TWO DISCRETE FRACTIONAL INTEGRAL OPERATORS REVISITED*

By

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In memory of Tom Wolff

1 Introduction

In this paper, we study discrete analogues of fractional integral operators. The key examples are given by the operators I_λ and J_λ, defined initially on functions on $\mathbb{Z}$ and $\mathbb{Z}^2$, by

$$I_\lambda f(m) = \sum_{n=1}^{\infty} f(m-n^2)\, n^{-\lambda} \quad \text{and} \quad J_\lambda f(m_1, m_2) = \sum_{n=1}^{\infty} f(m_1 - n, m_2 - n^2)\, n^{-\lambda}.$$

Let us deal with I_λ first. Its continuous analogue is the operator

$$F \longmapsto \int_1^\infty F(\lambda - t^2)\, \frac{dt}{t^\lambda} = \frac{1}{2}\int_1^\infty F(x-u)\, \frac{du}{u^{\frac{1}{2}+\frac{\lambda}{2}}}\,;$$

and, as is well-known, this operator is bounded from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$, wherever $1 < p < q < \infty$ and $1/p - 1/q = (1-\lambda)/2$. On the basis of this, it is a reasonable guess that I_λ is bounded from $\ell^p(\mathbb{Z})$ to $\ell^q(\mathbb{Z})$ wherever $1/p - 1/q = (1-\lambda)/2$ (to which one must add the further necessary conditions that $1/p > 1-\lambda$ and $1/q < \lambda$). While it is easy to make this conjecture, proving it seems quite difficult. There is a similar conjectural statement for the operator J_λ, but the numerology of the exponents is a little different.

In [SW$_2$], we began the study of these operators. Further advances were made by Oberlin [O]. These recent results have inspired us to return to the problem, and this work has allowed us to go a substantial way in completely resolving these questions. We now sketch some of the background.

The multiplier corresponding to I_λ is $m_\lambda(\theta) = \sum_{n=1}^{\infty} n^{-\lambda} e^{2\pi i n^2 \theta}$, that is, $\widehat{I_\lambda f}\,(\theta) = m_\lambda(\theta)\,\hat{f}(\theta)$. In [SW$_2$], we used the Hardy–Littlewood circle method to

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show that if $\frac{1}{2} < \lambda < 1$, $m_\lambda(\theta)$ is in $L^{r,\infty}$ with $r = 2/(1-\lambda)$. This implied the following.

Theorem A. *For $\frac{1}{2} < \lambda < 1$, I_λ is bounded from ℓ^p to ℓ^q provided $1/p - 1/q = (1-\lambda)/2$, $1/q < \lambda$, $1/p > 1 - \lambda$ and $p \leq 2 \leq q$.*

Interpolation with the trivial result that $I_{1+\epsilon}$ is bounded from ℓ^p to ℓ^p showed that for $\frac{1}{2} < \lambda < 1$, I_λ is bounded from ℓ^p to ℓ^q if $1/p - 1/q > (1-\lambda)/2$, $1/q < \lambda$ and $1/p > 1 - \lambda$. We obtained some results for $0 < \lambda \leq \frac{1}{2}$ by a further interpolation.

Simple examples show that for $0 < \lambda < 1$, the conditions $1/p - 1/q \geq (1-\lambda)/2$, $1/p > 1 - \lambda$ and $1/q < \lambda$ are necessary.

Oberlin's result is the following.

Theorem B. *For $1/p - 1/q > (1-\lambda)/2$, $1/p > 1 - \lambda$ and $1/q < \lambda$, $0 < \lambda < 1$, I_λ maps ℓ^p to ℓ^q.*

Theorem B substantially improves our results in the region $0 < \lambda < \frac{1}{2}$ (although it does not contain the case of equality for $1/2 < \lambda < 1$ in Theorem A).

In fact, after Theorem B the only remaining question is what happens in the case of equality when $1/p - 1/q = (1-\lambda)/2$. It should be noted that if $0 < \lambda \leq \frac{1}{3}$, these three conditions are equivalent to the two conditions $1/p > 1 - \lambda$ and $1/q < \lambda$. Figures 1, 2 and 3 illustrate this.

Theorem B shows that the open regions in the lower right-hand corner of the above figures correspond to p's and q's for which I_λ is bounded, improving our results for the case $0 < \lambda < \frac{1}{2}$ and solving the problem for $0 < \lambda < \frac{1}{3}$. Oberlin's arguments do not use the circle method, and his result motivated us to reexamine and substantially refine our arguments using that method. We prove the following result.

Theorem 1. *For $0 < \lambda \leq \frac{1}{2}$, $1/p - 1/q = (1-\lambda)/2$, $1/p > 1 - \lambda$ and $1/q < \lambda$,*

$$\|I_\lambda f\|_{\ell^q} \leq A\|f\|_{\ell^p}.$$

This conclusion goes beyond the previous results for λ's with $\frac{1}{3} < \lambda < \frac{1}{2}$. It completely settles the question for $0 < \lambda \leq 1/2$.

If one knew that $I_{1+i\gamma}$ were bounded on ℓ^p, $1 < p < \infty$, with suitable bounds depending on γ, interpolation with the results in [SW$_2$] would fully resolve the problem, i.e., the remaining cases when $1/2 < \lambda < 1$. We give a new result in this direction.

Theorem 2. *For $\frac{4}{3} < p < 4$,*

$$\|I_{1+i\gamma} f\|_{\ell^p} \leq A_\gamma \|f\|_{\ell^p}.$$

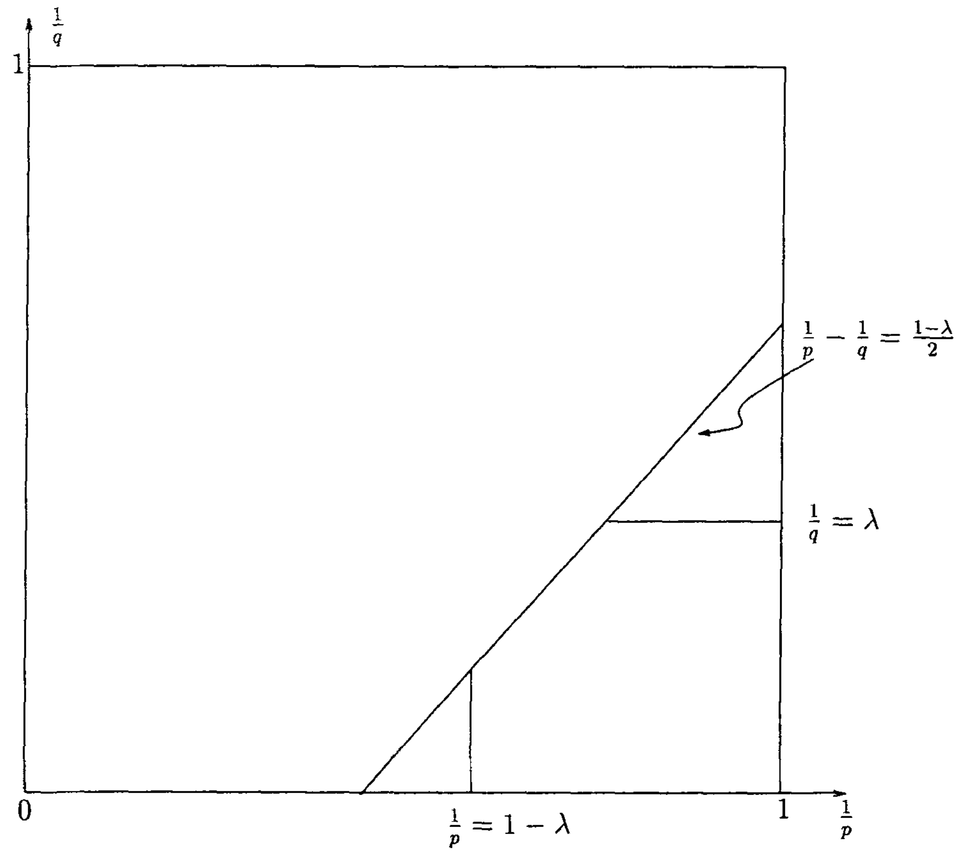

Figure 1. $\frac{1}{3} < \lambda < 1$

Theorem 2 is an improvement over the result in [SW_1]. Interpolation with the results of [SW_2] then shows the following.

Corollary of Theorem 2. *For* $\frac{1}{2} < \lambda < 1$,

$$\|I_\lambda f\|_{\ell^q} \leq A\|f\|_{\ell^p}$$

provided $1/p - 1/q = (1-\lambda)/2$, $1/p > 1-\lambda$, $1/q < \lambda$, *and* $1/p < \frac{3}{4}$ *and* $1/q > \frac{1}{4}$.

So the remaining boundedness question concerns the two half-open solid line segments on the line segment $1/p - 1/q = (1-\lambda)/2$ in Figure 4.

The situation for J_λ is similar with a different range of exponents. The multiplier is

$$m_\lambda(\theta,\phi) = \sum_{n=1}^{\infty} e^{2\pi i n^2\theta}\, e^{2\pi i n\phi}\, n^{-\lambda}.$$

In [SW$_2$], we showed that for $\frac{1}{2} < \lambda < 1$, $m_\lambda(\theta,\phi)$ is in $L^{r,\infty}$ with $r > 3/(1-\lambda)$. This implies Theorem C.

Theorem C. *Suppose $\frac{1}{2} < \lambda < 1$, $1/p - 1/q = (1-\lambda)/3$, $1/p > 1-\lambda$, $1/q < \lambda$; then J_λ is bounded from ℓ^p to ℓ^q provided p and q satisfy $p \le 2 \le q$.*

Interpolation shows that J_λ is bounded from ℓ^p to ℓ^q provided $\frac{1}{2} < \lambda < 1$, $1/p - 1/q > (1-\lambda)/3$, $1/p > 1-\lambda$ and $1/q < \lambda$.

Oberlin [O] proved the following result.

