ON THE DISTRIBUTION OF A QUADRATIC FORM IN A MULTIVARIATE NORMAL SAMPLE*

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1. Summary

The distribution of a quadratic form in a normal sample plays a very important role in multivariate statistical analysis. In many cases, statistics are functions of a quadratic form or special types of it.

In the univariate case, the distribution of a quadratic form was treated by many authors and was derived by using the Laguerre polynomials expansion or the Dirichlet series expansion, etc. [6], [7], [8]. In this paper, the distribution of a quadratic form in the multivariate case will be given in terms of zonal polynomials which were developed for multivariate analysis by A. T. James [3], [4] and A. G. Constantine [2]. Recently, the author's attention was called to C. G. Khatri [9] which deals with the same problem also by using zonal polynomials. However, the present paper treats the problem from another point of view.

The distribution of a quadratic form enables us to derive the distribution of a linear combination of several Wishart matrices. The distributions and probability functions of certain statistics of a quadratic form are given.

2. Introduction and notations

Let the $p\times N$ $(p\leqq N)$ matrix variate X be a sample matrix from a p-variate normal population with the density function

$$(1)\qquad \frac{1}{\pi^{pN/2}|2\Sigma|^{N/2}}\, etr\left[-\frac{1}{2}\Sigma^{-1}XX'\right],$$

where $etr(S)=\exp[TrS]$ and S is a square matrix. Let A be a real symmetric matrix of order N with full rank. The problem we are going to consider is to derive the density function of the quadratic form

* The first version of the paper, [0], was published in March, 1966.

$$Z=XAX' . \tag{2}$$

We shall also consider the density and the probability functions of some statistics related to it.

Before describing the procedure, it is convenient to list the following results which are useful for our argument.

(i) Let S and T be real symmetric matrices of order N and $C_\kappa(S)$ be a zonal polynomial corresponding to a partition $\kappa=\{k_1, \cdots, k_p\}$, $k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0$ of k. Let $d(H)$ be the invariant measure on the orthogonal group $O(N)$, normalized so that the measure of the whole group is unity. Then,

$$etr(S)=\sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(S)}{k!} , \tag{3}$$

$$\int_{O(N)} etr(SHTH')\, d(H)=\sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(S)C_\kappa(T)}{k!\, C_\kappa(I_N)} . \tag{4}$$

These were given by *A*. T. James [4].

The following notations are used:

$$(a)_\kappa=\prod_{i=1}^{p} \left(a-\frac{1}{2}(i-1)\right)_{k_i} ,$$

$$(a)_k=a(a+1) \cdots (a+k-1) ,$$

$$\Gamma_p(a)=\pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(a-\frac{1}{2}(i-1)\right) ,$$

$$\Gamma_p(a; \kappa)=\pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma\left(a+k_i-\frac{1}{2}(i-1)\right) .$$

Hence,

$$\Gamma_p(a; \kappa)=(a)_\kappa \Gamma_p(a) . \tag{5}$$

[See Constantine [2].]

(ii) Let A be a real positive definite symmetric matrix and B an arbitrary matrix. Then,

$$\int_{S>0} etr(-AS)|S|^{t-(p+1)/2} C_\kappa(SB)\, dS=\Gamma_p(t; \kappa)|A|^{-t} C_\kappa(A^{-1}B) , \tag{6}$$

where $t \geqq (p-1)/2$ [Constantine [2]].

(iii) If T is any positive definite symmetric matrix, then

$$\int_0^A |S|^{t-(p+1)/2} C_\kappa(TS)\, dS=\frac{\Gamma_p(t; \kappa)\Gamma_p((p+1)/2)}{\Gamma_p(t+(p+1)/2; \kappa)} |A|^t C_\kappa(AT) , \tag{7}$$

where the integral is over all S for which $0<S<A$ [Constantine [2]].

Constantine and James defined generalized hypergeometric functions in terms of zonal polynomials as.

$$(8) \qquad {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; S) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(S)}{k!},$$

$$(9) \qquad {}_pF_q^{(N)}(a_1, \cdots, a_p; b_1, \cdots, b_q; S, T) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(S) C_\kappa(T)}{k!\, C_\kappa(I_N)},$$

where I_N is the unit matrix of order N.

