Analysis of general second-order fluid flow in double cylinder rheometer

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Abstract The fractional calculus approach in the constitutive relationship model of second-order fluid is introduced and the flow characteristics of the viscoelastic fluid in double cylinder rheometer are studied. First, the analytical solution of which the derivative order is 1/2 is derived with the analytical solution and the reliability of Laplace numerical inversion based on Crump algorithm for the problem is verified, then the characteristics of second-order fluid flow in the rheometer by using Crump method is analyzed. The results indicate that the more obvious the viscoelastic properties of fluid are, the more sensitive the dependence of velocity and stress on fractional derivative order is.

Keywords: double cylinder rheometer, second-order fluid, fractional calculus.

In many fields, such as oil production, polymer chemistry and pipe line engineerings as well as biorheology, it is always an interesting issue to measure the material constants, in which the common equipment is the double cylinder rheometer (DCR). Combining the constitutive relationship and momentum equation, one can obtain the analytical solution to compute the characteristic variables of flow field. These characteristic variables may be compared with the experimental data, and through data fitting we can determine the material constants. In 1989, Liu Ciqun and Huang Junqi[1] derived the analytical solution based on linear constitutive relationship. Yan Zongyi *et al*.[2] numerically investigated Maxwell model also based on linear constitutive relationship. In the research reports of second-order fluid, the employed constitutive relationship has the following form:

$$\tau(t) = E_0\varepsilon(t) + E_1 \frac{\partial}{\partial t}\varepsilon(t), \tag{1}$$

where τ is stress, ε is strain, E_0 and E_1 are constants. Eq. (1) is a linear relation and derived based on the phenomenal theory, which may describe the relationship of stress and strain approximately. Since the 1960s, Slonimsky[3], Bagly[4], Rogers[5] and Friedrich[6] have sequentially introduced the fractional calculus approach into rheology to study various problems. For numerous fluids with character of both elastic and viscous materials the constitutive relationship model has advantages over customary linear model. Li Jian and Jiang Tiqian[7] used the nonlinear model to analyze the characteristics of sesbania gum and xanthan gum in their experiment and obtained satisfactory results

Generally the constitutive relationship of viscoelastic second-order fluids has the form as follows[4]:

$$\tau(t) = E_0\varepsilon(t) + E_1 \cdot \mathrm{D}^\alpha[\varepsilon(t)], \tag{2}$$

where D^α is the fractional calculus operator and may be defined as[4]

$$\mathrm{D}^\alpha[y(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{y(z)}{(t-z)^\alpha}\mathrm{d}z, \qquad 0 < \alpha < 1, \tag{3}$$

where $\Gamma(\cdot)$ is Gamma function. While $\alpha = 1$, eq. (2) may be simplified as eq. (1), and while $\alpha = 0$, the constitutive relationship describes the complete Newtonian fluid. As a matter of fact, $E_0\varepsilon(t)$ is an elastic term and $E_1\mathrm{D}^\alpha[\varepsilon(t)]$ is a viscoelastic term.

This paper will study the rotatory flow of the second-order fluid in DCR. By using Laplace and Hankel integral transform to the governing equations, we obtain the image function in Laplace domain. As a derivative order $\alpha = 1/2$ we obtain the analytical solution. There is excellent agreement between the analytical solution while $\alpha = 1/2$ and numerical inversion solution that is evaluated by Crump[8] method. For any values of α between zero and unit, with Crump method one may obtain satisfactory results. In similar engineering application and experimental analyses, the illustration of this report is helpful. In this paper we also investigate the basic properties of the nonlinear constitutive relationship and sensitivity of related parameters.

1 Model formulation and solution

1.1 Basic equation

The flow in DCR is an axial symmetry. In cylinder coordinate system let z be a vertical axis which coincides with the axis of DCR and r be the radial distance. We have the following equations:

constitutive relationship:

$$\tau_{r\theta} = \eta_0 r\frac{\partial}{\partial r}\left(\frac{V}{r}\right) + \beta\mathrm{D}^\alpha\left[r\frac{\partial}{\partial r}\left(\frac{V}{r}\right)\right]; \tag{4}$$

momentum equation:

$$\rho\frac{\partial V'}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}). \tag{5}$$

By letting

$$u = \frac{V}{V_0}, x = \frac{r}{r_i}, t = \eta_0 t'/\rho r_i^2, \mathrm{R_c} = \beta/\rho r_i^2, \tag{6}$$

we have dimensionless form

$$\frac{\partial u}{\partial t} = (1 + R_c\mathrm{D}^\alpha)\left[\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - \frac{u}{x^2}\right], \tag{7}$$

where $\tau_{r\theta}$ is component of stress, V is velocity, t' is time, η_0 is viscous parameter, β is viscoelastic parameter, ρ is density, r_i is radius of inner cylinder.

