0}F(r,u(r))r^2=0. \tag{3.49}$$

If not, there exists a constant $C>0$ and $\varepsilon>0$ such that

$$F(r,u(r))r^2\geqslant C\quad\text{for all } 0<r<\varepsilon.$$

Hence

$$\infty=\int_0^\varepsilon\frac{C}{r}\,\mathrm{d}r\leqslant\int_0^\varepsilon F(r,u(r))r\,\mathrm{d}r<\infty$$

which is a contradiction.

Now using (3.49), (3.48) becomes

$$\tfrac{1}{2}u'(1)^2=2\int_0^1 F(r,u)r\,\mathrm{d}r+\int_0^1\frac{\partial F}{\partial r}(r,u)r^2\,\mathrm{d}r. \tag{3.50}$$

Since $u'(1)=\lim_{n\to\infty}u_n'(1)$, and hence from (3.47) and (3.50) we have

$$2\int_0^1 F(r,u)r\,\mathrm{d}r+\int_0^1\frac{\partial F}{\partial r}(r,u)r^2\,\mathrm{d}r$$
$$=\lim_{n\to\infty}\left\{(1+\alpha_n)\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r+\int_0^1\frac{\partial F}{\partial r}(r,u_n)r^{1+\alpha_n}\,\mathrm{d}r\right\}. \tag{3.51}$$

By Fatou's and using (ii) of Definition (2.1) we have

$$\begin{aligned}&2\int_0^1 F(r,u)r\,\mathrm{d}r\leqslant\underline{\lim}(1+\alpha_n)\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r\\&\int_0^1\frac{\partial F}{\partial r}(r,u)r^2\,\mathrm{d}r\leqslant\underline{\lim}\int_0^1\frac{\partial F}{\partial r}(r,u_n)r^{\alpha_n+1}\,\mathrm{d}r.\end{aligned} \tag{3.52}$$

By going to a subsequence, we conclude from (3.51) and (3.52) that

$$\lim_{n\to\infty}(1+\alpha_n)\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r=2\int_0^1 F(r,u)r\,\mathrm{d}r$$

and

$$\lim_{n\to\infty}\int_0^1 \frac{\partial F}{\partial r}(r,u_n)r^{\alpha_n+1}\,dr = \int_0^1 \frac{\partial F}{\partial r}(r,u)r^2\,dr.$$

Lemma 3.7. Let f in A' be critical. Then

$$\frac{2}{b(0)} = \sup\left\{c^2;\ \sup_{\|w\|_1\leqslant 1}\int_0^1 f(r,cw)wr\,dr<\infty\right\} \tag{3.53}$$

Proof. $f = h(r,t)\exp[b(r)t^2]$ for (r,t) in Q_{δ_1}.
Let

$$C_0^2 = \sup\left\{c^2;\ \sup_{\|w\|_1\leqslant 1}\int_0^1 f(r,cw)wr\,dr<\infty\right\}$$

Step 1. $C_0^2 \not> 2/b(0)$.
If not, then choose $\varepsilon>0$, $c>0$ and a $\delta<(\delta_1,\delta_0)$ such that

$$\frac{2}{b(0)} < c^2 < (c+\varepsilon)^2 < C_0^2. \tag{3.54}$$

For $r_0\in[0,\delta_1]$, define

$$\begin{aligned} W_{r_0}(r) &= \frac{\log\frac{1}{r}}{\left(\log\frac{1}{r_0}\right)^{1/2}} \quad \text{for } r_0\leqslant r\leqslant 1\\ W_{r_0}(r) &= \left(\log\frac{1}{r_0}\right)^{1/2} \quad \text{for } 0\leqslant r\leqslant r_0. \end{aligned} \tag{3.55}$$

Then $\|w_{r_0}\|_1 = 1$. Since $(\partial f/\partial r)(r,t)\geqslant 0$ in Q_{δ_0}, we have

$$h(0,t)\exp[b(0)t^2] \leqslant h(r,t)\exp[b(r)t^2] \quad \text{in } Q_{\delta_0}.$$

