Chaos and asymptotical stability in discrete-time recurrent neural networks with generalized input-output function

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Abstract We theoretically investigate the asymptotical stability, local bifurcations and chaos of discrete-time recurrent neural networks with the form of

$$u_i(t+1) = ku_i(t) + \Delta t\left(\sum_{j=1}^{n} a_{ij}v_j(t) + a_i\right), \quad i = 1,2,\cdots,n,$$

where the input-output function is defined as a generalized sigmoid function, such as $v_i = \tanh(\mu_i u_i)$, $v_i = \frac{2}{\pi}\arctan\left(\frac{\pi}{2}\mu_i u_i\right)$ and $v_i = \frac{1}{1+e^{-u/\varepsilon}}$, etc. Numerical simulations are also provided to demonstrate the theoretical results.

Keywords: **chaos, asymptotical stability, bifurcation, neural network, snap-back repeller.**

Recently the models of nervous system and brain which are called artificial neural networks[1—4] have been widely applied to various information processing problem with considerable success. For the continuous-time Hopfield neural networks, Li and Gopalsamy et al.[5,6] investigated the dynamic properties by studying the qualitative behavior of equilibrium points. For the discrete-time neural networks with input-output function as $x_i = \frac{1}{1+e^{-y_i/\varepsilon}}$ or $y_i = \varepsilon \ln \frac{x_i}{1-x_i}$, Chen et al.[3,7,8] have both numerically analyzed and theoretically proven asymptotical stability, bifurcations and existence of topological chaos, and further shown that the neural models with chaotic structure do have globally searching ability.

This paper aims to extend Chen and Aihara's works to the generalized input-output functions such as $v_i = \tanh(\mu_i u_i)$, $v_i = \frac{2}{\pi}\arctan\left(\frac{\pi}{2}\mu_i u_i\right)$ and $v_i = \frac{1}{1+e^{-u_i/\varepsilon}}$ which is continuously differentiable and monotonically increasing. Define the region $S = \{(v_1,\cdots,v_n) \mid \alpha \leqslant v_i \leqslant \beta;\ i = 1,\cdots,n\}$, then S can be divided into vertices $S_v = \{(v_1,\cdots,v_n) \mid v_i = \alpha \text{ or } \beta;\ i = 1,\cdots,n\}$, internal points $S_I = \{(v_1,\cdots,v_n) \mid \alpha < v_i < \beta;\ i = 1,\cdots,n\}$ and boundary points $= S - S_v - S_I$, respectively. $x \propto y$ means that x is directly proportional to y.

1 Asymptotical stability of discrete-time recurrent neural networks

1.1 Discrete-time recurrent neural networks

The continuous-time recurrent neural networks or Hopfield neural networks[4] can generally be written as

$$\frac{\mathrm{d}u_i(t)}{\mathrm{d}t} = -bu_i(t) + \sum_{j=1}^{n} a_{ij}v_j(t) + a_i, \quad i = 1,\cdots,n, \tag{1.1}$$

where $b \geqslant 0$ and input-output function $v_i(t) = s(u_i(t))$ is monotonically increasing with assumptions (1) $s(x) \in C^1$, $0 < s'(x) \leqslant M$ and (2) $s(x) = \alpha + o\left(\frac{1}{x}\right)$ $(x \to -\infty)$ and $s(x) = \beta + o\left(\frac{1}{x}\right)$ $(x \to +\infty)$. Obviously, $\alpha \leqslant s(x) \leqslant \beta$ and inverse function $s^{-1}(y)$ is also monotonically increasing. Note that assumptions (1) and (2) imply that $s'(x) = o\left(\frac{1}{x}\right)$ as $x \to \infty$. Discretizing eq. (1.1) by Euler method, we have the discrete-time recurrent neural networks as

$$u_i(t+1) = ku_i(t) + \Delta t\left(\sum_{j=1}^{n} a_{ij}v_j(t) + a_i\right), \quad i = 1,2,\cdots,n, \tag{1.2}$$