3. Distribution function of a quadratic form

THEOREM 1. *If A is a positive definite symmetric matrix and X is distributed with density function (1), then the density function of*

$$Z = XAX'$$

is given by

$$(10) \qquad \frac{1}{\Gamma_p(N/2)|2\Sigma|^{N/2}|A|^{p/2}} |Z|^{(N-p-1)/2} {}_0F_0^{(N)}\left(A^{-1}, -\frac{1}{2}\Sigma^{-1}Z\right).$$

PROOF. The density function of Z can be expressed in the form of multiple integral:

$$(11) \qquad \frac{1}{\pi^{pN/2}|2\Sigma|^{N/2}} \iint_{Z=XAX'} etr\left[-\frac{1}{2}\Sigma^{-1}XX'\right] dX$$

$$= \frac{1}{\pi^{pN/2}|2\Sigma|^{N/2}|A|^{p/2}} \iint_{Z=YY'} etr\left[-\frac{1}{2}\Sigma^{-1}YA^{-1}Y'\right] dY.$$

Since A is a symmetric matrix, the integral is invariant under the transformation $A^{-1} \to HA^{-1}H'$, $H \in O(N)$ and the integration with respect to H over the orthogonal group $O(N)$. Hence, by use of (i),

$$(12) \qquad \int_{Z=YY'} dY \int_{O(N)} etr\left[-\frac{1}{2}\Sigma^{-1}YHA^{-1}H'Y'\right] d(H)$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(A^{-1})}{k!\, C_\kappa(I_N)} \iint_{Z=YY'} C_\kappa\left(-\frac{1}{2}\Sigma^{-1}YY'\right) dY.$$

Thus, from the Wishart integral, the density function of Z is given by

$$(13) \qquad \frac{1}{\Gamma_p(N/2)|2\Sigma|^{N/2}|A|^{p/2}} |Z|^{(N-p-1)/2} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(A^{-1}) C_\kappa(-\frac{1}{2}\Sigma^{-1}Z)}{k!\, C_\kappa(I_N)},$$

which completes the proof.

If we set $A=I_N$, the density function of $Z=XX'$ becomes

$$\frac{1}{\Gamma_p(N/2)|2\Sigma|^{N/2}}|Z|^{(N-p-1)/2}\, etr\left[-\frac{1}{2}\Sigma^{-1}Z\right], \tag{14}$$

which is just the density function of a central Wishart matrix of N degrees of freedom.

Note. C. G. Khatri [9] obtained the density function of Z and expressed it as the product of a Wishart density function and a generalized hypergeometric function. However, his form is not always convenient for studying other properties of Z.

Remark. In the derivation of the distribution we have assumed that A is a positive definite symmetric matrix. By this, we do not lose any generality: in fact, if A is positive semi-definite and of rank n ($\leqq N$), there exists an orthogonal matrix P such that

$$PAP'=\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}, \tag{15}$$

where Λ is a diagonal matrix of order n and of full rank. Thus the quadratic form of X becomes

$$XAX'=Y_1\Lambda Y_1' \tag{16}$$

and Y_1 is a submatrix of $Y=[Y_1Y_2]$ where Y_1 and Y_2 are $p\times n$ and $p\times(N-n)$ matrices, respectively. Thus we have only to replace A by Λ and N by n which is the rank of A.

An interesting result on Wishart matrices can be derived from the density function (10).

Since A is a (positive definite) symmetric matrix, A can be decomposed into the following form.

$$A=\alpha_1P_1+\alpha_2P_2+\cdots+\alpha_sP_s, \tag{17}$$

where

1 $\alpha_1, \alpha_2, \cdots, \alpha_s$ are all the eigenvalues of A,

2 P_i $(i=1, \cdots, s)$ is a projection matrix, that is,

$$P_i^2=P_i \quad \text{and} \quad P_i=P_i',$$

3 $P_iP_j=P_jP_i=0$ if $i\neq j$,

4 $I_N=P_1+P_2+\cdots+P_s$.