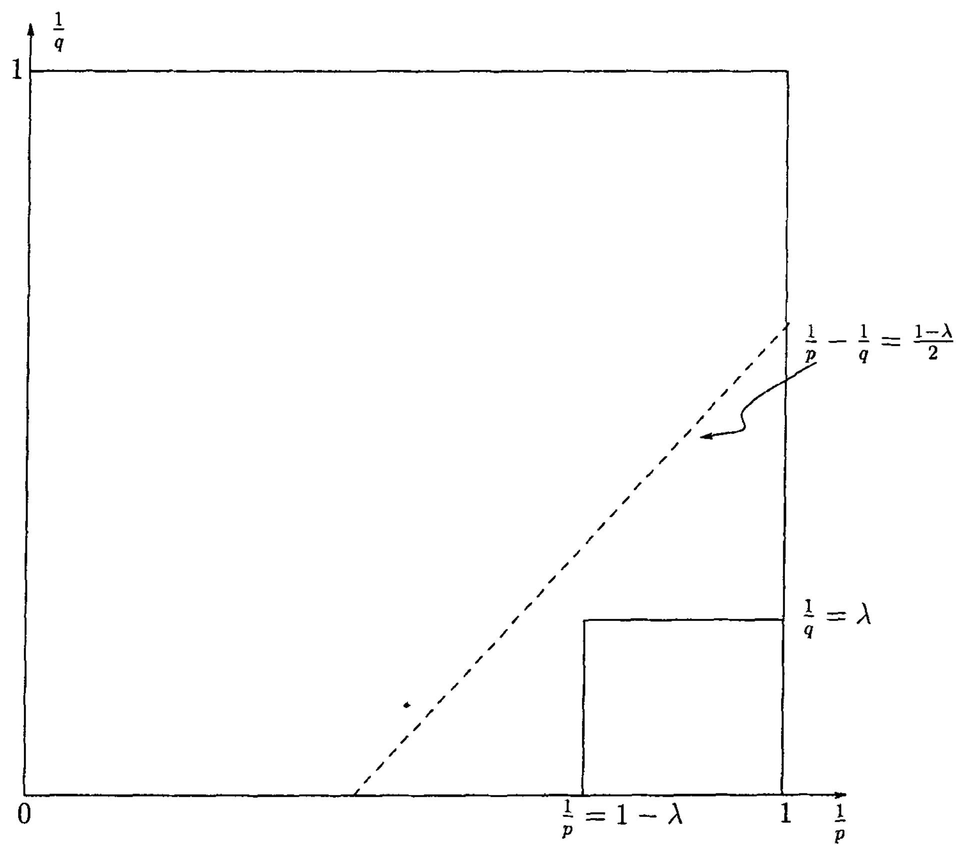

Figure 2. $0 < \lambda < \frac{1}{3}$

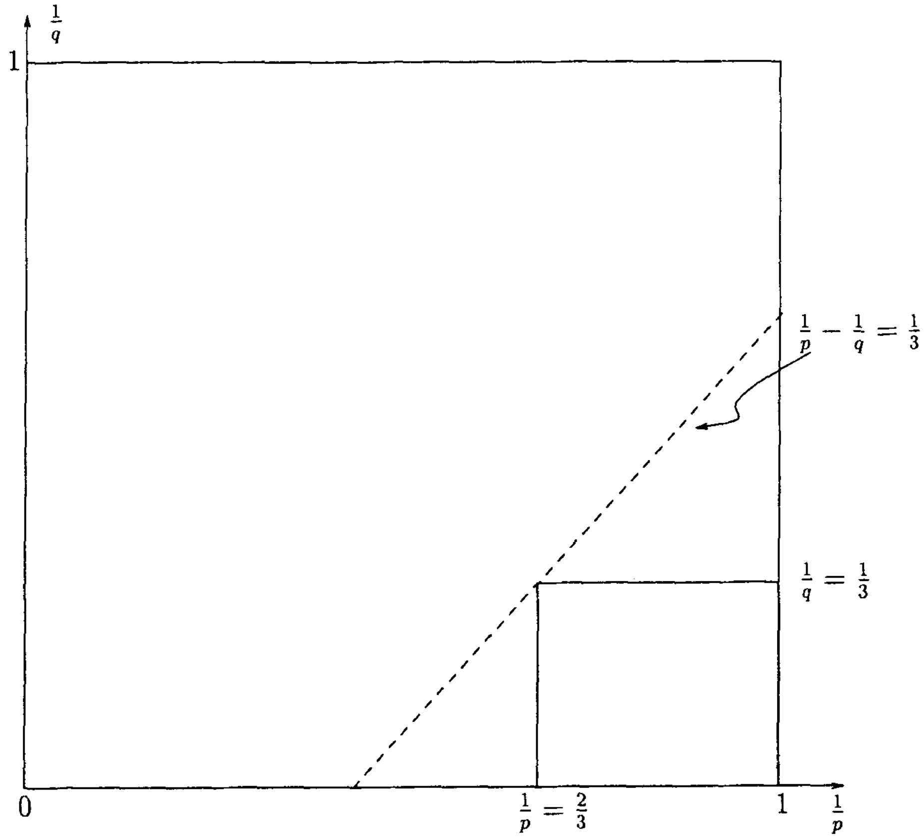

Figure 3. $\lambda = \frac{1}{3}$

Theorem D. *Suppose* $0 < \lambda < 1$, $1/p - 1/q > (1-\lambda)/3$, $1/p > 1 - \lambda$ *and* $1/q < \lambda$; *then* J_λ *maps* ℓ^p *to* ℓ^q.

Again, simple examples show $1/p - 1/q \geq (1-\lambda)/3$, $1/p > 1-\lambda$ and $1/q < \lambda$ are necessary conditions for J_λ to map ℓ^p to ℓ^q. Oberlin's result solves the $\ell^p \to \ell^q$ boundedness question for p and q satisfying $1/p - 1/q > (1-\lambda)/3$, but not in the case where $>$ is replaced by $=$. We settle this question for $0 < \lambda \leq \frac{1}{2}$.

Theorem 3. *For* $0 < \lambda \leq \frac{1}{2}$,

$$\|J_\lambda f\|_{\ell^p} \leq A_p \|f\|_{\ell^p}$$

provided $1/p - 1/q = (1-\lambda)/3$, $1/p > 1 - \lambda$ *and* $1/q < \lambda$.

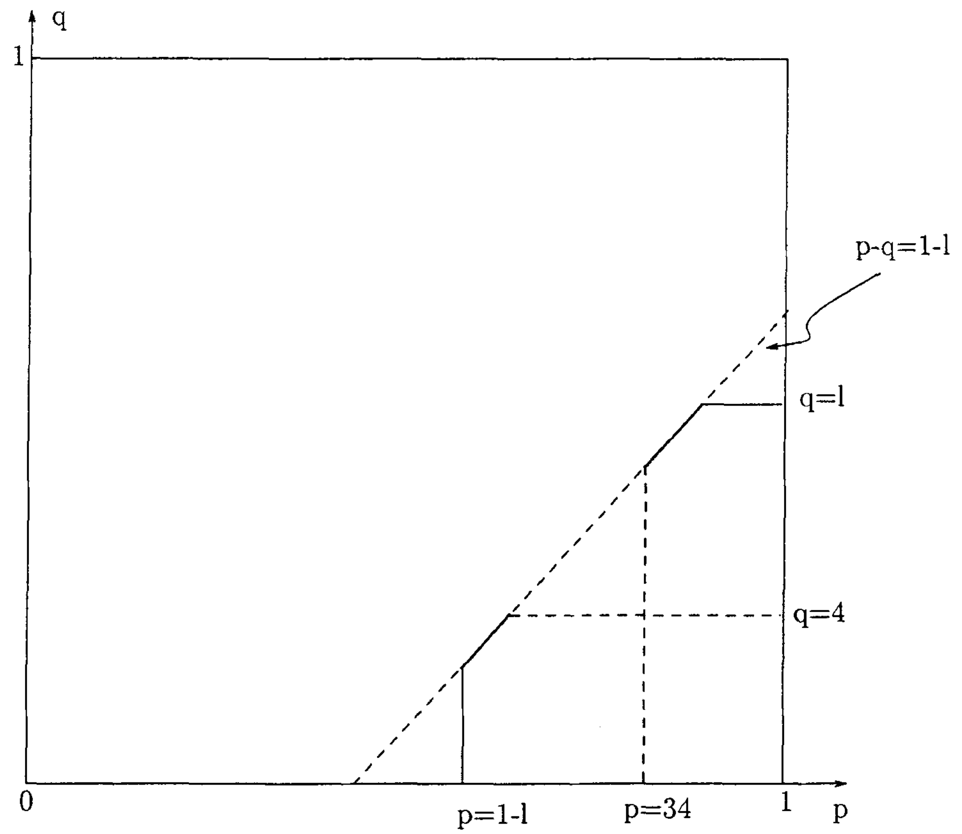

Figure 4.

Note that Theorem 3 goes beyond Theorem D only when $\lambda > \frac{2}{5}$.

We shall also show that the method of the proof of Theorem 2 yields

Theorem 4. $\|J_{1+i\gamma} f\|_{\ell^p} \leq A_\gamma \|f\|_{\ell^p}$ *for* $\frac{3}{2} < p < 3$.

(This result was obtained by a different method in [SW$_1$].) We then have the following.

Corollary of Theorem 4. *For* $\frac{1}{2} < \lambda < 1$,

$$\|J_\lambda f\|_{\ell^q} \leq A\|f\|_{\ell^p}$$

provided $1/p - 1/q = (1-\lambda)/3$, $1/p > 1-\lambda$, $1/q < \lambda$, and $1/p < \frac{2}{3}$ and $1/q > \frac{1}{3}$.

We now describe somewhat imprecisely the new ideas involved in the proofs. As before, we begin by writing the multiplier of the operator I_λ, $m_\lambda(\theta)$, as approximately

$$\sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} \mathcal{F}(y - i\theta)\, y^{\lambda/2}\, dy \tag{1.1}$$

with

$$\mathcal{F}(z) = \sum_{\eta=-\infty}^{\infty} n^2\, e^{-2\pi n^2 z},$$

where $\mathcal{F}$ is a derivative of a Θ function. Next, consider a Farey dissection (of level $2^{j/2}$) in terms of fractions a_j/q_j, $(a_j, q_j) = 1$, $0 < a_j \leq q_j$, and $q_j \leq 2^{j/2}$. We apply this dissection to the jth summand in (1.1); and, using a well-known approximation to Θ which results from its transformation formula, we get modulo error terms

$$m_\lambda(\theta) \sim$$

$$\sum_{q \lesssim 2^{j/2}} \sum_{\substack{a/q = a_j/q_j \\ (a,q)=1}} \frac{1}{q_j} S(a_j, q_j)\, \chi_{q_j}(\theta - a_j/q_j) \int_{2^{-j}}^{2^{-j+1}} (y - i(\theta - a_j/q_j))^{-3/2}\, y^{\lambda/2}\, dy.$$

Here $\chi_{q_j}(\theta - a_j/a_\lambda) = \chi(q_j 2^{j/2}(\theta - a_j/q_j))$, with χ the characteristic function of the unit interval. The $S(a, q)$ are the Gauss sums.