Now $(c+\varepsilon)^2 < C_0^2$ implies that there exists an absolute constant M depending only on $(c+\varepsilon)$ and f such that

$$\begin{aligned} M &\geqslant \int_0^1 f(r,(c+\varepsilon)w_{r_0})w_{r_0}r\,dr \geqslant \int_0^\delta f(r,(c+\varepsilon)w_{r_0})w_{r_0}r\,dr\\ &\geqslant \int_0^{r_0} f\left(0,(c+\varepsilon)\left(\log\frac{1}{r_0}\right)^{1/2}\right)\left(\log\frac{1}{r_0}\right)^{1/2} r\,dr\\ &= \frac{1}{2}\left(\log\frac{1}{r_0}\right)^{1/2} h\left(0,(c+\varepsilon)\left(\log\frac{1}{r_0}\right)^{1/2}\right)\exp\left[b(0)(c+\varepsilon)^2\log\frac{1}{r_0}\right]r_0^2\\ &\geqslant \frac{\frac{1}{2}\left(\log\frac{1}{r_0}\right)^{1/2} h\left(0,(c+\varepsilon)\left(\log\frac{1}{r_0}\right)^{1/2}\right)\exp\left[\varepsilon^2\left(\log\frac{1}{r_0}\right)\right]}{r_0^{(c^2b(0)-2)}} \to \infty \end{aligned}$$

as $r_0\to 0$.

Hence $C_0^2 \leqslant 2/b(0)$.

Step 2. $C_0^2 = 2/b(0)$.
Suppose not, then choose $\varepsilon > 0$, $\delta > 0$ such that $\delta \leqslant \min(\delta_1, \delta_0)$ and for all r in $[0, \delta]$,

$$C_0^2 < (C_0 + \varepsilon)^2 < \frac{2 - \varepsilon}{b(r)}.$$

Let $\|w\|_1 \leqslant 1$, then

$$\int_0^1 f(r, (C_0 + \varepsilon)w)wr\,dr = \int_0^\delta + \int_\delta^1. \tag{3.56}$$

Since $\|w\|_1 = 1$ implies from Lemma (3.1)

$$|w(r)| \leqslant \log\frac{1}{r},$$

hence there exists a constant M_1 such that

$$\sup_{\|w\|_1 \leqslant 1} \int_\delta^1 f(r, (C_0 + \varepsilon)w)wr\,dr \leqslant M_1 \tag{3.57}$$

and

$$\begin{aligned}
\int_0^\delta f(r, (C_0 + \varepsilon)w)wr\,dr &\leqslant \int_0^\delta h(r, (C_0 + \varepsilon)w)[\exp(C_0 + \varepsilon)^2 b(r) w^2]wr\,dr \\
&\leqslant \int_0^\delta h(r, (C_0 + \varepsilon)w)[\exp(2 - \varepsilon)w^2]wr\,dr \\
&\leqslant M_2 \int_0^\delta [\exp(2 - \varepsilon/2)w^2]r\,dr \\
&\leqslant M_2 \int_0^\delta r^{\varepsilon/2 - 1}\,dr \leqslant M_3
\end{aligned} \tag{3.58}$$

where

$$M_2 = \sup_{(r,t) \in Q_\delta} h(r, t)t \exp -\frac{\varepsilon}{2}t^2.$$

This implies $C_0 > (C_0 + \varepsilon)$ which is a contradiction. Hence $C_0^2 = 2/b(0)$.

Lemma 3.8. Let f in A' be critical and suppose there exists a $t_0 > 0$ satisfying

$$\exp -t_0^2 < \delta_1$$

$$h\left(0, \left(\frac{2}{b(0)}\right)^{1/2}\right)t_0 > 2\left(\frac{2}{b(0)}\right)^{1/2} \tag{3.59}$$

Let $a \geqslant 0$ such that

$$\sup_{\|w\|_1 \leqslant 1} \int_0^1 f(r, aw) wr \, \mathrm{d}r \leqslant a \tag{3.60}$$

then $a^2 < 2/b(0)$.