This decomposition is called a spectral decomposition of a symmetric matrix and the decomposition is unique.

Thus, the quadratic form is decomposed into

$$XAX'=\alpha_1 XP_1X'+\alpha_2 XP_2X'+\cdots+\alpha_s XP_sX' , \tag{18}$$

and each XP_iX' $(i=1,2,\cdots,s)$ is of degrees of freedom equal to rank P_i. If rank P_i is not smaller than p—the dimension of the normal distribution under consideration—XP_iX' is a Wishart matrix; if it is not the case, XP_iX' is a so-called pseudo Wishart matrix. Since P_i and P_j are orthogonal, XP_iX' and XP_jX' are mutually independent. Thus (18) means that any quadratic form can be decomposed to a linear combination of independent Wishart or pseudo Wishart matrices with coefficients equal to the eigenvalues of A.

Conversely, if A is a diagonal matrix such that

$$A=\begin{pmatrix}\alpha_1 I_{n_1} & & & & \\ & \alpha_2 I_{n_2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \alpha_t I_{n_t}\end{pmatrix} , \tag{19}$$

where all $\alpha_1,\cdots,\alpha_t$ are not equal to zero, the quadratic form is also a linear combination of Wishart or pseudo Wishart matrices. From this, the density function of a linear combination of Wishart or pseudo Wishart matrices can be calculated.

THEOREM 2. *If $Z=XAX'$ is distributed with density function (10), then*

$$\begin{aligned}(20)\quad & Pr\{XAX'\leqq\Omega\}\\ &=\frac{\Gamma_p((p+1)/2)}{\Gamma_p((N+p+1)/2)}\frac{|\Omega|^{N/2}}{|2\Sigma|^{N/2}|A|^{p/2}}{}_1F_1^{(N)}\Big(\frac{N}{2};\frac{N+p+1}{2};\\ &\qquad A^{-1},-\frac{1}{2}\Sigma^{-1}\Omega\Big) .\end{aligned}$$

PROOF. From (10),

$$\begin{aligned}(21)\quad Pr\{XAX'\leqq\Omega\}&=\frac{1}{\Gamma_p(N/2)|2\Sigma|^{N/2}|A|^{p/2}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(A^{-1})(-\frac{1}{2})^k}{k!\,C_\kappa(I_N)}\\ &\quad\times\int_0^{\Omega}|Z|^{(N-p-1)/2}C_\kappa(\Sigma^{-1}Z)\,dZ .\end{aligned}$$

By the use of (iii) and (5), (21) becomes

$$\frac{\Gamma_p((p+1)/2)}{\Gamma_p((N+p+1)/2)}\frac{|\Omega|^{N/2}}{|2\Sigma|^{N/2}|A|^{p/2}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(N/2)_\kappa}{((N+p+1)/2)_\kappa}\frac{C_\kappa(A^{-1})C_\kappa(-\frac{1}{2}\Omega\Sigma^{-1})}{k!\,C_\kappa(I_N)},$$

which is (20).

If we set $A=I$, this is the same as (62) of Constantine [2].

4. Certain statistics of a quadratic form

THEOREM 3. *Let $Z=XAX'$ be distributed as (10). The joint density function of latent roots $\lambda_1>\cdots>\lambda_p>0$ of the determinantal equation*

$$|XAX'-\lambda I|=0 \tag{22}$$

is given by

$$\frac{\pi^{p^2/2}}{\Gamma_p(N/2)\Gamma_p(p/2)|2\Sigma|^{N/2}|A|^{p/2}}\left(\prod_{i=1}^{p}\lambda_i\right)^{(N-p-1)/2}\prod_{i<j}(\lambda_i-\lambda_j) \times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\Sigma^{-1})C_\kappa(A^{-1})C_\kappa(-\frac{1}{2}\Lambda)}{k!\,C_\kappa(I_N)\,C_\kappa(I_p)}, \tag{23}$$

where $\Lambda=\operatorname{diag}\{\lambda_1,\lambda_2,\cdots,\lambda_p\}$.