Generally, the inner cylinder is fixed and the outer cylinder makes simple harmonic motion. In this case, the initial and boundary conditions are

$$u(x,0) = 0, \qquad 1 \leqslant x \leqslant b, \tag{8}$$

$$u(1,t) = 0, \qquad t > 0, \tag{9}$$

$$u(b,t) = \cos(\omega t), \qquad t > 0, \tag{10}$$

where $b = r_0/r_i$, r_0 = radius of outer cylinder, $\omega = \omega' r_i^2 \rho/\eta_0$, ω' is the frequency factor of simple harmonic motion.

1.2 Solution

Making Laplace transform to problems (7)—(10) we have

$$s\bar{u} = (1 + R_c s^\alpha)\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{1}{x}\frac{\partial \bar{u}}{\partial x} - \frac{\bar{u}}{x^2}\right), \tag{11}$$

$$\bar{u}(1,s) = 0, \tag{12}$$

$$\bar{u}(b,s) = \frac{s}{s^2 + \omega^2}, \tag{13}$$

where $\bar{u}$ is image of u in Laplace domain, s is Laplace transform parameter.

The solution of eqs. (11)—(13) is

$$\bar{u}(x,s) = \frac{s}{s^2 + \omega^2}\frac{I_1(\Delta)K_1(\Delta x) - K_1(\Delta)I_1(\Delta x)}{I_1(\Delta)K_1(\Delta b) - K_1(\Delta)I_1(\Delta b)}. \tag{14}$$

With Crump method one can numerically inverse eq. (14) and obtain the distribution of velocity, where $\Delta = \sqrt{s/(1 + R_c s^\alpha)}$. Obviously, it is difficult to analytically inverse eq. (14), even for some special values of α. In order to gain easily the inversed solution, we make Weber transform to eqs. (11)—(13). If the kernel function is taken as

$$H(\rho_i, x) = J_1(\rho_i x)Y_1(\rho_i) - J_1(\rho_i)Y_1(\rho_i x), \tag{15}$$

Weber transform of $\bar{u}$ may be defined as

$$\tilde{\bar{u}} = \int_1^b x\bar{u}\,(x,s)H(\rho_i, x)\mathrm{d}x, \tag{16}$$

where ρ_i is roots of equation $H(\rho_i, b) = 0$.

Through Weber transform, we have

$$\tilde{\bar{u}} = \frac{\rho_i^2 + R_c s^\alpha \rho_i^2}{s + R_c \rho_i^2 s^\alpha + \rho_i^2}\tilde{\bar{u}}_0, \tag{17}$$

where

$$\tilde{\bar{u}}_0 = \frac{2}{\pi}\frac{s}{s^2 + \omega^2}\frac{J_1(\rho_i)}{\rho_i^2\, J_1(\rho_i b)}. \tag{18}$$

The inversion transform formula of Weber transform is

$$\bar{u}\,(x,s)=\sum_{i=1}^{\infty}\tilde{\bar{u}}\,(\rho_i,s)\,\frac{H(\rho_i,x)}{N(\rho_i)},\tag{19}$$

where

$$\frac{1}{N(\rho_i)}=\frac{\pi^2}{2}\,\frac{\rho_i^2\,J_1^2(\rho_i b)}{J_1^2(\rho_i)-J_1^2(\rho_i b)}.\tag{20}$$

Substituting eq. (17) into (19) we have

$$\bar{u}\,(x,s)=\bar{u}_0(x,s)-\sum_{i=1}^{\infty}E(s)\,\frac{\pi J_1(\rho_i)J_1(\rho_i b)}{J_1^2(\rho_i)-J_1^2(\rho_i b)}H(\rho_i,x),\tag{21}$$

where

$$E(s)=\frac{s}{s+R_c\rho_i^2 s^{\alpha}+\rho_i^2}\,\frac{s}{s^2+\omega^2},\tag{22}$$

$$\bar{u}_0(x,s)=\frac{b}{b^2-1}\,\frac{x^2-1}{x}\,\frac{s}{s^2+\omega^2}.\tag{23}$$

1.3 Special solution for different α

By defining the inversion of $E(s)$ by

$$f(t)=L^{-1}[E(s)],\tag{24}$$

we obtain the inversion of eq. (21):

$$u(x,t)=\frac{b}{b^2-1}\,\frac{x^2-1}{x}\cos(\omega t)-\sum_{i=1}^{\infty}f(t)\,\frac{\pi J_1(\rho_i)J_1(\rho_i b)}{J_1^2(\rho_i)-J_1^2(\rho_i b)}H(\rho_i,x).\tag{25}$$

It is easy to derive the solution for $\alpha=0,1$ and $1/2$.