In our previous paper, we used essentially this formula to get an estimate of the size of $|m_\lambda(\theta)|$, which was enough to prove our restricted results for $\ell^p \to \ell^q$ boundedness. If we want to go further, we must analyze not only the size of m_λ, but in effect its cancellation properties which will ultimately determine the boundedness of I_λ. We do this in two steps.

First we rearrange the sum above by focusing on the size of the denominators q_j which occur. That is, for each s we consider those q_j's where $2^s \leq q_j < 2^{s+1}$ and define, using the above summation, $B_\lambda(s, \theta)$ by

$$B_\lambda(s, \theta) = \sum_{2^s \leq q < 2^{s+1}} \sum_j \sum_{q_j = q} \cdots .$$

Thus, essentially,

$$m_\lambda(\theta) = \sum_{s=0}^{\infty} B_\lambda(s, \theta).$$

We let $\mathcal{B}_\lambda(s)$ denote the convolution operator whose multiplier is $B_\lambda(s, \theta)$. It turns out that the crucial estimates for $\mathcal{B}_\lambda(s)$ (and hence I_λ) are for λ near $\frac{1}{3}$. We obtain these by interpolation of two estimates. First

$$\|\mathcal{B}_\lambda(s)\|_{\ell^2 \to \ell^2} \leq A_\lambda\, 2^{-s/2}, \quad \text{when } \Re(\lambda) = 1,$$

which again is merely a size estimate of $B_\lambda(s,\theta)$. The second estimate needed for the interpolation is

$$\|\mathcal{B}_\lambda(s)\|_{\ell^1\to\ell^\infty} \leq A_{\lambda,\eta}\, 2^{(1+\eta)s}, \quad \text{for } \Re(\lambda) = -1,$$

and any $\eta > 0$.

This is not an estimate of the size of the multiplier $B_\lambda(s,\theta)$, but the size of its Fourier coefficients, and is the most delicate part of the proof. Very curiously it involves $\lambda = -1$, and for this value of λ we know of no simple interpretation of the operator I_λ. Once these estimates have been proved, we get the desired results for the main contributions to $m_\lambda(\theta)$. There are also two error terms, but these can be handled by somewhat similar but cruder estimates, which will then complete the proof of Theorem 1.

2 The proof of Theorem 1

For any complex number λ, define the operator I_λ as

$$I_\lambda f(m) = \sum_{n=1}^{\infty} \frac{f(m-n^2)}{n^\lambda}.$$

The main job in proving Theorem 1 is to prove that for $\frac{1}{3} < \lambda \leq \frac{1}{2}$ (especially when λ is close to $\frac{1}{3}$)

$$\|I_\lambda f\|_{\ell^{p'}} \leq A\|f\|_{\ell^p}, \tag{2.1}$$

whenever $1/p + 1/p' = 1$ and $1/p - 1/p' = (1-\lambda)/2$. Theorem 1 then follows from the first Main Theorem of [SW$_2$], Theorem A above, by interpolation. Now

$$\widehat{I_\lambda f}(\theta) = m_\lambda(\theta)\hat{f}(\theta) \quad \text{with } m_\lambda(\theta) = \sum_{n=1}^{\infty} e^{2\pi i n^2\theta} n^{-\lambda}.$$

If $\Re\lambda \leq \frac{1}{2}$, $m_\lambda(\theta)$ does not exist as a function but is a well-defined distribution. Then slightly modifying the discussion in [SW$_2$], we write

$$n^{-\lambda} = c(\lambda)n^2 \int_0^\infty e^{-2\pi n^2 y}\, y^{\lambda/2} dy \quad \text{with } c(\lambda) = (2\pi)^{(2+\lambda)/2} \Big/ \Gamma\left(\frac{\lambda+2}{2}\right).$$

Then

$$m_\lambda(\theta) = c(\lambda) \int_0^\infty y^{\frac{\lambda}{2}}\, \mathcal{F}(y - i\theta)\, dy \quad \text{with } \mathcal{F}(z) = \sum_{n=1}^{\infty} n^2\, e^{-2\pi n^2 z}$$

for $\Re z > 0$. It is clear that the contribution of $\int_1^\infty$ to $m_\lambda(\theta)$ has an absolutely convergent Fourier series and thus corresponds to an operator that is bounded form ℓ^p to ℓ^q whenever $q \geq p \geq 1$. Thus it suffices to consider the operator with multiplier

$$\nu_\lambda(\theta) = \int_0^1 y^{\lambda/2}\,\mathcal{F}(y - i\theta)\,dy = \sum_{j=1}^{\infty} \nu_{\lambda,j}\,(\theta),$$

where

$$\nu_{\lambda,j}(\theta) = \int_{2^{-j}}^{2^{-j+1}} y^{\lambda/2}\,\mathcal{F}(y - i\theta)\,dy.$$

Following the lines in [SW_2] we shall prove

Proposition 2.1. *For* $2^{-j} \leq y \leq 2 \cdot 2^{-j}$, $q \leq 2^{-j/2}$, $(a,q) = 1$, *and* $|\theta - a/q| \leq 1/q2^{j/2}$,

$$\mathcal{F}(y - i\theta) = \frac{1}{2\pi\, 2^{3/2}} \cdot \frac{S(a,q)}{q(y - i(\theta - a/q))^{3/2}} + E_{y,q}\,(\theta - a/q), \tag{2.2}$$

where

$$E_{y,q}(\theta) = \sum_{m\neq 0} \frac{S(a,q,m)}{q}\, h_{m,q,y} \quad \textit{and} \quad |h_{m,q,y}(\theta)| \leq \frac{Aq^{1/2}}{|y|^{5/4}}\, e^{-\delta m^2}$$

with $\delta > 0$, *for* $|\theta| \leq 1/q2^{j/2}$.

Here

$$S(a,q) = \sum_{r=1}^{q} e^{2\pi i r^2 a/q} \quad \text{and} \quad S(a,q,m) = \sum_{r=1}^{q} e^{2\pi i r^2 a/q}\, e^{2\pi i m r/q}.$$

We postpone the proof of Proposition 2.1.

Motivated by Proposition 2.1, we define major arcs, for a fixed j, as follows. For $0 \leq s \leq j/2 - 10$, $2^s \leq q < 2^{s+1}$, and $(a,q) = 1$, set

$$I_j(a,q,s) = \{\theta : |\theta - a/q| \leq 1/2^s 2^{j/2}\}.$$

We also define

$$I_j(1,1) = \{\theta : 0 \leq \theta \leq 1/22^{j/2}\} \cup \{\theta : 1 - 1/22^{j/2} \leq \theta \leq 1\}.$$

The contribution from $I_j(1,1)$ to our operators may be treated in the same manner as the other intervals; however, the discussion of $I_j(1,1)$ causes some notational inconvenience, so we omit it. Note that since $s \leq j/2 - 10$, the major arcs corresponding to a fixed j are disjoint. For if

$$I_j(a_1, q_1, s_1) \cap I_j(a_2, q_2, s_2) \neq \emptyset,$$

we would have

$$\frac{1}{q_1 q_2} \le \left| \frac{a_1}{q_1} - \frac{a_2}{q_2} \right| \le \frac{2}{2^{j/2}} \max\left(\frac{1}{2^{s_1}}, \frac{1}{2^{s_2}} \right).$$

So

$$2^{-s_1} 2^{-s_2} < 2^{-j/2} \max\left(2^{-s_1}, 2^{-s_2}\right),$$

which contradicts the fact that s_1 and s_2 are less than or equal to $j/2 - 10$. We then decompose $\nu_\lambda(\theta)$ into three parts:

$$\nu_\lambda(\theta) = P_{\lambda,1}(\theta)/2\pi 2^{3/2} + P_{\lambda,2}(\theta) + P_{\lambda,3}(\theta).$$

$P_{\lambda,1}(\theta)$ is the contribution to the major arcs from the main term in (2.2), that is, $S(a,q)/q(y - i(\theta - a/q))^{3/2}$. The term $P_{\lambda,2}(\theta)$ is the contribution from $E_{y,q}(\theta - a/q)$ in the major arcs, and $P_{\lambda,3}(\theta)$ denotes the contribution from the complement of the major arcs.

In analyzing the three components of $\nu_\lambda(\theta)$, we deal with a double summation: in the index j arising from the decomposition

$$\nu(\theta) = \sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} y^{\lambda/2} \mathcal{F}(y - i\theta)\, dy$$

and the index s, which measures the size of q for the fraction a/q. For the term $P_{\lambda,1}$, we carry out the summation in j first and then the summation in s. For both $P_{\lambda,2}$ and $P_{\lambda,3}$, the summation is in the reverse order, first in s and then in j.

Let χ denote the characteristic function of $[-1,1]$. Then, for $0 \le \theta \le 1$,

$$\begin{aligned} P_{\lambda,1}(\theta) &= \sum_{j=1}^{\infty} \sum_{s=1}^{\frac{j}{2}-10} \sum_{q(j)=2^s}^{2^{s+1}} \cdot \sum_{\substack{a(j)=1 \\ (a(j),q(j)=1}}^{q(j)} \frac{S(a(j),q(j))}{q(j)} \chi\left(2^{\frac{j}{2}} 2^s \left(\theta - \frac{a(j)}{q(j)}\right)\right) \\ &\qquad \cdot \int_{2^{-j}}^{2\cdot 2^{-j}} \frac{y^{\lambda/2}}{\left(y - i\left(\theta - \frac{a(j)}{q(j)}\right)\right)^{3/2}} dy \\ &= \sum_{s=1}^{\infty} \sum_{j \ge 2s+20} \\ &= \sum_{s=1}^{\infty} B_\lambda(s,\theta), \end{aligned}$$

where

$$B_\lambda(s,\theta) = \sum_{q=2^s}^{2^{s+1}} \frac{1}{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(a,q) \cdot \sum_{\substack{j \\ j \geq 2s+20 \\ a(j)=a \\ q(j)=q}} \chi\left(2^{j/2} 2^s \left(\theta - \frac{a(j)}{q(j)}\right)\right) \cdot$$

$$\int_{2^{-j}}^{2 \cdot 2^{-j}} \frac{y^{\lambda/2}}{\left(y - i\left(\theta - \frac{a(j)}{q(j)}\right)\right)^{3/2}} \, dy.$$

We remark that because of the cut-off function $\chi(2^{j/2} 2^s (\theta - a(j)/q(j)))$, the functions $P_{\lambda,1}(\theta)$ and $B_\lambda(s,\theta)$ are supported in $(0,1)$. To be precise, we should consider these functions extended to all of R to be periodic with period 1. Similar remarks apply to other functions of θ below.