RROOF. Since $Z=XAX'$ is a positive definite symmetric matrix, there exists an orthogonal matrix H of the first column having positive elements such that $Z=XAX'=H\Lambda H'$. By replacing Z by $H\Lambda H'$ in (10) and using the formula

$$\frac{1}{2^p}\int_{O(p)}C_\kappa(\Sigma^{-1}H(-\tfrac{1}{2}\Lambda)H')d^*(H)=\frac{\pi^{p^2/2}}{\Gamma_p(p/2)}\frac{C_\kappa(\Sigma^{-1})C_\kappa(-\frac{1}{2}\Lambda)}{C_\kappa(I_p)}, \tag{24}$$

we have (23) as the joint density function of latent roots, since $dZ=\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i=1}^{p}d\lambda_i d^*(H)$. It should be noted that $d^*(H)=\frac{2^p\pi^{p^2/2}}{\Gamma_p(p/2)}d(H)$.

Remark. If we set $\Sigma=I_p$, (23) is equal to the density function of latent roots of determinantal equation $|XAX'-\lambda\Sigma|=0$, i.e.,

$$\frac{\pi^{p^2/2}}{2^{pN/2}\Gamma_p(N/2)\Gamma_p(p/2)}\left(\prod_{i=1}^{p}\lambda_i\right)^{(N-p-1)/2}\prod_{i<j}(\lambda_i-\lambda_j)\,{}_0F_0^{(N)}(A^{-1},-\tfrac{1}{2}\Lambda). \tag{25}$$

If, in addition, we set $A=I_N$, we have a well known density function [Anderson [1], p. 320].

THEOREM 4. *Let X be distributed with density function (1). Then, the density function of*

$$z=TrXAX' \tag{26}$$

is

$$\frac{1}{\Gamma(Np/2)|2\Sigma|^{N/2}|A|^{p/2}} z^{Np/2-1} {}_0F_0^{(Np)}(-\tfrac{1}{2}z\,;\ \Sigma^{-1}\otimes A^{-1})\ , \tag{27}$$

where $\Sigma^{-1}\otimes A^{-1}$ is a Kronecker's product of Σ^{-1} and A^{-1}.

PROOF. We can write z as

$$z = TrXAX' = \sum_{i=1}^{p} \sum_{\alpha,\beta=1}^{N} a_{\alpha\beta} x_{i\alpha} x_{i\beta} \tag{28}$$

and $E(x_{i\alpha}x_{j\beta}) = \delta_{\alpha\beta}\sigma_{ij}$.

Now we denote the row vectors of X by $x^{(1)}, x^{(2)}, \cdots, x^{(p)}$, i.e., $x^{(1)} = (x_{11}, x_{12}, \cdots, x_{1N})$, $x^{(2)} = (x_{21}, x_{22}, \cdots, x_{2N})$, $\cdots$, $x^{(p)} = (x_{p1}, x_{p2}, \cdots, x_{pN})$, and set

$$x = (x^{(1)}, x^{(2)}, \cdots, x^{(p)})\ .$$

The x is distributed as Np-dimensional normal distribution with mean 0 and covariance matrix $\Sigma\otimes I_N$ and z is rewritten as

$$z = x(I_p\otimes A)x'\ . \tag{29}$$

Since $(I_p\otimes A)^{-1/2}(\Sigma\otimes I_N)^{-1}(I_p\otimes A)^{-1/2}$ is symmetric, we obtain (27) in the same way as theorem 1 was proved.

THEOREM 5. *Let $z = TrXAX'$ be distributed as (27). Then*

$$Pr\{TrXAX' \leqq u\} = \frac{1}{\Gamma(Np/2+1)|2\Sigma|^{N/2}|A|^{p/2}} u^{Np/2} \times {}_1F_1^{(Np)}\left(\frac{Np}{2}\,;\ \frac{Np}{2}+1\,;\ -\frac{1}{2}u,\ \Sigma^{-1}\otimes A^{-1}\right)\ . \tag{30}$$