Proof. From Lemma (3.7), $a^2 \leqslant 2/b(0)$. Suppose $a^2 = 2/b(0)$, then take $r_0 = \exp -t_0^2$, w_{r_0} as in (3.55) and from (3.60) we have

$$\begin{aligned}
\left(\frac{2}{b(0)}\right)^{1/2} = a &\geqslant \int_0^{r_0} f(r, aw_{r_0}) w_{r_0} r \, \mathrm{d}r \\
&\geqslant \int_0^{r_0} f(0, aw_{r_0}) w_{r_0} \, \mathrm{d}r \\
&= f(0, at_0) t_0 \frac{r_0^2}{2} \\
&= t_0 h(0, at_0) \exp 2\left(\log \frac{1}{r_0}\right) \frac{r_0^2}{2} \\
&= \tfrac{1}{2} t_0 h\left(0, \left(\frac{2}{b(0)}\right)^{1/2} t_0\right) > \left(\frac{2}{b(0)}\right)^{1/2}
\end{aligned}$$

which is a contradiction. Hence the result.

Lemma 3.9. For any $\varepsilon > 0$, $0 \leqslant \alpha < 1$,

$$\sup_{0 \leqslant r \leqslant 1} r^{\varepsilon} \left(\frac{1 - r^{1-\alpha}}{1 - \alpha}\right) \leqslant \frac{1}{\varepsilon}. \tag{3.61}$$

Proof. Let $g(r) = r^{\varepsilon}(1 - r^{1-\alpha}/1 - \alpha)$. Then $g(0) = g(1) = 0$. Let $0 < r_0 < 1$ such that

$$g(r_0) = \sup_{0 \leqslant r \leqslant 1} g(r)$$

then

$$0 = g'(r_0) = \varepsilon r_0^{\varepsilon - 1} \left(\frac{1 - r_0^{1-\alpha}}{1 - \alpha}\right) - r_0^{\varepsilon - \alpha}.$$

Hence

$$\frac{1 - r_0^{1-\alpha}}{1 - \alpha} = \frac{r_0^{1-\alpha}}{\varepsilon}.$$

Therefore

$$g(r) \leqslant g(r_0) \leqslant \frac{r_0^{1-\alpha+\varepsilon}}{\varepsilon} \leqslant \frac{1}{\varepsilon}.$$

Lemma 3.10. Let f in A′ be critical, then

$$\inf_{B_1 \cup B_1^*} I_1 = \inf_{\partial(B_1 \cup B_1^*)} I_1 = \inf_{B_{01}} I_1 \tag{3.62}$$

Proof. u is in $B_1 \cup B_1^*$ implies $|u|$ also in $B_1 \cup B_1^*$ and $I_1(u) = I_1(|u|)$. Let $u \in B_1 \cup B_1^*$; choose a $\gamma < 1$ such that

$$\|u\|_1^2 = \frac{1}{\gamma}\int_0^1 f(r, \gamma u) ur \, \mathrm{d}r.$$

Then γu is in $\partial(B_1 \cup B_1^*)$ and $I_1(\gamma u) \leqslant I_1(u)$. Hence

$$\inf_{B_1 \cup B_1^*} I_1 = \inf_{\partial(B_1 \cup B_1^*)} I_1.$$

Now let $u \geqslant 0$ is in $\partial(B_1 \cup B_1^*)$. Since f is critical, we have for any $s > 1$

$$\int_0^1 f(r, su) ur \, \mathrm{d}r < \infty.$$

Let $v = su$, then

$$\begin{aligned} \|v\|_1^2 = s^2 \|u\|_1^2 &= s^2 \int_0^1 f(r, u) ur \, \mathrm{d}r \\ &= s \int_0^1 f\left(r, \frac{v}{s}\right) vr \, \mathrm{d}r < \int_0^1 f(r, v) vr \, \mathrm{d}r \end{aligned} \tag{3.63}$$

because $s > 1$ and $f(r, t)/t$ is increasing.