Next note that for a fixed s (but varying j with $j \geq 2s + 20$) the supports of the functions $\chi(2^{j/2}2^s(\theta - a(j_1)/q(j_1)))$ and $\chi(2^{j/2}2^s(\theta - a(j_2)/q(j_2)))$, with $a(j_1)/q(j_1) \neq a(j_2)/q(j_2)$, are disjoint. For otherwise we would have

$$\frac{1}{4\, 2^{2s}} \leq \left| \frac{a(j_1)}{q(j_1)} - \frac{a(j_2)}{q(j_2)} \right| \leq \frac{2}{2^s} \max\left(\frac{1}{2^{j_1/2}}, \frac{1}{2^{j_2/2}}\right).$$

But $j_1, j_2 \geq 2s + 20$, so this cannot happen. Thus,

$$B_\lambda(s,\theta) = \sum_{q=2^s}^{2^{s+1}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S(a,q)}{q} \cdot \int_{\rho(s,\theta)}^{2^{-2s-19}} y^{\lambda/2} \frac{dy}{\{y - i(\theta - a/q)\}^{3/2}}.$$

Here $\rho(s,\theta) = 2^{-j}$, where j is the largest integer such that $2^{-j} \geq 2^{2s}\theta^2$. So $2^{2s}\theta^2 \leq \rho(s,\theta) < 2 \cdot 2^{2s}\theta^2$. If $\rho(s,\theta) \geq 2^{-2s-19}$, the integral is interpreted to be 0. Note that as a result we can always insert the factor $\chi(c2^{2s}(\theta - a/q))$ in front of the integral in the above sum for an appropriate c, e.g., $c = 10$.

We set

$$\mathcal{B}_{\lambda,s} f = (B_\lambda(s,\theta)\, \hat{f}(\theta))^\vee.$$

Our goal is now to show that for $\frac{1}{3} < \lambda < 1$, $1/p + 1/p' = 1$, $1/p - 1/p' = (1-\lambda)/2$,

$$\|\mathcal{B}_{\lambda,s} f\|_{\ell^{p'}} \leq A 2^{-\delta(\lambda)s} \|f\|_{\ell^p} \tag{2.3}$$

for some positive δ. For then we can add in s to show that the operator corresponding to $P_{\lambda,1}$ satisfies the estimate of (2.1).

We prove (2.3) by complex interpolation. In fact, we show that $\mathcal{B}_{\lambda,s}$ extends to an analytic family of operators in $-1 \leq \Re\lambda \leq 1$.

For $\lambda = 1 + i\gamma$, we have

$$\|\mathcal{B}_{\lambda,s} f\|_{\ell^2} \leq A\, 2^{-s/2} (1/|\gamma| + 1)\|f\|_{\ell^2}; \tag{2.4}$$

and for $\lambda = -1 + i\lambda$, and any $\eta > 0$, we have

$$\|\mathcal{B}_{\lambda,s} f\|_{\ell^\infty} \leq A_\eta\, 2^{s(1+\eta)}\|f\|_{\ell^1}. \tag{2.5}$$

The estimate (2.3) will follow from (2.4) and (2.5) by complex interpolation where we interpolate the operators $f \to (\lambda - 1)\, \mathcal{B}_{\lambda,s} f$.

To deal with $B_\lambda(s, \theta)$, we consider first

$$U_\lambda(s,\theta) = \chi(c\, 2^{2s}\theta) \int_{\rho(s,\theta)}^{2^{-2s-19}} y^{\lambda/2} (y - i\theta)^{-3/2}\, dy.$$

Here $U_\lambda(s,\theta) = 0$ if $\rho(s,\theta) \geq 2^{-2s-19}$.

It is clear that $U_\lambda(s,\theta)$ is integrable as a function of θ for $\Re\lambda > -1$ and that if $g_1(\theta)$ and $g_2(\theta)$ are trigonometric polynomials, $\int U_\lambda(s,\theta)\, g_1(\theta)\, g_2(\theta)\, d\theta$ is analytic in $\Re\lambda > -1$.

In order to continue U_λ to $\Re\lambda \geq -1$, we split U_λ into two parts. Let $\psi(\theta)$ be a smooth function supported in $|\theta| \leq 2$ with $\psi(\theta) = 1$ for $|\theta| \leq 1$ and choose a constant c_1 ($c_1 = 2c$ will do) so that

$$\psi(c_1\theta)\, \chi(c\theta) = \psi(c_1\theta).$$

Then

$$U_\lambda(s,\theta) = U_{\lambda,1}(s,\theta) + U_{\lambda,2}(s,\theta),$$

where

$$U_{\lambda,1}(s,\theta) = \psi(c_1\, 2^{2s}\theta) \int_0^{2^{-2s-19}} y^{\lambda/2} (y - i\theta)^{-3/2}\, dy,$$

and

$$U_{\lambda,2}(s,\theta) = \chi(c\, 2^{2s}\theta)\, \left(1 - \psi(c_1\, 2^{2s}\theta)\right) \int_{\rho(s,\theta)}^{2^{-2s-19}} y^{\lambda/2} (y - i\theta)^{-3/2}\, d\theta$$

$$- \psi(c_1\, 2^{2s}\theta) \int_0^{\rho(s,\theta)} y^{\lambda/2} (y - i\theta)^{-3/2}\, dy$$

$$= U_{\lambda,3}(s,\theta) + U_{\lambda,4}(s,\theta).$$

As with U_λ, it is clear that $U_{\lambda,1}$ and $U_{\lambda,2}$ are integrable if $\Re\lambda > -1$ and, for g_1 and g_2 trigonometric polynomials, $\int U_{\lambda,j}\, g_1 g_2$ are analytic functions of λ in $\Re\lambda > -1$ for $j = 1, 2$.

We need the following lemma.

Lemma 2.1. (i) *For* $j = 1,2$ *and* $\Re\lambda > -1$, $U_{\lambda,j}(s,\theta)$ *is supported in* $|\theta| \le 1/c\,2^{2s} = \frac{1}{10}\,2^{-2s}$.

(ii) *For* $j = 1,2$ *and* $\Re\lambda = 1 + i\gamma$, *the functions* $U_{\lambda,j}(s,\theta)$ *satisfy*

$$|U_{1+i\gamma,j}\,(s,\theta)| \le A\,(1/|\gamma| + 1).$$

(iii) *For each* ℓ, $c_\ell(U_{\lambda,1}(s,\theta))$ *(originally defined for* $\Re\lambda > -1$*) extends to a function analytic in* $\Re\lambda > -2$ *and for* $\lambda \ge -1$ *satisfies*

$$|c_\ell(U_{\lambda,1}(s,\theta))| \le A\,.$$

(Here c_ℓ *denotes the* ℓ*th Fourier coefficient.)*

(iv) *For* $\theta \ne 0$,

$$U_{-1+i\gamma,2}\,(s,\theta) = \lim_{\lambda\to -1+i\gamma} U_{\lambda,2}(s,\theta)$$

exists, and there is an L^1 *function* $h(\theta)$ *such that, for* $\Re\lambda > -1$,

$$|U_{\lambda,2}(\theta)| \le A\,h(\theta),$$

where the constant A *is independent of* λ.

Proof of Lemma 2.1. (i) is clear. To prove (ii), make the change of variables $y = y'\theta$ in each of the integrals for $U_{\lambda,j}$, $j = 1,3,4$ to see that

$$|U_{\lambda,j}\,(s,\theta)| \le A\left|\int_{A_j(\theta)}^{B_j(\theta)} y^{\frac{1}{2}+i\gamma}\,\frac{dy}{(y\pm i)^{3/2}}\right|.$$

To estimate the integral, it suffices to observe that

$$\int_1^A y^{(1+i\gamma)/2}\,\frac{dy}{(y\pm i)^{3/2}} = \int_1^A y^{-1+i\gamma/2}\,dy + O\left(\int_1^A y^{-2}\,dy\right),$$

and

$$\int_1^A y^{-1+i\frac{\gamma}{2}}\,dy = \frac{2}{i\gamma}\,\left(A^{i\gamma} - 1\right).$$

Therefore,

$$|U_{1+i\gamma,j}\,(s,\theta)| \le A\,(1/|\gamma| + 1)\,.$$

We turn to (iii). Since $U_{\lambda,1}$ has small support, the integral over $[-\frac{1}{2}, \frac{1}{2}]$, defining its Fourier coefficients, may be replaced by an integration from $-\infty$ to ∞. Then

$$c_\ell(U_{\lambda,1}) = d_{\ell,s} \star e^{\lambda}_{\ell,s},$$

where $d_{\ell,s}$ is the Fourier transform of $\psi(c_1\, 2^{2s}\cdot)$ evaluated at ℓ, and

$$e^{\lambda}_{\ell,s} = \int_{-\infty}^{\infty} e^{-2\pi i\ell\theta} \int_0^{2^{-2s-19}} y^{\lambda/2}\,(y-i\theta)^{-3/2}\,dy\,.$$

Since $\sum_\ell |d_{\ell,s}| \leq A$, it suffices to show $e^{\lambda}_{\ell,s}$ extends in λ to a function analytic in $\Re\lambda > -2$ and, for $\Re\lambda \geq -1$,

$$\left|e^{\lambda}_{\ell,s}\right| \leq A.$$

To study $e^{\lambda}_{\ell,s}$, we use the identity

$$\int_{-\infty}^{\infty} e^{-2\pi i\xi\theta}\,(y-i\theta)^{-3/2}\,d\theta = \begin{cases} \dfrac{(2\pi)^{3/2}}{\Gamma\left(\frac{3}{2}\right)}\,\xi^{1/2}\,e^{-2\pi\xi y}, & \xi > 0, \\[2ex] 0, & \xi \leq 0. \end{cases}$$