(i) $\alpha=0$,

$$f(t)=-(1+R_c)\rho_i^2 e^{-(1+R_c)\rho_i^2 t}*\cos(\omega t);\tag{26}$$

(ii) $\alpha=1$,

$$f(t)=-\frac{\rho_i^2}{(1+R_c\rho_i^2)^2}\exp\left[-\frac{\rho_i^2}{1+R_c\rho_i^2}t\right]*\cos(\omega t);\tag{27}$$

(iii) $\alpha=1/2$,

$$f(t)=\frac{1}{r_2-r_1}[r_2e^{r_2^2 t}\mathrm{erfc}(r_2\sqrt{t})-r_1e^{r_1^2 t}\mathrm{erfc}(r_1\sqrt{t})]*[\delta(t)-\omega\sin(\omega t)],\tag{28}$$

where the sign $*$ represents convolution and $\delta(\cdot)$ is Dirac delta.

$$r_1=\frac{1}{2}(R_c\rho_i^2-\sqrt{R_c^2\rho_i^4-4\rho_i^2}),\quad r_2=\frac{1}{2}(R_c\rho_i^2+\sqrt{R_c^2\rho_i^4-4\rho_i^2}).\tag{29}$$

2 Analyses and discussion

2.1 $R_c=0$

In this case, eq. (22) can be simplified as

$$E(s)=\frac{s}{s+\rho_i^2}\,\frac{s}{s^2+\omega^2}.\tag{30}$$

To make inverse transform we have

$$f(t) = -\rho_i^2 e^{-\rho_i^2 t} * \cos(\omega t). \tag{31}$$

Substituting eq. (31) into eq. (25), we get the velocity formula for Newtonian fluid.

2.2 The formula of stress

Dimensionless stress can be represented as

$$F(x,t) = \frac{r_i}{V_0 \eta_0}\tau_{r\theta} = (1 + R_c D^\alpha)\left[x \frac{\partial}{\partial x}\left(\frac{u}{x}\right)\right]. \tag{32}$$

The Laplace transform of eq. (32) is

$$\overline{F}(x,s) = (1 + R_c s^\alpha)\left[x \frac{\partial}{\partial x}\left(\frac{\bar{u}}{x}\right)\right]. \tag{33}$$

Substituting eq. (21) into eq. (33) and letting $x = 1$, we have

$$\overline{F}(1,s) = F_0(s) + \sum_{i=1}^{\infty} E'(s) G(\rho_i), \tag{34}$$

where

$$F_0(s) = \frac{2b}{b^2 - 1}\frac{s(1 + R_c s^\alpha)}{s^2 + \omega^2}, \tag{35}$$

$$E'(s) = \frac{s(1 + R_c s^\alpha)}{s + R_c \rho_i^2 s^\alpha + \rho_i^2}\frac{s}{s^2 + \omega^2}, \tag{36}$$

$$G(\rho_i) = \frac{2J_1(\rho_i)J_1(\rho_i b)}{J_1^2(\rho_i) - J_1^2(\rho_i b)}. \tag{37}$$

In the same way we can obtain the inversion of $\overline{F}$ for the same values of α. Here we need not give unnecessary details.

2.3 Character of velocity at the beginning

From the theory of Laplace transform we know that as $s \to \infty$, the image function represents the initial state in the real space. As $s \to \infty$, we can rewrite eq. (19) as

$$\bar{u}(x,s) = \frac{1}{s^{2-\alpha}} \cdot \frac{2R_c}{\pi}\sum_{i=1}^{\infty}\frac{J_1(\rho_i)}{J_1(\rho_i b)}\frac{H(\rho_i, x)}{N(\rho_i)}. \tag{38}$$

Inversing eq. (38), we have

$$u(x,t) = \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} \cdot \frac{2R_c}{\pi}\sum_{i=1}^{\infty}\frac{J_1(\rho_i)}{J_1(\rho_i b)}\frac{H(\rho_i, x)}{N(\rho_i)}. \tag{39}$$

It can be seen from eq. (39) that as $\alpha < 1$, which is the constitutive relationship with fractional derivative, the velocity increases gradually from at zero, which indicates the viscoelastic character. While $\alpha = 1$, that is, with the linear constitutive relationship, the velocity has a step change in the initial period.