Choose an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$

$$\|v\|_1^2 < \int_\varepsilon^1 f(r, v) vr \, \mathrm{d}r \leqslant \int_0^1 f(r, v) vr \, \mathrm{d}r \tag{3.64}$$

and define

$$v_\varepsilon = \begin{cases} v(\varepsilon) & \text{if } 0 \leqslant r \leqslant \varepsilon \\ v(r) & \text{if } \varepsilon \leqslant r \leqslant 1. \end{cases} \tag{3.65}$$

Then from (3.64) v_ε is in B_{01}.

Now we claim that $I_1(v_\varepsilon) \to I_1(v)$ as $\varepsilon \to 0$.

Case 1. If v is in B_1, then $\|v_\varepsilon\|_\infty \leqslant \|v\|_\infty$ and hence by dominated convergence theorem $I_1(v_\varepsilon) \to I_1(v)$.

Case 2. If v is in B_1^*, then $v_\varepsilon \uparrow v$ and hence by Monotone convergence theorem, $I_1(v_\varepsilon) \to I_1(v)$. Hence

$$\inf_{B_{01}} I_1 \leqslant I_1(v_\varepsilon) \to I_1(v) \quad \text{as } \varepsilon \to 0. \tag{3.66}$$

f is critical and is in A', we have for $1 \leqslant s \leqslant 2$

$$f(r, su)su - 2F(r, su) \leqslant 2f(r, 2u)u - 2F(r, 2u)$$

and is in L^1. Hence by dominated convergence theorem,

$$I_1(v) \to I_1(u) \quad \text{as } s \to 1. \tag{3.67}$$

Combining (3.66) and (3.67) we have

$$\inf_{B_{01}} I_1 \leqslant \inf_{\partial(B_1 \cup B_1^*)} I_1 \leqslant \inf_{B_{01}} I_1$$

and hence the result.

Proof of theorem (2.1). From Lemma (3.2), there exists $\alpha_0 < 1$ such that Σ_α is non-empty for $\alpha_0 \leqslant \alpha < 1$ and $\{l_\alpha\}$ is bounded by $2m^2$ where m is given by (3.3). Let $l = \lim_{\alpha \to 1} l_\alpha$.

Let f satisfies (2.10). Let $\eta > 0$, $\gamma > 0$, $\alpha_n \to 1$, u_n in Σ_{α_n} such that

$$\begin{aligned} &\text{(i) } l_{\alpha_n} \to l \quad \text{as } \alpha_n \to 1 \\ &\text{(ii) } l_{\alpha_n} \leqslant \bar{I}_{\alpha_n}(u_n) < \left(l_{\alpha_n} + \frac{\eta}{2} \right). \\ &\text{(iii) } (l_{\alpha_n} + \eta) b \leqslant \gamma < 1. \end{aligned} \tag{3.68}$$

We claim that

$$\lim_{\alpha_n \to 1} u_n'(1) \neq 0. \tag{3.69}$$

If not, then $u_n'(1) \to 0$. Since $u_n \in \Sigma_{\alpha_n}$, we have

$$u_n'(1) = -\int_0^1 f(r, u_n) r^{\alpha_n} \, dr \to 0 \quad \text{as } \alpha_n \to 1.$$

Since for any $0 \leqslant r \leqslant 1$ we have

$$r^\alpha u_n'(r) = u_n'(1) + \int_r^1 f(t, u_n) t^{\alpha_n} \, dt$$

we have

$$\sup_{r \in [0,1]} |r^\alpha u_n'(r)| \to 0 \quad \text{as } \alpha_n \to 1.$$

This shows for any $0 < r_0 \leqslant 1$,

$$\sup_{r_0 \leqslant r \leqslant 1} |u_n'(r)| \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.70}$$

This in turn implies

$$\sup_{r_0 \leqslant r \leqslant 1} |u_n(r)| \leqslant \int_{r_0}^1 |u_n'(t)| \, dt \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.71}$$