where $k = 1 - b\Delta t$, $\Delta t > 0$, or

$$u(t+1) = ku(t) + \Delta t(Av(u(t)) + I) \triangleq F(u(t)), \tag{1.3}$$

where $u(t) = (u_1(t),\cdots,u_n(t))^{\mathrm{T}}$, $A = (a_{ij})_{n\times n}$, $I = (a_1,\cdots,a_n)^{\mathrm{T}}$ and $v(u(t)) = (v_1(t),\cdots,v_n(t))^{\mathrm{T}} = (s(u_1(t)),\cdots,s(u_n(t)))^{\mathrm{T}}$. Obviously $F \in C^1$, and eq. (1.3) can be rewritten as

$$v(t+1) = \begin{pmatrix} v_1(t+1) \\ \vdots \\ v_n(t+1) \end{pmatrix} = \begin{pmatrix} s\left(ks^{-1}(v_1) + \Delta t\left(\sum_{j=1}^{n} a_{1j}v_j + a_1\right)\right) \\ \vdots \\ s\left(ks^{-1}(v_n) + \Delta t\left(\sum_{j=1}^{n} a_{nj}v_j + a_n\right)\right) \end{pmatrix}, \tag{1.4}$$

where $v_i = v_i(t)$, and $i = 1,2,\cdots,n$.

1.2 Conditions of asymptotical stability

Synchronously updating is that all neurons are updated in parallel, using only old values as input, and asynchronously updating is that neurons are updated one by one, using fresh values of previously updated neurons (see refs. [3,7]).

Theorem 1.1. Assume that $A^{\mathrm{T}} = A$ and (1) $0 \leqslant k \leqslant \frac{1}{3}$, $\frac{1-k}{M} > -\Delta t\lambda_{\min}$ or (2) $\frac{1}{3} \leqslant k \leqslant 1$, $\frac{2k}{M} > -\Delta t\lambda_{\min}$, where $\lambda_{\min}$ is the smallest eigenvalue of A. Then $v(t)$ of eq. (1.4) asymptotically converges to a fixed points, as far as eq. (1.4) is synchronously updated.

Proof. The Lyapunov function $f(v)$ is used to prove the theorem. Define

$$f(v) = -\frac{1}{2}\Delta t\sum_{i,j=1}^{n} a_{ij}v_iv_j - \Delta t\sum_{i=1}^{n} a_iv_i - (k-1)\sum_{i=1}^{n}\int_0^{v_i} s^{-1}(x)\mathrm{d}x. \tag{1.5}$$

Then we have $f(v(t+1)) - f(v(t)) \leqslant -\frac{1}{2}(\Delta v)^{\mathrm{T}}\left(\Delta tA + \frac{1-k}{M}E\right)\Delta v$, or $f(v(t+1)) - f(v(t)) \leqslant -\frac{1}{2}(\Delta v)^{\mathrm{T}}\left(\Delta tA + \frac{2k}{M}E\right)\Delta v$, where $\Delta v = (\Delta v_1,\cdots,\Delta v_n)^{\mathrm{T}}$. For condition (1), the symmetric matrix $\Delta tA + \frac{1-k}{M}E$ is positively definite. Hence, using the first inequality we have $f(v(t+1)) - f(v(t)) \leqslant 0$, where equality holds only for $\Delta v = 0$, which means that $v(t)$

has reached a fixed point of the network dynamics. For condition (2), using the last one, we have the same conclusion.

Theorem 1.2. Assume that $A^{\mathrm{T}} = A$ and (1) $0 \leqslant k \leqslant \frac{1}{3}$, $\frac{1-k}{M} > -\Delta t \min_i a_{ii}$, or (2) $\frac{1}{3} \leqslant k \leqslant 1$, $\frac{2k}{M} > -\Delta t \min_i a_{ii}$. Then $v(t)$ of eq. (1.4) asymptotically converges to a fixed point, as long as eq. (1.4) is asynchronously updated.

Proof. Without loss of generality, let neuron l be the updated neuron at the iteration $-(t+1)$. Then for condition (1), $f(v(t+1)) - f(v(t)) \leqslant \frac{1}{2}\left(\Delta t a_{ll} + \frac{1-k}{M}\right)(\Delta v_l)^2 < 0$ $(\Delta v_l \neq 0)$. We have condition (2) similar to the derivation of (1).

Next we investigate the correlations between asymptotically stable fixed points and minima of the computational energy function $f(v)$ (eq. (1.5)).