(This identity can be checked by taking the Fourier transform of the right-hand side and using the Fourier inversion formula.) Thus

$$\begin{aligned} e^{\lambda}_{\ell,s} &= \int_0^{2^{-2s-19}} y^{\lambda/2} \int_{-\infty}^{\infty} e^{-2\pi i\ell\theta}\,(y\,-\,i\theta)^{-3/2}\,d\theta\,dy \\ &= \begin{cases} A\ell^{1/2}\,\int_0^{2^{-2s-19}} y^{\lambda/2}\,e^{-2\pi\ell y}dy, & \text{if } \ell > 0, \\ 0, & \text{if } \ell \leq 0. \end{cases} \end{aligned}$$

It is clear that $e^{\lambda}_{\ell,s}$ is analytic in $\Re\lambda > -2$, and a change of variables shows that

$$\left|e^{\lambda}_{\ell,s}\right| \leq A \quad \text{for } \Re\lambda \geq -1\,.$$

We consider next (iv). The limit clearly exists for $\theta \neq 0$, and we turn to the domination of $U_{\lambda,2}$ by an L^1 function $h(\theta)$. In the support of $(1-\psi(c_1 2^{2s}\theta))$, $\rho(s,\theta)$ (which is roughly $2^{2s}\theta^2$) is greater than $c_2 2^{-2s}$ for a positive c_2. Thus for $\Re\lambda \geq -1$,

$$\begin{aligned} |\,U_{\lambda,3}(\theta)\,| &\leq A\,\chi(c2^{2s}\theta)\,(1-\psi(c_1 2^{2s}\theta))\int_{c_2 2^{-2s}}^{2^{-2s-19}} y^{-1/2}\,|y-i\theta|^{-3/2}\,dy \\ &\leq A\,\chi(c2^{2s}\theta)2^{2s}\,, \end{aligned}$$

and the L^1 norm of this function is bounded.

Next, for $\Re\lambda \geq -1$,

$$\begin{aligned}|U_{\lambda,4}(s,\theta)| &\leq |\psi(c_1 2^{2s}\theta)| \int_0^{\rho(s,\theta)} y^{-1/2}\,|y - i\theta|^{-3/2}\,dy \\ &\leq A\,|\psi(c_1 2^{2s}\theta)|\,\theta^{-3/2}\,(\rho(s,\theta))^{1/2} \\ &\leq A\,\psi(c_1 2^{2s}\theta)\,\theta^{-1/2}\,2^s.\end{aligned}$$

The right-hand side of the above inequality clearly has bounded L^1 norm, and the proof of the lemma is complete.

From Lemma 2.1, we see that $U_{\lambda,s}$, defined originally for $\Re\lambda > -1$, extends to a family of distributions in $\Re\lambda \geq -1$ and, in fact, defines an analytic family of operators in $-1 \leq \Re\lambda \leq 1$. For if g_1 and g_2 are trigonometric polynomials, the function

$$\int U_{\lambda,s}\, g_1\, g_2,$$

which is an analytic function of λ for $\Re\lambda > -1$, is seen to be continuous up to $\Re\lambda = -1$ by using Lemma 2.1 and the dominated convergence theorem. To prove that (2.4) and (2.5) are satisfied, we need the following proposition.

Proposition 2.2. (i) $|S(a,q,m)| \leq A\,q^{1/2}$.

(ii) *For any $\eta > 0$, there is a constant A_η such that*

$$\sum_{1\leq q\leq M} \left| \sum_{\substack{a=1\\(a,q)=1}}^{q} S(a,q,m)\, e^{2\pi i \ell \frac{a}{q}} \right| \leq A_\eta\, M^{2+\eta},$$

for all M, m and ℓ.

Here we only need conclusion (ii) when $m = 0$, so that $S(a,q,m) = S(a,q)$; later we require conclusion (ii) for any m. The assertion (i) is well-known, see [W]; we prove conclusion (ii) later.

Now

$$B_\lambda(s,\theta) = \sum_{q=2^s}^{2^{s+1}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{S(a,q)}{q}\, U_\lambda\left(s, \theta - \frac{a}{q}\right).$$

So since $B_\lambda(s,\theta)$ is a finite sum of translates of $U_\lambda(s,\theta)$, it follows that $\mathcal{B}_{\lambda,s}$ is an analytic family of operators in $-1 \leq \Re\lambda \leq 1$. Conclusion (i) of Lemma 2.1 implies that the supports of the $U_\lambda(s, \theta - a/q)$ are disjoint, so conclusion (ii) of Lemma 2.1 and Proposition 2.2 imply

$$|B_{1+i\gamma}(s,\theta)| \leq A\,(1 + 1/|\gamma|)2^{-s/2}.$$

Thus

$$\|\mathcal{B}_{1+i\gamma} f\|_{\ell_2} \leq A(1 + 1/|\gamma|)\, 2^{-s/2} \|f\|_{\ell^2},$$

which is (2.4). Also,

$$c_\ell (B_{-1+i\gamma}(s,\theta) = c_\ell (U_{-1+i\gamma}(s,\theta)) \cdot \sum_{q=2^s}^{2^{s+1}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S(a,q)}{q}\, e^{2\pi i \ell \frac{a}{q}}.$$

Thus conclusions (iii) and (iv) of Lemma 2.1 together with Proposition 2.2, part (ii) imply, for any $\eta > 0$,

$$|c_\ell (B_{-1+i\gamma}(\ell,\theta))| \leq A_\eta\, 2^{(1+\eta)s}.$$

Thus

$$\|\mathcal{B}_{-1+i\gamma} f\|_{\ell^\infty} \leq A_\eta\, 2^{(1+\eta)s} \|f\|_{\ell^1},$$

giving (2.5). This concludes the proof of (2.3) and completes our discussion of $P_{\lambda,1}$.

We now turn to $P_{\lambda,2}$:

$$P_{\lambda,2}(\theta) = \sum_{j=1}^{\infty} P^j_{\lambda,2}(\theta),$$

where

$$P^j_{\lambda,2}(\theta) = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \sum_{s=1}^{j/2-10} \sum_{q=2^s}^{2^{s+1}} \frac{1}{q}$$

$$\times \sum_{a=1}^{q} S(a,q,m)\, \chi\left(2^{j/2}\, 2^s \left(\theta - \frac{a}{q}\right)\right) \int_{2^{-j}}^{2\cdot 2^{-j}} y^{\lambda/2}\, h_{m,q,y}\left(\theta - \frac{a}{q}\right) dy,$$

with $h_{m,q,y}(\theta)$ as in Proposition 2.1. We shall show that for $\Re\lambda = \frac{1}{2} + \epsilon$,

(2.6) $$\|P^j_{\lambda,2}(\theta)\|_{L^\infty} \leq A\, 2^{-\frac{\epsilon}{2} j},$$

so that

$$\|(P^j_{\lambda,2}\, \hat{f})^\vee\|_{\ell^2} \leq A\, 2^{-\frac{\epsilon}{2} j} \|f\|_{\ell^2},$$

and for $\Re\,\lambda = \epsilon$, and any $\eta > \epsilon$,

(2.7) $$|c_\ell (P^j_{\lambda,2})| \leq A_\eta\, 2^{-\frac{\epsilon}{2} j}\, 2^{\eta j},$$

so that

$$\|(P^j_{\lambda,2} \hat{f})^\vee\|_{\ell^\infty} \leq A_\eta\, 2^{-\frac{\epsilon}{2} j}\, 2^{\eta j} \|f\|_{\ell^1}.$$

We can then choose $\eta = \epsilon/4$. By interpolation, this will show

$$\|(P^j_{\lambda,2}f)^\vee\|_{\ell^{p'}} \le A\,2^{-\frac{\epsilon}{4}j}\|f\|_{\ell^p}$$

whenever $\lambda > 1/p'$. So, adding in j, we see that the operator corresponding to $P_{\lambda,2}$ satisfies the estimate of (2.1).

To prove (2.6), we note again that the supports of the functions $\chi(2^{j/2}\,2^s(\theta - a/q))$ (with j fixed) are disjoint. So

$$\begin{aligned}\left|P^j_{\lambda,2}(\theta)\right| &\le \frac{A}{q^{1/2}}\left|\int_{2^{-j}}^{2\cdot 2^{-j}} y^{\frac14+\epsilon/2}\sum_{m\neq 0}|h_{m,q,y}(\theta)|\,dy\right.\\ &\le A\,2^{-\frac{\epsilon}{2}j},\end{aligned}$$

since by Proposition 2.1

$$|h_{m,q,y}(\theta)| \le A\,\frac{q^{1/2}\,e^{-\delta m^2}}{|y|^{5/4}}.$$

To prove (2.7), observe that

$$\begin{aligned}\left|c_\ell\left(P^j_{\lambda,2}\right)\right| \le &\sum_{\substack{m\\ m\neq 0}}\sum_{s=1}^{j/2-10}\sum_{q=2^s}^{2^{s+1}}\frac{1}{q}\left|\sum_{\substack{a=1\\(a,q)=1}}^{q} S(a,q,m)\,e^{2\pi i\ell\frac{a}{q}}\right|\\ &\left\|\chi\left(2^{j/2}2^s(\theta)\right)\int_{2^{-j}}^{2\cdot 2^{-j}} y^{\epsilon/2}\,|h_{m,q,y}(\theta)|\,dy\right\|_{L^1(d\theta)}.\end{aligned}$$

So, using Propositions 2.1 and 2.2, we see that with $A(q)$ denoting a bound for

$$\sum_{(a,q)=1} S(a,q,m)\,e^{2\pi i\ell a/q},$$

we have

$$\begin{aligned}\left|c_\ell\left(P^j_{\lambda,2}\right)\right| &\le \sum_{s\le j/2-10}\ \sum_{q\le 2^{s+1}} q^{-3/2}\,A(q)\,2^{-j/4}\,2^{-\epsilon j/2}\\ &\le A_\eta\,2^{-j/4}\,2^{-\frac{\epsilon}{2}j}\sum_{s\le j/2} 2^{s/2}\,2^{\eta s}\\ &\le A_\eta\,2^{\eta j}\cdot 2^{-\frac{\epsilon}{2}j},\end{aligned}$$

which is (2.7). This finishes the discussion of $P_{\lambda,2}$.