2.4 Shear stress on inner cylinder at the beginning

In the same way as in analyzing velocity, as $s \to \infty$, the stress on inner cylinder can be expressed as

$$F(1,t) = -\frac{1}{\Gamma(2-2\alpha)} t^{1-2\alpha} \sum_{i=1}^{\infty} R_c^2 \rho_i^2 G(\rho_i). \tag{40}$$

From eq. (40) we know that as $\alpha < 1/2$, the stress on inner cylinder changes slowly; as $\alpha > 1/2$, the stress will be infinite. So in the latter case there is a drawback in describing the constitutive relationship with the present formula; at least it is so at the beginning. This problem will also be involved in the following computing analyses.

3 Computing results

It is difficult to obtain the real time space solution. So we employ the Crump method[8] to evaluate the velocity. We write down the formula of the numerical inversion as follows[9]:

$$u(x,t) = \frac{e^{at}}{T}\left\{\frac{1}{2}\bar{u}(x,a) + \sum_{k=1}^{\infty}\left[\mathrm{Re}\left(\bar{u}\left(x, a+\frac{k\pi i}{T}\right)\right)\cos\left(\frac{k\pi}{T}\right) - \mathrm{Im}\left(\bar{u}\left(x, a+\frac{k\pi i}{T}\right)\right)\sin\left(\frac{k\pi}{T}\right)\right]\right\} + E, \tag{41}$$

where

$$a = \lambda - \frac{\ln E'}{T}, \tag{42}$$

where λ is the selected parameter greater than the real part of singular point of the inverted function. T is half of the maximum computing time, E is the error given by

$$E = \sum_{n=1}^{\infty} e^{-2naT} f(2nT + t). \tag{43}$$

The stress is computed by the same formula as eq. (41). Sometimes eq. (41) converges slowly. In order to save the CPU time we employ the Epsilon algorithm to accelerate the convergence.

Figure 1 is the history curves of velocity with which we compare the exact and numerical inversion solution while $\alpha = 0, 1, 1/2$. The agreement between both solutions is excellent. In addition, from the figure we can see that the effect of α on the velocity is obvious only at the beginning. With time going on the effect of α disappears.

Figure 2 is the velocity history at several selected parameters α. We find that the closer the parameter α approaches a unit, the larger the step of velocity is at the beginning.

Figure 3 is history of velocity for selected parameters R_c and fixed space point. The figure represents the relationship between parameter R_c and velocity.

Figure 4 is the velocity distribution between the gap of rheometer for several selected R_c. The larger the parameter α is, the more uniformly the velocity distributes.

Figure 5 is the stress histories on the inner cylinder. In the figure, several selected α are investigated respectively. Initially, the stress increases with α, which verified the former conclusion about the step change of velocity at the beginning. This behavior is characterized by general constitutive relationship. In addition, the phase of stress advances with α.

Figure 6 is the stress history curves for different R_c. The stress increases with R_c, and its phase also advances with α.

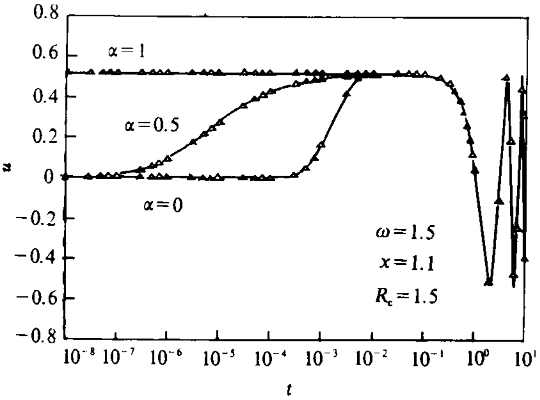

Fig. 1. Velocity history at fixed point. The comparison of exact and numerical inversion solution. —, exact; Δ, numerical.