PROOF. If we replace t and p in formula (iii) by $Np/2$ and 1, respectively, then we have

$$\int_0^u z^{Np/2-1} C_\kappa(z)\,dz = \frac{\Gamma(Np/2;\kappa)\Gamma(1)}{\Gamma(Np/2+1;\kappa)} u^{Np/2} C_\kappa(u)\ . \tag{31}$$

Hence,

$$Pr\{TrXAX' \leqq u\} = \frac{1}{\Gamma(Np/2)|2\Sigma|^{N/2}|A|^{p/2}} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(-\frac{1}{2}\Sigma^{-1}\otimes A^{-1})}{k!\,C_\kappa(I_{Np})} \times \int_0^u z^{Np/2-1} C_\kappa(z)\,dz \tag{32}$$

$$=\frac{1}{\Gamma(Np/2)|2\Sigma|^{N/2}|A|^{p/2}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(-\frac{1}{2}\Sigma^{-1}\otimes A^{-1})}{k!\,C_\kappa(I_{Np})}$$
$$\times\frac{\Gamma(Np/2;\kappa)}{\Gamma(Np/2+1;\kappa)}u^{Np/2}C_\kappa(u)\ ,$$

which can be reduced to the form of (29) by simple calculation.

THEOREM 6. *Let $Z=XAX'$ be distributed as (10) and let W be a Wishart matrix with m degrees of freedom and independent of Z. Then the density function of*

(33) $$R=(XAX')^{1/2}(XAX'+W)^{-1}(XAX')^{1/2}$$

is given by

(34) $$\frac{\Gamma_p((N+m)/2)}{\Gamma_p(N/2)\Gamma_p(m/2)|A|^{p/2}}|R|^{(N-p-1)/2}|I-R|^{-(N+p+1)/2}$$
$$\times{}_1F_0^{(N)}\left(\frac{N+m}{2};\ -R(I-R)^{-1},\ A^{-1}\right),$$

and the density function of latent roots $\lambda_1>\lambda_2>\cdots>\lambda_p$ of the determinantal equation

(35) $$|R-\lambda I|=0$$

is given by

(36) $$\frac{\pi^{p^2/2}\Gamma_p((N+m)/2)}{\Gamma_p(N/2)\Gamma_p(m/2)\Gamma_p(p/2)|A|^{p/2}}\left(\prod_{i=1}^{p}\lambda_i\right)^{(N-p-1)/2}\left(\prod_{i=1}^{p}(1-\lambda_i)\right)^{-(N+p+1)/2}$$
$$\times\prod_{i<j}(\lambda_i-\lambda_j){}_1F_0^{(N)}\left(\frac{N+m}{2};\ -\Lambda(I-\Lambda)^{-1},\ A^{-1}\right).$$

PROOF. Since R is invariant under the simultaneous transformation $XAX'\to\frac{1}{2}\Sigma^{-1/2}XAX'\Sigma^{-1/2}$ and $W\to\frac{1}{2}\Sigma^{-1/2}W\Sigma^{-1/2}$, we may assume the joint density function of XAX' and W is

(37) $$\frac{1}{\Gamma_p(m/2)}|W|^{(m-p-1)/2}etr[-W]\cdot\frac{1}{\Gamma_p(N/2)|A|^{p/2}}|Z|^{(N-p-1)/2}{}_0F_0^{(N)}(A^{-1},-Z).$$

If we take the transformation $G=Z+W$, $R=G^{-1/2}ZG^{-1/2}$, the density function of G and R is

(38) $$\frac{1}{\Gamma_p(N/2)\Gamma_p(m/2)|A|^{p/2}}|R|^{(N-p-1)/2}|I-R|^{(m-p-1)/2}$$
$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(A^{-1})}{k!\,C_\kappa(I_N)}etr[-G(I-R)]|G|^{(N+m-p-1)/2}C_\kappa(-RG)\ .$$

Integrating out G (>0) from (38) with the help of (ii), we have

$$\frac{\Gamma_p((N+m)/2)}{\Gamma_p(N/2)\Gamma_p(m/2)|A|^{p/2}}|R|^{(N-p-1)/2}|I-R|^{-(N+p+1)/2} \times \sum_{k=0}^{\infty}\sum_{\kappa}\left(\frac{N+m}{2}\right)_{\kappa}\frac{C_\kappa(A^{-1})C_\kappa(-R(I-R)^{-1})}{k!\,C_\kappa(I_N)} . \tag{39}$$

Hence the first part of the theorem is obtained.