We turn now to $P_{\lambda,3}$. For a fixed j, let

$$V_j = \bigcup_{s=1}^{j/2-10}\ \bigcup_{2^s\le q<2^{s+1}}\ \bigcup_{\substack{a=1\\(a,q)=1}}^{q} I(j,a,q,s).$$

and let W_j be the complement of V_j in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Then

$$P_{\lambda,3}(\theta) = \sum_{j=1}^{\infty} P^j_{\lambda,3}(\theta), \quad \text{where } P^j_{\lambda,3}(\theta) = \chi_{W_j}(\theta) \int_{2^{-j}}^{2\cdot 2^{-j}} y^{\lambda/2}\, \mathcal{F}(y - i\theta)\, dy.$$

We prove that for $\lambda = 1 + i\gamma$,

$$\left|P^j_{1+i\gamma,3}(\theta)\right| \le A\, 2^{-j/4}, \tag{2.8}$$

and for $\lambda = -1 + i\gamma$, and any $\eta > 0$,

$$|c_\ell(P^j_{-1+i\gamma,3})| \le A_\eta 2^{\frac{j}{2}}\, 2^{\eta j/2}. \tag{2.9}$$

Interpolation and summation in j then shows that the operator corresponding to $P_{\lambda,3}$ satisfies the estimate of (2.1) and the proof will be complete. To prove (2.8), we note that by Dirichlet's principle, for any θ there are a and q with $(a,q) = 1$ so that $|\theta - a/q| \le 1/q2^{j/2}$ and $q \le 2^{j/2}$. If $\chi_{W_j}(\theta) \neq 0$, $q > A\,2^{-j/2}$. Thus for $\lambda = 1 + i\gamma$, we use Proposition 2.1 to see that

$$\begin{aligned} \left|P^j_{\lambda,3}(\theta)\right| &\le A\, 2^{-j/4} \sup_\theta \left| \int_{2^{-j}}^{2\cdot 2^{-j}} y^{(1+i\gamma)/2} \frac{dy}{(y - i\theta)^{3/2}} \right| \\ &\le A\, 2^{-j/4} \int_{2^{-j}}^{2^{-j+1}} \frac{dy}{y} \\ &\le A\, 2^{-j/4}. \end{aligned}$$

To prove (2.9), we note that

$$\int_{2^{-j}}^{2\cdot 2^{-j}} y^{\lambda/2}\, \mathcal{F}(y - i\theta)\, d\theta = \sum_n e^{-2\pi i n^2 \theta} \int_{2^{-j}}^{2\cdot 2^{-j}} n^2\, y^{\lambda/2}\, e^{-2\pi n^2 y}\, dy,$$

which has Fourier coefficient bounded by $A2^{j/2}$ when $\lambda = -1 + i\gamma$. Thus to prove (2.9), it suffices to estimate the Fourier coefficients of $I_j(\theta) + II_j(\theta)$, where

$$I_j(\theta) = \sum_{s \le \frac{j}{2} - 10} \sum_{2^s \le q < 2^{s+1}} \frac{1}{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(a,q) \chi\left(2^{j/2}\, 2^s\, (\theta - a/q)\right) \cdot \left\{ \int_{2^{-j}}^{2\cdot 2^{-j}} y^{(-1+i\gamma)/2} \frac{dy}{[y - i(\theta - a/q)]^{3/2}} \right\}$$

and

$$II_j(\theta) = \sum_{\substack{m \\ m \neq 0}} \sum_{s \le j/2 - 10} \sum_{2^s \le q < 2^{s+1}} \frac{1}{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(a,q,m) \int_{2^{-j}}^{2\cdot 2^{-j}} y^{(-1+i\gamma)/2}\, h_{m,q,y}\left(\theta - \frac{a}{q}\right) dy.$$

Here the argument is similar to that used in estimating $P_{\lambda,1}$.

By Propositions 2.1 and 2.2, we have, for any $\eta > 0$,

$$|c_\ell(I_j)| \le A_\eta \sum_{s\le j/2} 2^{-s}\cdot 2^{(2+\eta)s}\cdot \|\chi(2^{j/2}2^s\theta)\|_{L^1}\cdot \int_{2^{-j}}^{2^{-j+1}} \frac{dy}{y^2}$$
$$\le A_\eta \sum_{s\le j/2} 2^{-s}\cdot 2^{(2+\eta)s}\cdot 2^{-s}\cdot 2^{-j/2}\cdot 2^j$$
$$\le A_\eta\, 2^{j/2}\, 2^{\eta j/2}.$$

If we use Proposition 2.1 and Proposition 2.2 again, we see that

$$|c_\ell(II_j)| \le A_\eta \sum_{s\le j/2} 2^{-s}\cdot 2^{(2+\eta)s}\cdot \|\chi(2^{j/2}2^s\theta)\|_{L^1}\, 2^{s/2} \int_{2^{-j}}^{2^{-j+1}} \frac{dy}{y^{7/4}}$$
$$\le A_\eta \sum_{s\le j/2} 2^{-s}\cdot 2^{(2+\eta)s}\cdot 2^{-s}\cdot 2^{s/2}\cdot 2^{-j/2}\, 2^{3/4j}$$
$$\le A_\eta\, 2^{j/2}\, 2^{\eta j/2}.$$

With this we have proved (2.9). This finishes the proof of Theorem 1 except for the propositions, to which we now turn.

Proof of Proposition 2.1. The proposition is obtained by a small modification of arguments in [SW$_2$].

$$\mathcal{F}(y+i\theta) = \frac{d}{dy}G(y+i\theta), \quad \text{where } G(y-i\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-2\pi n^2(y-i\theta)}.$$

Write $\theta = a/q + \beta$ and $n = mq + r$ to see that

$$G(y-i\theta) = \sum_{r=1}^{q} e^{2\pi i r^2 a/q}\, G_r(y-i\beta)$$

with

$$G_r(y-i\beta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{-2\pi(mq+r)^2\,(y-i\beta)}.$$

Now apply the Poisson summation formula

$$G_r(y-i\beta) = \frac{1}{2\pi}\,\frac{1}{\sqrt{2(y-i\beta)}}\sum_{m=-\infty}^{\infty} e^{2\pi i m\, r/q}\, e^{-m^2/2q^2(y-i\beta)}.$$

Arguing as is [SW$_2$], §2, allows us then to complete the proof of Proposition 2.1.

In this regard, notice that if $|\beta| \le A/q y^{1/2}$,

$$\Re\left(\frac{1}{q^2(y+i\beta)}\right) \ge \eta, \quad \text{with } \eta > 0.$$

Proof of Proposition 2.2. To prove conclusion (ii), we show that for some A_1 and A_2,

$$\sum_{q=1}^{M}\Big|\sum_{\substack{a=1\\(a,q)=1}}^{q} S(a,q,m)\, e^{-2\pi i\ell\frac{a}{q}}\Big| \le A_1\, 2^{A_2 \log M/\log\log M}\, M^2. \tag{2.10}$$

Let

$$\sigma(q,m,\ell) = \sum_{\substack{a=1\\(a,q)=1}}^{q} S(a,q,m)\, e^{2\pi i\ell\frac{a}{q}}.$$

First note that if q is a prime,

$$|\sigma(q,m,\ell)| \le 3q. \tag{2.11}$$

In fact,

$$\sigma(q,m,\ell) = \sum_{r=1}^{q} e^{2\pi i r\frac{m}{q}} \sum_{\substack{a=1\\(a,q)=1}}^{q} e^{2\pi i r^2\frac{a}{q}}\, e^{2\pi i\ell\frac{a}{q}}.$$

So

$$\begin{aligned}
|\sigma(q,m,\ell)| &\le \sum_{r=1}^{q}\Big|\sum_{a=1}^{q-1} e^{2\pi i r^2\frac{a}{q}}\, e^{2\pi i\ell\frac{a}{q}}\Big| \\
&= \sum_{r=1}^{q}\Big|\sum_{a=1}^{q} e^{2\pi i r^2\frac{a}{q}}\, e^{2\pi i\ell\frac{a}{q}} - 1\Big| \\
&\le q + \sum_{r=1}^{q}\Big|\sum_{a=1}^{q} e^{2\pi i r^2\frac{a}{q}}\, e^{-2\pi i\ell\frac{a}{q}}\Big| \\
&= q + \sum_{r=1}^{q} n(r,\ell),
\end{aligned}$$

where $n(r,\ell) = q$ if $r^2 \equiv \ell \,(\mathrm{mod}\, q)$ and 0 otherwise. For a fixed ℓ, $n(r,\ell) \ne 0$ for at most two values of r, since if $r_1^2 \equiv r_2^2 \,(\mathrm{mod}\, q)$, then

$$r_1^2 - r_2^2 = (r_1 - r_2)(r_1 + r_2) \equiv 0.$$

Then since the prime q divides the product, it divides one of the factors. Hence, $r_1 \equiv \pm r_2 \,(\mathrm{mod}\, q)$. This proves (2.11).

Next, we note that if $(q_1, q_2) = 1$,

$$\sigma(q_1 q_2, m, \ell) = \sigma(q_1, m_1, \ell)\, \sigma(q_2, m_2, \ell) \tag{2.12}$$

for appropriate integers m_1 and m_2. The proof of (2.12) is essentially contained in, e.g., [V], Chap. 2.

Now let $n = n_1 n_2 \cdots n_r$, with n_j distinct. Then $r \leq A_2 \log n / \log\log n$. This follows since the smallest integer with r distinct factors is $r!$. Thus, from (2.11) and (2.12), we see that if q is the product of r distinct primes,

$$\sigma(q, m, \ell) \leq 3^{A_2 \frac{\log q}{\log\log q}}\, q.$$

Let $\sigma^*(q) = \sup_{m,\ell} |\sigma(q, m, \ell)|$. For $1 \leq q \leq M$, write $q = q_1 q_2$, where q_2 is square free, and each prime occurring as a factor of q_1 occurs at least to the power 2. Then since $\sigma^*(q) \leq \sigma^*(q_1)\sigma^*(q_2)$,

$$\sum_{q=1}^{M} \sigma^*(q) \leq \sum_{q_1=1}^{M} \sigma^*(q_1) \sum_{q_2=1}^{M/q_1} \sigma^*(q_2) \leq A_1\, 3^{A_2 \frac{\log M}{\log\log M}}\, M^2 \sum_{q_1=1}^{M} \frac{\sigma^*(q_1)}{q_1^2}.$$

Conclusion (i) implies $\sigma^*(q_1) \leq A q_1^{3/2}$. So it suffices to prove

$$\sum_{\substack{q=1 \\ \text{each prime factor of } q \\ \text{occurs to the power of at least 2}}}^{M} \frac{1}{q^{1/2}} \leq A(\log M)^N \tag{2.13}$$

for some N.