Lemma 1.1. Assume $A^{\mathrm{T}} = A$. Then (1) fixed points of $v(t)$ for eq. (1.4) are composed of stationary points of $f(v)$ and all of vertices of $v(t)$, regardless of synchronously or asynchronously updating; (2) as far as $k \neq 1$ or $b \neq 0$, there is no stationary points of $f(v)$ on vertices or boundary points of v.

Proof. The proof is similar to that in ref. [7].

Theorem 1.3. Under the conditions of Theorems 1.1 and 1.2, the fixed point of $v(t)$ is asymptotically stable for eq. (1.4) if and only if this point is a local minimum of $f(v)$ for $\alpha P_1 \leqslant v \leqslant \beta P_1$.

Proof. By using the definition of the asympotical stability for discrete-time system[7,9] and Lemma 1.1, the proof is easy.

Corollary 1.1. Assume the same conditions of Theorems 1.1 and 1.2, and assume that $a_{ii} = 0$ for $i = 1, 2, \cdots, n$ and $k = 1$. Then a fixed point $v(t)$ is asymptotically stable for eq. (1.4) if and only if this point is a local minimum of $f(v)$ for $\alpha P_1 \leqslant v \leqslant \beta P_2$ on vertices.

Let both $\lim_{v_i \to \alpha} \frac{s'(u_i(t+1))}{s'(u_i(t))}$ and $\lim_{v_i \to \beta} \frac{s'(u_i(t+1))}{s'(u_i(t))}$ be finite numbers or ∞, we give the following theorem to evaluate the local stability of the fixed points in an exact manner.

Theorem 1.4. Assume that $A^{\mathrm{T}} = A$,

(1) for synchronous updating: a fixed points v^* is asymptotically stable if the largest and smallest eigenvalues $\lambda'_{\max}$ and $\lambda'_{\min}$ of matrix $\mathrm{diag}(k_1, \cdots, k_n) + \Delta t \,\mathrm{diag}(s'(u_1^*), \cdots, s(u_n^*))A$ satisfy $|\lambda'_{\max}| < 1$ and $|\lambda'_{\min}| < 1$; on the other hand, v^* is unstable if either $\lambda'_{\max}$ or $\lambda'_{\min}$ satisfies $|\lambda'_{\max}| > 1$ or $|\lambda'_{\min}| > 1$, where $v_i^* = s(u_i^*)$ and

$$k_i = \begin{cases} k \lim_{v_i \to v_i^*} \dfrac{s'(u_i(t+1))}{s'(u_i(t))}, & \text{if } v_i^* = \alpha \quad \text{or} \quad v_i^* = \beta, \\ k, & \text{otherwise;} \end{cases}$$

(2) for asynchronous updating: a fixed point v^* is asymptotically stable if $|\max\{k_i + \Delta t a_{ii} s'(u_i^*)\}| < 1$ and $|\min\{k_i + \Delta t a_{ii} s'(u_i^*)\}| < 1$.

Proof. Examining whether the absolute values of all eigenvalues of $\frac{\mathrm{d}v(t+1)}{\mathrm{d}v(t)}$ at u^* are less than 1 or more than 1, we get the conditions of the theorem.

For input-output function $v_i = \tanh(\mu u_i)$ $(\mu > 0)$, we have the similar theorem.

Theorem 1.5. Assume $A^{\mathrm{T}} = A$,

(1) when $-1 < k < 1$ and $k \neq 0$, no fixed point on vertices or boundary points is asymptotically stable. While $k = 0$, the fixed point on vertices or boundary points is asymptotically stable. Asymptotically stable conditions for internal fixed points follow Theorem 1.4 for both synchronous updating and asynchronous updating;

(2) for synchronous updating: when $k \leqslant -1$ or $k = 1$, a fixed point $v^* \in S$ is asymptotically stable if the largest and the smallest eigenvalues $\lambda''_{\max}$ and $\lambda''_{\min}$ of matrix $\mathrm{diag}(k_1, \cdots, k_n) + \mu\Delta t\,\mathrm{diag}(1 - v_1^*, \cdots, 1 - v_n^*)A$ satisfy $|\lambda''_{\max}| < 1$ and $|\lambda''_{\min}| < 1$; on the other hand, a fixed point $v^* \in S$ is unstable if either $\lambda''_{\max}$ or $\lambda''_{\min}$ satisfies $|\lambda''_{\max}| > 1$ or $|\lambda''_{\min}| > 1$, where

$$k_i = \begin{cases} \mathrm{sign}k\, e^{-2\mathrm{sign}(kv_i^*)\mu\Delta t\left(\sum_{j=1}^{n} a_{ij}v_j^* + a_i\right)} & k = \pm 1,\ v_i^* = \pm 1, \\ 0 & k < -1,\ v_i^* = \pm 1, \\ k & \text{otherwise}; \end{cases}$$