From (ii) of definition (2.1) and (3.33) we have

$$\begin{aligned}\tfrac{1}{2}\delta_0^2 u_n'(\delta_0)^2 &= (1+\alpha_n)\int_0^{\delta_0} F(r,u_n)r^{\alpha_n}\,\mathrm{d}r + \int_0^{\delta_0} \frac{\partial F}{\partial r}(r,u_n)r^{1+\alpha_n}\,\mathrm{d}r \\ &\quad + \frac{(1-\alpha)}{2}\int_0^{\delta_0} u_n'(r)^2 r^{\alpha_n}\,\mathrm{d}r - \delta_0^{1-\alpha_n}F(\delta_0,u_n(\delta_0)) \\ &\geqslant (1+\alpha)\int_0^{\delta_0} F(r,u_n)r^{\alpha_n}\,\mathrm{d}r - \delta_0^{1+\alpha_n}F(\delta_0,u_n(\delta_0))\end{aligned} \tag{3.72}$$

Hence by (3.70) and (3.72) we have

$$\int_0^{\delta_0} F(r,u_n)r^{\alpha_n}\,\mathrm{d}r \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.73}$$

From (3.71) and by dominated convergence theorem

$$\int_{\delta_0}^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.74}$$

Combining (3.73) and (3.74) we have

$$\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.75}$$

Let N_0 be such that for all $n \geqslant N_0$,

$$\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r < \frac{\eta}{2}. \tag{3.76}$$

From (ii) and (iii) of (3.68) and (3.76)

$$\begin{aligned}\tfrac{1}{2}\|u_n\|_{\alpha_n}^2 &= \bar{I}_{\alpha_n}(u_n) + \int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r \\ &< \left(l_{\alpha_n} + \frac{\eta}{2}\right) + \frac{\eta}{2} = (l_{\alpha_n} + \eta) \\ &\leqslant \frac{\gamma}{b}.\end{aligned}$$

Hence

$$\begin{aligned}|u_n(r)|^2 &\leqslant \|u\|_1^2 \log\frac{1}{r} \\ &< 2(l_{\alpha_n} + \eta)\log\frac{1}{r} \\ &\leqslant \frac{2\gamma}{b}\log\frac{1}{r}.\end{aligned} \tag{3.77}$$

From (3.32), (3.70) and (3.77) we have

$$
\begin{aligned}
u_n(0) &= \int_0^1 t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right) f(t,u_n)\,\mathrm{d}t \\
&= \int_0^{\delta_1} t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right) f(t,u_n)\,\mathrm{d}t + \int_{\delta_1}^1 t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right) f(t,u_n)\,\mathrm{d}t \\
&\leqslant M\int_0^{\delta_1} t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right)\exp(bu_n^2)\,\mathrm{d}t + M_1 \\
&\leqslant M\int_0^{\delta_1} t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right)\exp\left(2\gamma\log\frac{1}{t}\right)\mathrm{d}t + M_1 \\
&\leqslant M\int_0^{\delta_1} t^{\alpha_n-2\gamma}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right)\mathrm{d}t + M_1
\end{aligned} \tag{3.78}
$$

Now choose $\varepsilon > 0$ such that

$$\alpha_n > 2\gamma - 1 + \varepsilon \quad \text{for all } n \text{, large.}$$

Then from (3.61) and (3.78) we have

$$
\begin{aligned}
u_n(0) &\leqslant M\int_0^{\delta_1} t^{\alpha_n-2\gamma-\varepsilon/2}t^{\varepsilon/2}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right)\mathrm{d}t + M_1 \\
&\leqslant \frac{2M}{\varepsilon}\frac{1}{\left(\alpha_n-2\gamma+1-\frac{\varepsilon}{2}\right)} + M_2 \leqslant \frac{4M}{\varepsilon^2} + M_1.
\end{aligned} \tag{3.79}
$$