But the sum on the left-hand side of (2.13) is dominated by

$$\prod_{\substack{p \\ p \text{ prime} \\ p \leq M}} \left(1 + \frac{1}{p} + \frac{1}{p^{3/2}} + \frac{1}{p^2} + \cdots\right) \leq \prod_{\substack{p \\ p \text{ prime} \\ p \leq M}} \left(1 + \frac{A}{p}\right).$$

Since $\sum_{p \leq M} 1/p \leq A \log\log M$, (2.13) follows by taking the logarithm of the products.

3 The proof of Theorem 2

To prove Theorem 2, we need several lemmas concerning multiplier operators.

Lemma 3.1. *Let I_k be a family of disjoint intervals on $\mathbb{R}$ and denote their characteristic functions by χ_k. Define projection operators E_k by putting*

$$(E_k f)^\wedge = \chi_k \hat{f}.$$

Then for $2 \leq p < \infty$, $\|(\sum |E_k f|^2)^{1/2}\|_{L^p(\mathbb{R})} \leq C(p)\, \|f\|_{L^p(\mathbb{R})}$.

Lemma 3.1 is due to Rubio de Francia [R].

Here and in what follows, C and $C(p)$ denote constants (the second depending on p), which need not be the same at all occurrences.

Lemma 3.2. *For some p, with $1 < p < \infty$, let $m(\theta)$ be a bounded multiplier on $L^p(\mathbb{R})$ with norm A. Suppose also that $\psi(\theta)$ is a smooth function supported in $[-2, 2]$, and that $I_1, I_2, \ldots, I_N$ are disjoint intervals on $\mathbb{R}$ of equal length d with centers r_k. Assume finally that $B_1, \ldots, B_N$ are complex numbers with $|B_k| \leq B$. Set*

$$M(\theta) = \sum_{k=1}^{N} B_k \, \psi\left(c\,\frac{\theta - r_k}{d}\right) m(\theta - r_k),$$

where $c \geq 4$. Then $M(\theta)$ is a bounded Fourier multiplier on $L^p(\mathbb{R})$ with norm at most $C(p)ABN^{1/2}$.

Proof of Lemma 3.2. We may assume $2 \leq p < \infty$. Let $\mathcal{M}$ be the operator corresponding to $M(\theta)$, T_k be the operator corresponding to $B_k\psi\left(c\left(\frac{\theta - r_k}{d}\right)\right) m(\theta - r_k)$, S the operator corresponding to the multiplier $\psi(c\theta)\, m(\theta)$, E_k as in Lemma 3, and finally $V_k f(x) = e^{-2\pi i r_k x} f(x)$. Then $T_k f = T_k E_k f$, so

$$
\begin{aligned}
|\mathcal{M}f(x)| &= \left|\sum_{k=1}^{N} T_k f(x)\right| \\
&= \left|\sum_{k=1}^{N} T_k E_k f(x)\right| \\
&\leq \sqrt{N}\left(\sum_{k=1}^{N} |T_k E_k f(x)|^2\right)^{1/2} \\
&= \sqrt{N}\left(\sum_{k=1}^{N} |B_k|^2 \left|V_k S V_k^{-1} E_k f\right|^2\right)^{1/2} \\
&\leq B\sqrt{N}\left(\sum_{k=1}^{N} \left|S\left(V_k^{-1} E_k f\right)\right|^2\right)^{1/2}.
\end{aligned}
$$

So, by the Marcinkiewicz–Zygmund Theorem,

$$\|\mathcal{M}f\|_{L^p} \le B\sqrt{N}\,C(p)A\left\|\left(\sum_{k=1}^{N}\left|V_k^{-1}\,E_kf\right|^2\right)^{1/2}\right\|_{L^p}$$

$$= B\sqrt{N}\,C(p)\,A\left\|\left(\sum_{k=1}^{N}|E_kf|^2\right)^{1/2}\right\|_{L^p}$$

$$\le AB\sqrt{N}\,C(p)\,\|f\|_{L^p}$$

by Lemma 3.1, and Lemma 3.2 is proved.

Next, suppose $v(\theta)$ is a function on R supported in $-\frac{1}{2} < \theta \le \frac{1}{2}$. Define

$$v_{\mathrm{per}}(\theta) = \sum_{n=-\infty}^{\infty} v(\theta+n),$$

so that v_{per} is the periodic extension of v.

Lemma 3.3. *Suppose for some $p, 1 \le p \le \infty$, $v(\theta)$ is a bounded Fourier multiplier on $L^p(\mathbb{R})$ with norm A. Then $v_{\mathrm{per}}(\theta)$ is a bounded multiplier on $\ell^p(Z)$ with norm at most CA.*

See [MSW], §2 for the proof of and discussion of Lemma 3.3.

Using this, we get the discrete analogue of Lemma 3.2.

Lemma 3.4. *Fix p with $1 < p < \infty$. Suppose that $m(\theta)$ is a bounded Fourier multiplier on $L^p(\mathbb{R})$ and that $\psi(\theta)$ is a C^∞ function supported on $[-2,2]$. Further assume we are given disjoint subintervals $I_1,\dots,I_N$ of $[0,1]$ having equal length d and centers $r_1,\dots,r_N$. Let c be appropriately large, e.g., $c \ge 4$. Set*

$$M(\theta) = \sum_{k=1}^{N} B_k\,\psi(c(\theta - r_k))\,m(\theta - r_k),$$

with $B_1,\dots,B_N$ complex numbers. Then $M(\theta)$, given on $[0,1]$ but extended by periodicity to the whole line $\mathbb{R}$, is a bounded Fourier multiplier on $\ell^p(Z)$ with norm of most $AB\sqrt{N}\,C(p)$. (Here A is the multiplier norm of m on $L^p(R)$ and $B = \sup_{1\le k\le N}|B_k|$.)

We wish to deduce Lemma 3.4 from Lemmas 3.2 and 3.3. However, $M(\theta)$ is supported in $[0,1]$, so we are speaking of the periodic extension of a function defined on $[0,1]$, while Lemma 3.3 applies to the periodic extension of a function supported in $[-\frac{1}{2},\frac{1}{2}]$. To deal with this point, set $\bar{M}(\theta) = M(\theta - \frac{1}{2})$. Then to obtain the desired bound on the norm of the multiplier M on $\ell^p(Z)$, it suffices to obtain the bound for $M(\theta - \frac{1}{2})$ on $\ell^p(Z)$. By Lemma 3.3, this is dominated by the norm of the multiplier $M(\theta - \frac{1}{2})$ on $L^p(R)$, which is the same as the multiplier norm of $M(\theta)$ on $L^p(R)$; finally we apply Lemma 3.2 to $M(\theta)$ as a multiplier on $L^p(R)$.

In this section, we take $\lambda = 1 + i\gamma$. As in Section 2, it suffices to deal with the operator corresponding to the multiplier $\nu_\lambda(\theta)$, where

$$\nu_\lambda(\theta) = \int_0^1 y^{\lambda/2}\,\mathcal{F}(y - i\theta)\,dy.$$

Let $\rho > 1$ be such that $\rho^{i\gamma/2} = 1$. We change the definition of $\nu_{\lambda,j}$ slightly by setting

$$\nu_{\lambda,j} = \int_{\rho^{-j}}^{\rho^{-j+1}} y^{(1+i\gamma)/2}\,\mathcal{F}(y - i\theta)\,dy.$$

For simplicity of notation, we assume $\frac{1}{2}\gamma\log 2 = 2\pi$, so that $\rho = 2$. Then $\nu_{\lambda,j}$ is as in Section 2. The argument for general $\rho > 1$ is then almost the same as for $\rho = 2$, which we now give.

Since again

$$\mathcal{F}(y - i\theta) = \sum_{n=-\infty}^{\infty} n^2\, e^{-2\pi n^2 y}\, e^{2\pi i n^2\theta},$$

the Fourier coefficients, $c_\ell(\nu_{\lambda,j})$, satisfy

$$|c_\ell(\nu_{\lambda,j})| \le A\,\frac{\ell^2}{2^{3j/2}}\, e^{-\delta\,\ell^2/2^j}$$

for some $\delta > 0$. Hence

$$\sum_\ell |c_\ell(\nu_{\lambda,j})| \le A,$$

and the operators corresponding to the multipliers $\nu_{\lambda,j}$ are uniformly bounded on ℓ^p, for $1 \le p \le \infty$.

We wish to show $\sum_{j=1}^{\infty} \nu_{\lambda,j}$ is bounded on ℓ^p for $\frac{4}{3} < p < 4$, when $\lambda = 1 + i\gamma$. Put

$$\mu_{\lambda,j}(\theta) = \frac{1}{2\pi\sqrt{2}} \int_{2^{-j}}^{2^{-j+1}} y^{\lambda/2}\,\frac{1}{(y - i\theta)^{3/2}}\,dy,$$

and

$$H_{\lambda,j}(\theta) = \sum_{s \le \frac{j}{2} - 10}\ \sum_{2^s \le q < 2^{s+1}}\ \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{S(a,q)}{q}\,\psi\left(c\,2^{2s}\left(\theta - \frac{a}{q}\right)\right)\mu_{\lambda,j}\left(\theta - \frac{a}{q}\right).$$

As in Section 2, c is an appropriate large number. Further, ψ is an appropriate smooth function which is one for $-1 \leq \theta \leq 1$ and is supported in $[-2,2]$. $H_{\lambda,j}(\theta)$ differs from $I_j(\theta)$ of Section 2 in that in $H_{\lambda,j}$ we have the cutoff of function $\psi(c\,2^{2s}(\theta - a/q))$ rather than $\chi(2^{j/2}2^s(\theta - a/q))$ of I_j. First observe that

$$\|\nu_{\lambda,j}(\theta) - H_{\lambda,j}(\theta)\|_{L^\infty} \leq A\,2^{-\delta j} \tag{3.1}$$

for some $\delta > 0$. Using Proposition 2.1, we see that for θ in a major arc, $|\nu_{\lambda,j}(\theta) - I_j(\theta)| \leq A\,2^{-j/4}$.