(3) for asynchronous updating: if $k \leqslant -1$ or $k = 1$, a fixed point $v^* \in S$ is asymptotically stable if

$$|\max\{k_i + \mu\Delta t a_{ii}(1 - v_i^{*2})\}| < 1 \quad \text{and} \quad |\min\{k_i + \mu\Delta t a_{ii}(1 - v_i^{*2})\}| < 1.$$

Proof. The proof is similar to that in ref. [7].

2 Local bifurcations

This section examines the generic bifurcations of the fixed points, in particular the stationary points of $f(v, I)$, by taking I as bifurcation parameters, where $I = (a_1, \cdots, a_n)^{\mathrm{T}}$ and

$$f(v, I) = -\frac{1}{2}\Delta t \sum_{i,j=1}^{n} a_{ij}v_iv_j - \Delta t \sum_{i=1}^{n} a_iv_i - (k-1)\sum_{i=1}^{n}\int_0^{v_i(t)} s^{-1}(x)\mathrm{d}x. \tag{2.1}$$

From eq. (2.1), we can easily obtain

$$\frac{\partial f}{\partial v} = -\Delta t(Av + I) - (k-1)u(t) = -[u(t+1) - u(t)], \tag{2.2}$$

and $\dfrac{\partial^2 f}{\partial v^2} = -(k-1)\mathrm{diag}\left(\dfrac{1}{s'(u_1)}, \cdots, \dfrac{1}{s'(u_n)}\right) - \Delta t A$ where

$$\frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial v_1}, \cdots, \frac{\partial f}{\partial v_n}\right)^{\mathrm{T}}, \quad \frac{\partial^2 f}{\partial v^2} = \left(\frac{\partial f}{\partial v_i v_j}\right)_{n\times n}.$$

Eq. (2.2) can be rewritten as

$$u(t+1) = u(t) - \frac{\partial f}{\partial v}. \tag{2.3}$$

Then the Jacobian matrix $J_u = \dfrac{\mathrm{d}u(t+1)}{\mathrm{d}u(t)}$ of eq. (2.3) at (v, I) becomes $J_u = E - \dfrac{\partial^2 f}{\partial v^2} \times \mathrm{diag}(s'(u_1), \cdots, s'(u_n))$ and $J_v = \dfrac{\mathrm{d}v(t+1)}{\mathrm{d}v(t)} = k\,\mathrm{diag}\left(\dfrac{s'(u_1(t+1))}{s'(u_1(t))}, \cdots, \dfrac{s'(u_n(t+1))}{s'(u_n(t))}\right) + \Delta t\,\mathrm{diag}(s'(u_1(t+1)), \cdots, s'(u_n(t+1)))A$. Obviously, J_u and J_v have the same eigenvalues at the fixed point v^*, which are all real numbers according to appendix C in ref. [7].

Let u^* or v^* ($v_i^* = s(u_i^*)$ for $i = 1, 2, \cdots, n$) be an internal fixed point, which is in fact a stationary point of $f(v, I)$ for $A^{\mathrm{T}} = A$. Let I^c be the critical values for bifurcation. Then the

$f_v(v^*, I^c) = 0$, and at least one eigenvalue of $J_u(v^*, I^c)$ is $\lambda = 1$ or -1. By analyzing the fold and flip bifurcation[10], we have the following theorem.

Theorem 2.1. Let $A^T = A$ and assume that v^* is an internal fixed point of eq. (2.3) and $f_{vu}(v^*, I^c)$ has one zero-eigenvalue with normalized left eigenvector ξ^* and right eigenvector ξ. Then for synchronous updating, a fold bifurcation (saddle-node bifurcation) occurs if $\xi^* f_{vuu}(\xi, \xi) \neq 0$.

For the case that $J_u(v^*, I^c)$ has at least one eigenvalue $\lambda = -1$, we have the following theorem.