If we denote the matrix of latent roots $\Lambda=\text{diag}\{\lambda_1, \cdots, \lambda_p\}$, then from the invariance of $C_\kappa(S)$ under the orthogonal group,

$$C_\kappa(-R(I-R)^{-1})=C_\kappa(-\Lambda(I-\Lambda)^{-1}) . \tag{40}$$

(39) times dR is, therefore, written as the product of a function of Λ and $d^*(H)\,d\Lambda$ and hence the integration of it over the orthogonal group shows that the joint density function of λ's is given by (35).

If we set $A=I$ in (36) and use the formula

$$ {}_1F_0(a; z)=|I-Z|^{-a} , \qquad \text{(Constantine)} \tag{41}$$

(36) becomes

$$\frac{\pi^{p^2/2}\Gamma_p((N+m)/2)}{\Gamma_p(N/2)\Gamma_p(m/2)\Gamma_p(p/2)}\left(\prod_{i=1}^{p}\lambda_i\right)^{(N-p-1)/2}\left(\prod_{i=1}^{p}(1-\lambda_i)\right)^{(m-p-1)/2}\prod_{i<j}(\lambda_i-\lambda_j) ,$$

which is the density function in the central case [Constantine [2]].

THEOREM 7. *Let XAX' and W be independently distributed as (37). The density function of*

$$U=(XAX')^{-1/2}W(XAX')^{-1/2}, \tag{42}$$

the ratio of the quadratic form to the Wishart matrix, is given by

$$\frac{\Gamma_p((N+m)/2)}{\Gamma_p(N/2)\Gamma_p(m/2)|A|^{p/2}}|U|^{-(N+p+1)/2}\,{}_1F_0^{(N)}\left(\frac{N+m}{2}; U^{-1}, A^{-1}\right) . \tag{43}$$

PROOF. We can rewrite R as

$$\begin{aligned} R&=(XAX')^{1/2}(XAX'+W)^{-1}(XAX')^{1/2} \\ &=(I+(XAX')^{-1/2}W(XAX')^{-1/2})^{-1} \\ &=(I+U)^{-1} . \end{aligned} \tag{44}$$

Hence, if we put $R=(I+U)^{-1}$ in (34) it is easy to check that (34) is the same as (42), since Jacobian $J(R\to U)=|R|^{(p+1)}=|I+U|^{-(p+1)}$.

THEOREM 8. *Let U be distributed as (43). The probability of U is given by*

(45)
$$Pr\{(XAX')^{-1/2}W(XAX')^{-1/2}>B\} = \frac{\Gamma_p((N+m)/2)\Gamma_p((p+1)/2)}{\Gamma_p((N+p+1)/2)\Gamma_p(m/2)}\frac{|B|^{-N/2}}{|A|^{p/2}} \times {}_2F_1^{(N)}\left(\frac{N+m}{2}, \frac{N}{2}; \frac{N+p+1}{2}; B^{-1}, A^{-1}\right).$$

PROOF. In the inequality of Loewner's sense in matrix,

(46)
$$U>B \text{ is equivalent to } B^{-1}>U^{-1}>0 .$$

Hence, the integral is rewritten as

(47)
$$\int_{U>B}|U|^{-(N+p+1)/2}C_\kappa(U^{-1})\,dU=\int_0^{B^{-1}}|V|^{(N-p-1)/2}C_\kappa(V)\,dV .$$

According to the formula (iii), this definite integral is

(48)
$$\frac{\Gamma_p(N/2;\,\kappa)\Gamma_p((p+1)/2)}{\Gamma_p(N/2+(p+1)/2;\,\kappa)}|B|^{-N/2}C_\kappa(B^{-1}) .$$

Thus, we obtain (45) by integrating (43) with respect to U term by term.

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