Next, note that since $s \leq j/2 - 10$, on the support of $\chi(2^{j/2}\,2^s\theta) - \psi(c\,2^{2s}\theta)$, $|\theta| \geq 2^{-2s}$ so

$$|\mu_{\lambda,j}(\theta)| = \left| \int_{2^{-j}}^{2^{-j+1}} \frac{y^{\lambda/2}}{(y - i\theta)^{3/2}}\,dy \right|$$

$$= \left| \int_1^2 \frac{y^{\lambda/2}}{(y - i2^j\theta)^{3/2}} \right| \leq \frac{A}{(2^j\theta)^{3/2}}$$

$$\leq A\,2^{3s/2}/2^{3j/2}.$$

Since also $|\,S(a,q)/q\,| \leq A\,q^{-1/2}$, then on the major arcs,

$$|\,I_j(\theta - a/q) - H_{\lambda,j}(\theta - a/q)\,|$$

$$\leq A\,2^{-3j/4} \sum_{s \leq \frac{j}{2}} 2^{3j/2} \sum_{2^s \leq q < 2^{j+1}} \frac{1}{q^{1/2}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \chi(2^{j/2}2^s(\theta - a/q))$$

$$\leq A\,2^{-3j/4} \sum_{s \leq j/2} 2^s \leq A\,2^{-j/4}.$$

Thus, on the major arcs,

$$|\nu_{\lambda,j}(\theta) - H_{\lambda,j}(\theta)\,| \leq A\,2^{-j/4}.$$

On the complement of the major arcs, $|\nu_{\lambda,j}(\theta)\,| \leq A\,2^{-j/4}$, as we have seen in Section 2, and $H_{\lambda,j}(\theta) = 0$. Thus (3.1) is proved.

Let

$$\mathcal{H}_{\lambda,j}f = \left(H_{\lambda,j}(\theta)\,\hat{f}(\theta)\right)^{\vee}.$$

We also define

$$\mathcal{N}_{\lambda,j}\,f = \left(\nu_{\lambda,j}(\theta)\,\hat{f}(\theta)\right)^{\vee} \quad \text{and} \quad \mathcal{N}_\lambda\,f = \left(\nu_\lambda(\theta)\,\hat{f}(\theta)\right)^{\vee}.$$

We shall show that for $\lambda = 1 + i\gamma$,

$$\|(\mathcal{H}_{\lambda,j}f)\|_{\ell^p} \leq A_p(1 + 1/|\gamma|)\|f\|_{\ell^p}, \quad 4/3 < p < 4, \tag{3.2}$$

and

$$\|(\sum_j \mathcal{H}_{\lambda,j} f)\|_{\ell^p} \leq A_p(|\gamma| + 1/|\gamma|)\|f\|_{\ell^p}, \quad 4/3 < p < 4. \tag{3.3}$$

The estimate (3.2) together with (3.1) and the fact that the operator corresponding to $\nu_{\lambda,j}$ is uniformly bounded in ℓ^p shows by interpolation that for $\frac{4}{3} < p < 4$,

$$\left\| ((\mathcal{H}_{\lambda,j} - \mathcal{N}_{\lambda,j})\, f) \right\|_{\ell^p} \leq A 2^{-\delta(p)j} \|f\|_{\ell^p}.$$

Thus $\sum_j (\mathcal{N}_{\lambda,j} - \mathcal{H}_{\lambda,j})$ is a bounded multiplier on ℓ^p, $\frac{4}{3} < p < 4$. So matters are reduced to proving (3.3). We write

$$H_{\lambda,j} = \sum_{s \leq j/2 - 10} H_{\lambda,j,s},$$

where

$$H_{\lambda,j,s}(\theta) = \sum_{2^s \leq q < 2^{s+1}} \cdot \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{1}{q} S(a,q)\, \psi\left(c\, 2^{2s}\left(\theta - \frac{a}{q}\right)\right) \mu_j\left(\theta - \frac{a}{q}\right).$$

Also set

$$\mathcal{H}_{\gamma,j,s} f = \left(\mathcal{H}_{1+i\gamma,s}(\theta)\, \hat{f}(\theta)\right)^{\vee}.$$

As in Section 1, for s fixed, the supports of the functions $\psi(c\,2^{2s}(\theta - a/q))$ are disjoint. Thus

$$\|\mathcal{H}_{\lambda,j,s}\|_{L^\infty} \leq A(1 + 1/|\gamma|)\, 2^{-s/2} \tag{3.4}$$

and

$$\left\|\sum_j \mathcal{H}_{\lambda,j,s}\right\|_{L^\infty} \leq A(1 + 1/|\gamma|)\, 2^{-s/2}. \tag{3.5}$$

Thus, by interpolation with the ℓ^2 results given by (3.4) and (3.5), it suffices to prove that for $1 < \rho < \infty$,

$$\|(\mathcal{H}_{\gamma,j,s} f)\|_{\ell^p} \leq A(1 + 1/|\gamma|)\, 2^{s/2}\|f\|_{\ell^p} \tag{3.6}$$

and

$$\left\|\sum_j \mathcal{H}_{\gamma,j,s} f\right\|_{\ell^p} \leq A(|\gamma| + 1/|\gamma|) 2^{s/2} \|f\|_{\ell^p}. \tag{3.7}$$

The estimates (3.6) and (3.7) follow from Lemma 3.4. The intervals I_k are $(a/q - 2^{-2s}, a/q + 2^{-2s})$, $1 \le a \le q$, $(a,q) = 1$, $2^s \le q < 2^{s+1}$, and the B_k are $S(a,q)/q$, so that $N \le 2^{2s}$ and $B \le c/2^{s/2}$. In proving case (3.4), we take $m(\theta) = \mu_j(\theta)$, which is easily seen to be a bounded multiplier on $\ell^p(Z)$, $1 \le p \le \infty$, uniformly in j. To prove (3.5), we take $m(\theta) = \sum_j \mu_j(\theta)$, which for $\lambda = 1 + i\gamma$ is seen to have norm at most $c(p)(|\gamma| + 1/|\gamma|)$ by the Marcinkiewicz multiplier theorem. This concludes the proof of Theorem 2.

4 The operators J_λ

We indicate here the changes needed in the arguments of Section 2 and Section 3 to prove Theorems 3 and 4.

To prove Theorem 3, the critical result is that for $\frac{2}{5} < \lambda \le \frac{1}{2}$, $1/p' + 1/p = 1$ and $1/p - 1/p' = (1-\lambda)/3$, we have

$$\|J_\lambda f\|_{\ell^{p'}} \le A\|f\|_{\ell^p}.$$

Write

$$\widehat{J_\lambda f}\,(\theta,\phi) = \nu_\lambda(\theta,\phi)\,\hat{f}(\theta,\phi), \quad \text{with } \nu_\lambda(\theta,\phi) = \sum_{n=1}^{\infty} \frac{1}{n^\lambda} e^{2\pi i n^2\theta}\, e^{2\pi i n\phi}.$$

As in Section 3, write

$$\nu_\lambda(\theta,\phi) = \sum_{j=1}^{\infty} \nu_{\lambda,j}(\theta,\phi).$$

Further, for $s \le j/2 - 10$, $2^s \le q < 2^{s+1}$, $1 \le a \le q$, $1 \le b \le q$ and $(a,q) = 1$ we introduce major boxes $I_j(a,b,q)$. Here,

$$I_j(a,b,q) = \{(\theta,\phi) : |\theta - a/q| \le 1/2^s 2^{j/2}, |\phi - b/q| \le 1/2q\}.$$

For $(\theta,\phi) \in I_j(a,b,q)$,

(4.1)

$$\nu_{\lambda,j}(\theta,\phi) =$$
$$\frac{c(\lambda)}{q} S(a,q,b) \int_{2^{-j}}^{2\cdot 2^{-j}} \frac{y^{\lambda/2-1}}{(y - i(\theta - a/q))^{1/2}} \exp\left(-\frac{\pi}{2}\frac{(\phi - b/q)^2}{(y - i(\theta - b/q))}\right) dy$$

plus error terms. Instead of Proposition 2.2, we have

Proposition 4.1. *For ℓ_1 and ℓ_2 integers,*

$$\left| \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{b=1}^{q} S(a,q,b)\, e^{-2\pi i \ell_1 \frac{a}{q}}\, e^{-2\pi i \ell_2 b/q} \right| \leq Aq^2 .$$

The proof of Proposition 4.1 is simpler than the proof of Proposition 2.2, as is seen by first summing on b.

The interpolation is now an ℓ^2 to ℓ^2 estimate for $\Re\lambda = 1$ and an ℓ^1 to ℓ^∞ estimate for $\Re\lambda = -2$. The contribution from the error terms in (4.1) and from the complement of the major boxes is treated just as in Theorem 1. However, to study the contribution of the main term in (4.1) to the major boxes, we further decompose $I_j(a,q,b)$. We write

$$I_j(a,q,b) = \bigcup_{r=0}^{j/2-s} I_j^r\,(a,b,q),$$

where for $r \geq 1$

$$I_j^r(a,b,q) = \{(\theta,\phi) : 2^r/2^j \leq |\theta - a/q| \leq 2^{r+1}/2^j\,,\, |\phi - b/q| \leq 1/2q\}$$

and

$$I_j^0(a,b,q) = \{(\theta,\phi) : |\theta - a/q| \leq 1/2^j\,, |\phi - b/q| \leq 1/2q\}.$$

We must then consider for each fixed r and s the contribution for all q, with $2^s \leq q < 2^{s+1}$. Call the corresponding operator $D_{\lambda,r,s}$. We then obtain for $\Re\lambda = 1$

$$\|D_{\lambda,r,s} f\|_{\ell^2} \leq 2^{-r/2}\, 2^{-s/2}\, \|f\|_{\ell^2}\,,$$

and for $\Re\,\lambda = -2$

$$\|D_{\lambda,r,s} f\|_{\ell^\infty} \leq 2^r\, 2^{2s}\, \|f\|_{\ell^1}\,.$$

Thus, for $\Re\lambda > \frac{2}{5}$, we have by interpolation

$$\|D_{\lambda,r,s} f\|_{\ell^{p'}} \leq A 2^{-\theta r}\, 2^{-\theta s}\, \|f\|_{\ell^p}$$

for $1/p + 1/p' = 1$ and $1/p - 1/p' = (1-\lambda)/3$. We then add in r and s, and conclude as before.

The proof of Theorem 4 is similar to that of Theorem 2, except for the following change. In applying Lemma 3.1, we dealt with 2^{2s} intervals. Now we need the two-dimensional version of Lemma 3.1 (due to Journé [J]), which in this case we must apply to 2^{3s} boxes.

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