Theorem 2.2. Let $A^T = A$ and assume that v^* is an internal fixed point of eqs. (2.3) or (1.3) and $f_{vv}(v^*, I^c)\mathrm{diag}(s'(u_1^*), \cdots, s'(u_n^*))$ has at least one eigenvalue $\lambda = 2$ with normalized right eigenvector η and left eigenvector η^*. Then for synchronous updating, a flip bifurcation occurs if $\sigma = \frac{1}{3}\eta^* f_{vuuu}(\eta, \eta, \eta) - \frac{1}{2}(\eta^* f_{vuu}(\eta, \eta))^2 \neq 0$, and $\eta^* f_{vuu}(\eta, \eta) \neq 0$.

3 Chaos

This section intends to theoretically prove that there exists a chaotic structure in discrete-time recurrent neural networks when Δt is sufficiently large, and the chaotic structure is actually generated by a homoclinic bifurcation (global bifurcation).

3.1 Existence of fixed point

First, we show that eq. (1.3) possesses a fixed point for sufficient large Δt.

Theorem 3.1. Assume the following two conditions are satisfied:

(1) A is invertible, $\alpha P_1 < -A^{-1}I < \beta P_1$,

(2) $0 \leqslant b \leqslant M_0/\Delta t$, where $M_0(>0)$ is an arbitrary bounded number.

Then there exists a positive constant c_1, such that for any $\Delta t > c_1$ the discrete time system of eq. (1.3) has one fixed point u^* which is bounded. Furthermore, u^* is a unique bounded fixed point if Δt is sufficiently large, and $u_\infty^* = \lim_{\Delta t \to \infty} u^* = \{u \mid v(u) = -A^{-1}I\}$.

Proof. Define $\bar{u}$ such that $Av(\bar{u}) + I = 0$, i.e. $v(\bar{u}) = -A^{-1}I$. Obviously $\bar{u}$ is independent of Δt. Let ε_1 be a positive number and $U_1 = B^0(\bar{u}, \varepsilon_1)$. Now, we show that there exists $c_1(\varepsilon_1)(>0)$ such that eq. (1.3) has one fixed point $u^* \in U_1$ for any $\Delta t > c_1(\varepsilon_1)$.

Let

$$Q(u, \Delta t) = \frac{1}{\Delta t}[(k-1)u + \Delta t(Av(u) + I)]. \tag{3.1}$$

From condition (2), we have $\lim_{\Delta t \to \infty} Q(\bar{u}, \Delta t) = \lim_{\Delta t \to \infty} \frac{k-1}{\Delta t}\bar{u} = P_0$, i.e. $\lim_{\Delta t \to \infty} \| Q(\bar{u}, \Delta t) \| = 0$ which means that $u = \bar{u}$ is an approximate solution of $Q(u, \Delta t) = 0$ for sufficiently large number Δt.

Next, from eq. (3.1), the Jacobian matrix is $D_u Q(u, \Delta t) = \frac{k-1}{\Delta t}E + AZ(u)$ where $Z(u) = D_u v(u) = \mathrm{diag}(s'(u_1), \cdots, s'(u_n))$. Since $\lim_{\Delta t \to \infty} \frac{k-1}{\Delta t}E = O_{n\times n}$ according to condition (2), then for any bounded $u \in U_1$ including $\bar{u}$ we have $\lim_{\Delta t \to \infty} \det D_u Q(u, \Delta t) = \det A \cdot \prod_{i=1}^{n} s'(u_i) \neq 0$. Thus, there exists $c_2(\varepsilon_1)$ (>0) such that for a sufficiently large number M_1, $\| D_u^{-1} Q(\bar{u}, \Delta t) \| < M_1$

for any $\Delta t > c_2$. Furthermore, for positive number $\mu_1 < 1$, there is a positive number $\varepsilon_2 (< \varepsilon_1)$ such that $\| D_u Q(u, \Delta t) - D_u Q(\bar{u}, \Delta t) \| < \frac{\mu_1}{M_1}$ for any $u \in B(\bar{u}, \varepsilon_2)$.