Hence

$$\|u_n\|_\infty = u_n(0) \leqslant \frac{4M}{\varepsilon^2} + M_1.$$

Since u_n is in $\sum_{\alpha_n}$ and $\{\|u_n\|_\infty\}$ is bounded and hence u_n converges strongly in $C[0,1]$ and in H_0^1 to a function u. From (3.71) $u_n(r)\to 0$ as $\alpha_n\to\infty$ for every $r\neq 0$, we have $u\equiv 0$ and hence $u_n(0)\to 0$. Now choose N large such that $\|u_n\|_\infty \leqslant t_0$ for all $n\geqslant N$. From (iii) of Definition (2.1) we have

$$
\begin{aligned}
\lambda_{\alpha_n}\int_0^1 u_n\phi_{\alpha_n}r^{\alpha_n}\,\mathrm{d}r &= -\int_0^1 (r^{\alpha_n}\phi'_{\alpha_n})u_n\,\mathrm{d}r \\
&= -\int_0^1 (r^{\alpha_n}u'_n)'\phi_{\alpha_n}\,\mathrm{d}r \\
\lambda_{\alpha_n}\int_0^1 u_n\phi_{\alpha_n}r^{\alpha_n}\,\mathrm{d}r &= -\int_0^1 (r^{\alpha_n}u'_n)'\phi_{\alpha_n}\,\mathrm{d}r \\
&= \int_0^1 f(r,u_n)\phi_{\alpha_n}r^{\alpha_n}\,\mathrm{d}r \\
&< \lambda_{\alpha_n}\int_0^1 u_n\phi_{\alpha_n}r^{\alpha_n}\,\mathrm{d}r.
\end{aligned}
$$

and hence a contradiction. This proves the claim. Hence by going to a subsequence, we assume that

$$\lim_{\alpha_n \to 1} u_n'(1) \neq 0$$
$$l_{\alpha_n} \leqslant \bar{I}_{\alpha_n}(u_n) < 2l_{\alpha_n} \leqslant 4m^2. \tag{3.80}$$

Now

$$\begin{aligned} 4m^2 &\geqslant \frac{1}{2}\int_0^1 [f(r,u_n)u_n - 2F(r,u_n)]r^{\alpha_n}\,dr \\ &= \frac{1}{2}\int_0^1 [f(r,u_n)u_n - \beta F(r,u_n)]r^{\alpha_n}\,dr + \frac{\beta-2}{2}\int_0^1 F(r,u_n)r^{\alpha_n}\,dr \\ &\geqslant M_1 + \left(\frac{\beta-2}{2}\right)\int_0^1 F(r,u_n)r^{\alpha_n}\,dr \end{aligned}$$

where M_1 is constant independent of n. Hence $\exists M_2 > 0$ such that

$$\int_0^1 F(r,u_n)r^{\alpha_n}\,dr \leqslant M_2$$
$$\tfrac{1}{2}\|u_n\|_{\alpha_n}^2 = \bar{I}_{\alpha_n}(u_n) + \int_0^1 F(r,u_n)r^{\alpha_n}\,dr \leqslant 4m^2 + M_2. \tag{3.81}$$

Hence $\{\|u_n\|_{\alpha_n}\}$ is uniformly bounded. Hence from (3.80) and Lemma (3.6), u_n converges weakly to a non-zero solution u of (1.2).

From condition (i) of Theorem (2.1), we have for every $1 \leqslant p < \infty$, $f(u) \in L^p(D)$ (see Moser [6]). Hence by regularity of elliptic operators, $u \in w^{2,p}(D)$ and hence by Sobolev imbedding u is in $C^1(\bar{D})'$ and hence in $C^2(\bar{D})$. This proves the result.

Remark 3.1. From the proof of Theorem (2.1) it follows that if $m > 0$ is satisfying (2.11), then from Lemma (3.2) $l_\alpha \leqslant 2m^2$ and hence $l \leqslant 2m^2$. Therefore if $2m^2 b < 1$ implies $lb < 1$. This proves the criterion (2.10).