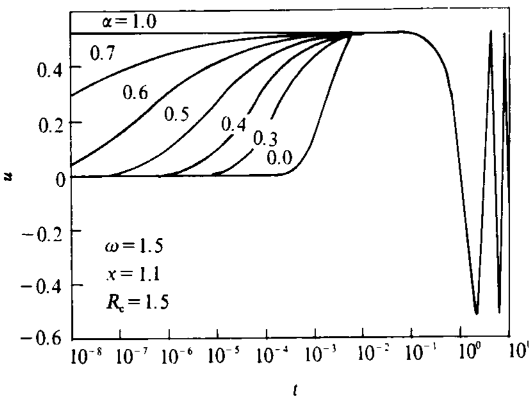

Fig. 2. Velocity history for a group of α.

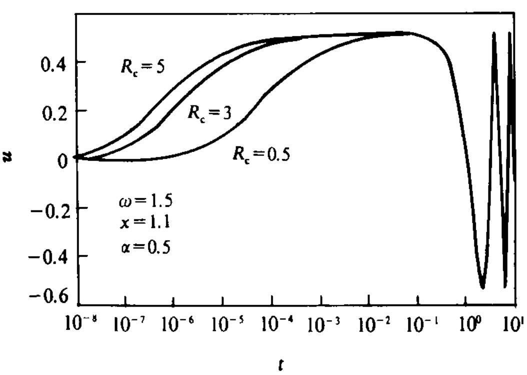

Fig. 3. Velocity history for a group of R_c.

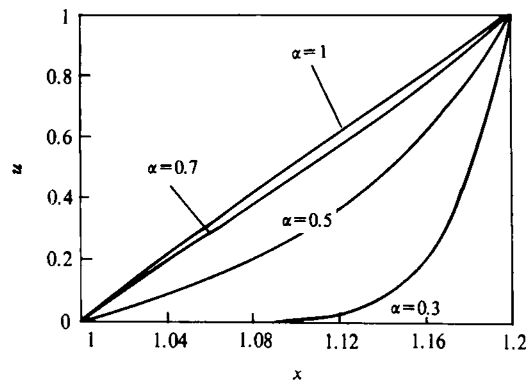

Fig. 4. Velocity distribution versus α. $t = 10^5$, $\omega = 1.5$, $R_c = 1.5$.

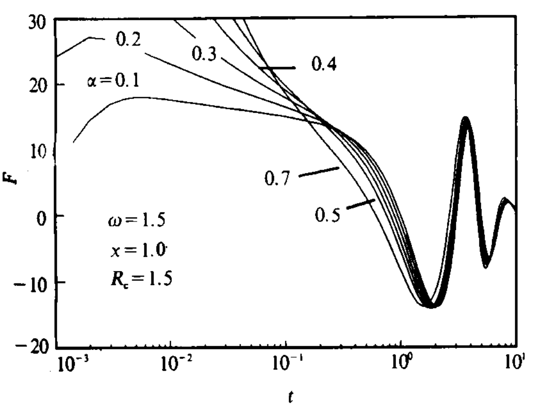

Fig. 5. Stress history versus α.

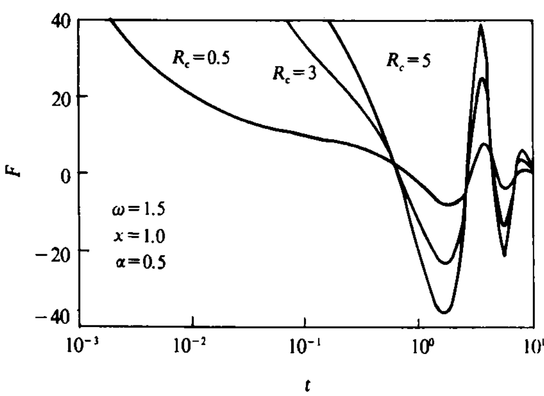

Fig. 6. Stress history versus R_c.

4 Conclusions

(i) We have obtained the numerical Laplace inversion solution for general second-order fluid flow in double cylinder rheometer, in which the fractional calculus approaching constitutive relationship was introduced. Especially while the derivative order $\alpha = 1/2$, we derived the analytical solution.

(ii) To compare the analytical and numerical inversion solution, we verified the reliability of Crump method to the flow model. With Crump method we analyzed the characteristics of second-order fluid flow in double cylinder rheometer.

(iii) The effect of fractional order α in constitutive relationship on flow field is very obvious. As $\alpha > 1/2$ the fluid already has obvious pseudo-solid character, and initially the stress on inner cylinder may be unlimited.

(iv) General constitutive relationship model is more useful than linear model for describing the properties of second-order fluid.

(v) The model and analytical method employed in the paper has been shown to be useful and reliable tools for engineering analyses.

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