Let r_1 satisfy $0 < r_1 < \frac{(1-\mu_1)\varepsilon_2}{M_1}$. Then there exists $c_3 > 0$ such that $\| Q(\bar{u}, \Delta t) \| < r_1$ for any $\Delta t > c_3$. Let $c_1(\varepsilon_1) = \max\{c_2(\varepsilon_1), c_3\}$. Then the following three conditions can be held for $\Delta t > c_1(\varepsilon)$: 1) $\Omega(\varepsilon_2) = \{u \mid \| u - \bar{u} \| < \varepsilon_2\} \subset U_1$; 2) $\| D_u Q(u, \Delta t) - D_u Q(\bar{u}, \Delta t) \| < \frac{\mu_1}{M_1}$; 3) $\frac{r_1 M_1}{1 - \mu_1} < \varepsilon_2$ where $\| Q(\bar{u}, \Delta t) \| < r_1$ and $\| D_u Q^{-1}(\bar{u}, \Delta t) \| < M_1$. According to Urabe's proposition (see Appendix in ref. [7]), $Q(u, \Delta t) = 0$ has a unique solution $u^* \in \Omega(\varepsilon_2) \subset U_1$ for all $\Delta t > c_1(\varepsilon_1)$. That is,

$$\frac{1}{\Delta t}[(k-1)u^* + \Delta t(Av(u^*) + I)] = 0$$

$$\text{or} \quad u^* = ku^* + \Delta t(Av(u^*) + I), \tag{3.2}$$

which means that eq. (1.3) has a unique fixed point $u^* \in U_1$ for any $\Delta t > c_1(\varepsilon_1)$.

Furthermore using the conclusion we have been obtained, we can show that u^* is also a unique bounded fixed point if Δt is sufficiently large.

3.2 Existence of topological chaos

This subsection will show the existence of the Marotto chaos by identifying a snap-back repeller. From eq. (1.3), it is easy to see that $F(u) = ku + \Delta t[Av(u) + I]$, then we get $D_u F(u) = kE + \Delta t AZ(u)$. Now we establish our principal theorem in this paper.

Theorem 3.2. Assume the same conditions as Theorem 3.1 and

(1) $\alpha P_1 < -\frac{1}{k}\alpha P_1 - A^{-1}I\left(1 + \frac{1}{k}\right) < \beta P_1$, $I + \alpha A P_1 > 0$, $k > 0$, or

(2) $\alpha P_1 < -\frac{1}{k}\alpha P_1 - A^{-1}I\left(1 + \frac{1}{k}\right) < \beta P_1$, $I + \alpha A P_1 < 0$, $k < 0$, or

(3) $\alpha P_1 < -\frac{1}{k}\beta P_1 - A^{-1}I\left(1 + \frac{1}{k}\right) < \beta P_1$, $I + \beta A P_1 > 0$, $k < 0$, or

(4) $\alpha P_1 < -\frac{1}{k}\beta P_1 - A^{-1}I\left(1 + \frac{1}{k}\right) < \beta P_1$, $I + \beta A P_1 < 0$, $k > 0$.

Then there exists a positive constant c_5 such that for any $\Delta t > c_5$, eq. (1.3) is chaotic in the sense of Marotto, and chaos is generated from a repeller.

Proof. For any $\Delta t > c_1$, define $A^{-1} = H = (H_1, \cdots, H_n)^{\mathrm{T}}$. For conditions (1) and (2) of Theorem 3.2, let $u^0 = (u_1^0, \cdots, u_n^0)^{\mathrm{T}}$ where $u_i^0 = s^{-1}\left(-\frac{1}{k}\alpha - H_i^{\mathrm{T}} I\left(1 + \frac{1}{k}\right)\right)$. Then $v^0 = v(u^0) = -\frac{1}{k}\alpha P_1 - A^{-1}I\left(1 + \frac{1}{k}\right)$. Note that by conditions (1) and (2), we have $\alpha P_1 < v^0 < \beta P_1$ and v^0 or u^0 has no relation with Δt. Since $F(u^0) = ku^0 - \frac{I + \alpha A P_1}{k}\Delta t$, then

$$F(u^0) \propto -\frac{I + \alpha A P_1}{k}\Delta t \tag{3.3}$$

which is negative because of conditions (1) and (2).

Therefore, $\lim_{\Delta t \to \infty} v(F(u^0)) = \lim_{\Delta t \to \infty} (s(F_1(u^0)), \cdots, s(F_n(u^0)))^{\mathrm{T}} = \alpha P_1$.