Proof of Theorem 2.2. From Lemma (3.2) there exists $\alpha_0 < 1$ such that $\sum_\alpha$ is non-empty and $\{a_\alpha\}$ is bounded for $\alpha_0 \leqslant \alpha < 1$. Lemma (3.3) gives (2.15).

Case (1). Let f be super critical and $\overline{\lim}_{\alpha\to 1} a_\alpha = a \neq 0$. Then from Lemma (3.4) we have

$$\sup_{\|w\|_1 \leqslant 1} \int_0^1 f(r, aw)wr\,dr \leqslant a.$$

contradicting the fact that f is super critical. Hence $a = 0$.

Case 2. If f is critical, let $a = \overline{\lim}_{\alpha\to 1} a_\alpha$. Then from (3.30) it follows that

$$\sup_{\|w\|_1 = 1} \int f(r, aw)wr\,dr \leqslant a$$

and from Lemma (3.8),

$$\frac{a^2}{2}b(0) < 1. \tag{3.82}$$

Now choose an ε and δ positive such that

$$\begin{aligned} &\text{(i) } f(r,t) \leqslant M\exp[(b(0)+\varepsilon)t^2] \quad \text{for all } (r,t)\in Q_\delta. \\ &\text{(ii) } \frac{a^2}{2}(b(0)+\varepsilon) < 1. \end{aligned} \tag{3.83}$$

Such a choice is possible because of (3.82) and the condition that f is critical.

Since $a_\alpha^2/2 = l_\alpha$, and hence f satisfies (2.10) of Theorem (2.1) with b replaced by $(b(0)+\varepsilon)$ and hence there exists a sequence u_n in $\sum_{\alpha_n}$ and a weak solution u of (1.2) such that

$$\begin{aligned} &\text{(iii) } I_{\alpha_n}(u_n) \to \frac{a^2}{2} \quad \text{as } \alpha_n \to 1 \\ &\text{(iv) } u_n \to u \quad \text{in } H_0^1. \\ &\text{(v) } \lim_{n\to\infty} \int_0^1 F(r,u_n)r^n\,\mathrm{d}r = \int_0^1 F(r,u)r\,\mathrm{d}r. \end{aligned} \tag{3.84}$$

In fact (iii) follows from Lemma (3.6). From weak lower semicontinuity of the norm we have

$$\|u\|_1^2 \leqslant \varliminf_{\alpha_n\to 1} \|u_n\|_{\alpha_n}.$$

and hence from (iii) we have

$$I_1(u) \leqslant \varliminf_{\alpha_n\to 1} I_{\alpha_n}(u_n) = \frac{a^2}{2}. \tag{3.85}$$

Let w be in B_{01}. Choose γ_α such that

$$\|w\|_\alpha^2 = \frac{1}{\gamma_\alpha}\int_0^1 f(r,\gamma_\alpha w)wr^\alpha\,\mathrm{d}r.$$

Such a γ_α exists and $\lim_{\alpha\to 1}\gamma_\alpha = \gamma_1$ exists and is $\leqslant 1$ because w is in B_{01} and $\gamma_\alpha w$ is in B_α. Hence

$$\frac{a_\alpha^2}{2} \leqslant I(\gamma_\alpha w).$$

Taking the $\overline{\lim}$ as $\alpha \to 1$, we get

$$\frac{a^2}{2} \leqslant I_1(\gamma w) \leqslant I_1(w)$$

This implies

$$\frac{a^2}{2} \leqslant \inf_{B_{01}} I_1. \tag{3.86}$$

From Lemma (3.10), (3.85) and (3.86) and using the fact that u is in B_1^*, we get

$$I_1(u) = \frac{a^2}{2} = \inf_{B_{01}} I_1$$

and $a \neq 0$ because $u \not\equiv 0$. This proves Theorem (2.2).

Remark 3.2. Suppose $f(r, t) \leqslant 0$ for $r \in [0, 1]$ and $0 \leqslant t \leqslant t_0$ and satisfying all other hypothesis on f, then also the Theorems (2.1) and (2.2) are valid.

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