Now, we prove that u^* is a snap-back repeller. Since $\lim_{\Delta t\to\infty} v(u^*) = v(u^*_\infty) = -A^{-1}I$ which is from Theorem 3.1 and $v(u^0) = -\frac{1}{k}\alpha P_1 - A^{-1}I\left(1+\frac{1}{k}\right)$, then $\lim_{\Delta t\to\infty}[v(u^*) - v(u^0)] = \frac{1}{k}(\alpha P_1 + A^{-1}I) \neq P_0$ i.e. $\lim_{\Delta t\to\infty} \| u^* - u^0 \| \neq 0$ because $v_i = s(u_i)$ is a monotonic function. Therefore, there exists $\varepsilon_3 > 0$ and $c_6(\varepsilon_3) > 0$ such that $u^* \notin B(u^0, \varepsilon_3)$ for any $\Delta t > c_6(\varepsilon_3)$. Let

$$Q_1(u,\Delta t) = \frac{1}{\Delta t}[F(F(u)) - u^*] = \frac{1}{\Delta t}[F^2(u) - u^*]. \tag{3.4}$$

Then

$$Q_1(u^0,\Delta t) = -\frac{u^*}{\Delta t} + \frac{k^2}{\Delta t}u^0 + A(v(F(u^0)) - \alpha P_1) \text{ and } \lim_{\Delta t\to\infty} \| Q_1(u^0,\Delta t) \| = 0, \tag{3.5}$$

which means that u^0 is an approximate solution of $Q_1(u,\Delta t) = 0$ for sufficiently large number Δt. Since $D_u Q_1(u,\Delta t) = [kE + \Delta t\, AZ(F(u))]\left[\frac{k}{\Delta t}E + AZ(u)\right]$ and note that $\Delta t Z(F(u^0)) = \mathrm{diag}(\Delta t s'(F_1(u^0)),\cdots,\Delta t s'(F_n(u^0)))$ and $\lim_{\Delta t\to\infty} \Delta t s'(F_i(u^0)) = \lim_{\Delta t\to\infty}\frac{\Delta t}{F_i(u^0)}\cdot F_i(u^0)s'(F_i(u^0)) = 0$ by using assumption (2) in sec. 1.1 and eq. (3.3), we have $\lim_{\Delta t\to\infty}\Delta t Z(F(u^0)) = O_{n\times n}$. Obviously, $\lim_{\Delta t\to\infty}\frac{k}{\Delta t}E = O_{n\times n}$. Hence, using conditions (1) and (2), we have

$$\lim_{\Delta t\to\infty}\det D_u Q_1(u^0,\Delta t) = k^n\cdot\det A\cdot\prod_{i=1}^{n} s'\left(s^{-1}\left(-\frac{1}{k}\alpha - H_i^{\mathrm{T}}I\left(1+\frac{1}{k}\right)\right)\right) \neq 0. \tag{3.6}$$

Let $U_2 = B^0(u^0,\varepsilon_3)$, from eq. (3.6), there exists a positive constant $c_7 > 0$ such that when $\Delta t > c_7$, for a sufficiently large constant $M_2 > 0$, $\| D_u^{-1}Q_1(u^0,\Delta t) \| < M_2$. Furthermore, for $0 < \mu_2 < 1$, there exists a positive number ε_4 $(<\varepsilon_3)$ such that for any $u \in B(u^0,\varepsilon_4)$, we have $\| D_u Q_1(u,\Delta t) - D_u Q_1(u^0,\Delta t) \| < \frac{\mu_2}{M_2}$. Selecting r_2 such that $0 < r_2 < \frac{(1-\mu_2)\varepsilon_4}{M_2}$ and according to eq. (3.5), there exists a positive constant $c_8 > 0$ such that when $\Delta t > c_8$, $\| Q_1(u^0, \Delta t) \| < r_2$. Let $c_9 = \max\{c_1, c_6, c_7, c_8\}$. Then the following three conditions hold for $\Delta t > c_9$: 1) $\Omega(\varepsilon_4) = \{u \mid \| u - u^0 \| < \varepsilon_4\} \subset U_2 = B^0(u^0,\varepsilon_3)$; 2) $\| D_u Q_1(u,\Delta t) - D_u Q_1(u^0,\Delta t) \| < \frac{\mu_2}{M_2}$; 3) $\frac{r_2 M_2}{1-\mu_2} < \varepsilon_4$; where $\| Q_1(u^0,\Delta t) \| < r_2$ and $\| D_u^{-1}Q_1(u^0,\Delta t) \| < M_2$.

According to Urabe's proposition, $Q_1(u,\Delta t) = 0$ has a unique solution $u^{0*} \in \Omega(\varepsilon_4) \subset B(u^0,\varepsilon_3)$ for all $\Delta t > c_9$. From the definition of $B(u^0,\varepsilon_3)$, we can see that $u^{0*} \neq u^*$. That is, $u^* = F(F(u^{0*})) = F^2(u^{0*})$. From Lemma 6.1 in ref. [7], there exists a positive constant c_4 such that all eigenvalues of $D_u F(u)$ exceed the unity in norm for any $\Delta t > c_4$ and for any bounded u. Let $c_5 = \max\{c_4, c_9\}$. Then u^* is a snap-back repeller. By Marotto's theorem[11], the proof for the existence of chaos is straightforward. In the same way, we can give the proof of the theorem for conditions (3) and (4). In this case, let $u^0 = (u_1^0, \cdots, u_n^0)^{\mathrm{T}}$ where $u_i^0 = s^{-1}\left(-\frac{1}{k}\beta - H_i^{\mathrm{T}}I\left(1+\frac{1}{k}\right)\right)$.

4 Example

In this section, we give the simulation results for a two-dimensional case of eq. (1.3) as follows to verify theoretical results, where the input-output function $v_i(t) = \tanh(\mu u_i(t))$, $b = 1$, $M_0 = 1$, $a_{11} = -1$, $a_{12} = a_{21} = -0.5$, $a_{22} = -2$, $a_1 = a_2 = 0.8$ and $\mu = 125$ (fig. 1).

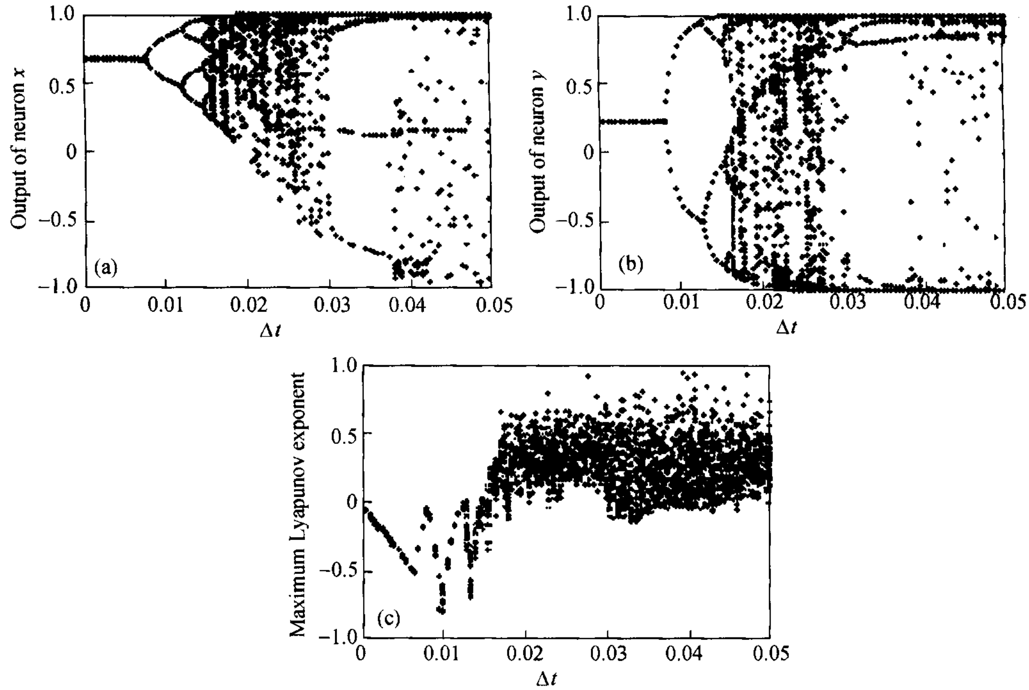

Fig. 1. (a)—(c) show the outputs of neurons and maximum Lyapunov exponents with increasing discrete time Δt.

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