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Failure of Palais–Smale condition and blow-up analysis for the critical exponent problem in $\mathbb{R}^2$

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Abstract. Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Let $f: \mathbb{R} \to \mathbb{R}$ be a smooth non-linearity behaving like $\exp\{s^2\}$ as $s \to \infty$. Let F denote the primitive of f. Consider the functional $J: H_0^1(\Omega) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(u)\mathrm{d}x.$$

It can be shown that J is the energy functional associated to the following nonlinear problem:

$$-\Delta u = f(u) \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

In this paper we consider the global compactness properties of J. We prove that J fails to satisfy the Palais–Smale condition at the energy levels $\{k/2\}$, k any positive integer. More interestingly, we show that J fails to satisfy the Palais–Smale condition at these energy levels along two Palais–Smale sequences. These two sequences exhibit different blow-up behaviours. This is in sharp contrast to the situation in higher dimensions where there is essentially one Palais–Smale sequence for the corresponding energy functional.

Keywords. Blow-up analysis; critical exponent problem in $\mathbb{R}^2$; Moser functions; Palais–Smale sequence; Palais–Smale condition.

1. Introduction

1.1 *Preliminaries*

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geqslant 2$. For $n \geqslant 3$, let $\mathscr{A}_n$ denote the subset of $C^1(\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_+)$ consisting of functions $g(s)$ which satisfy the following growth conditions:

$$\lim_{s \to \infty} g(s)\, s^{((n+2)/(n-2))} = \infty,$$

$$\lim_{s \to \infty} g(s)\, s^{-((n+2)/(n-2))} = 0.$$

When $n = 2$, let $\mathscr{B}$ denote the subset of $C^1(\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_+)$ consisting of functions $h(s)$ which vanish only at $s = 0$ and which satisfy the following growth conditions:

For every $\delta > 0$,

$$\lim_{s \to \infty} h(s)\exp\{\delta s^2\} = \infty,$$

$$\lim_{s \to \infty} h(s)\exp\{-\delta s^2\} = 0.$$

In view of the Sobolev and the Trudinger [22] imbeddings, we call f a function of critical growth in $\mathbb{R}^n$, $n \geqslant 2$, if

$$f(s) = \begin{cases} s^{((n+2)/(n-2))} + g(s), & g \in \mathscr{A}_n \quad \text{if} \quad n \geqslant 3, \\ h(s)\exp\{4\pi s^2\}, & h \in \mathscr{B} \quad \text{if} \quad n = 2. \end{cases}$$

For questions related to the achievement of the best constant in the Trudinger imbedding, see [11, 16, 13, 15].

We now consider the following non-linear problem with f having critical growth in $\mathbb{R}^n$, $n \geqslant 2$:

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in} \quad \Omega, \\ u &> 0 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{P}$$

Let F denote the primitive of f. Then, the energy functional $J: H_0^1(\Omega) \to \mathbb{R}$ associated to the problem (P) is given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x. \tag{1.1}$$

It is not difficult to show that every solution of the non-linear problem (P) is a critical point of J and vice-versa.

1.2 *A brief review of past developments*

Since problem (P) lacks compactness, J will not satisfy Palais–Smale condition at all energy levels. It is of great interest to characterize the energy levels at which J fails to satisfy the Palais–Smale condition. We explain briefly why this characterization is important. Using Rellich's identity [8], Pohazaev [19] was able to show that when $n \geqslant 3$, (P) does not admit a solution if $f(s) = s^{(n+2)/(n-2)}$ and Ω is star-shaped with respect to anyone of its points. In order to reverse this non-existence and obtain some existence results for the problem (P) there are therefore two simple prescriptions:

(i) add a perturbation $g(s)$ from $\mathscr{A}_n$ to the critical non-linearity $s^{(n+2)/(n-2)}$, or,
(ii) pose the problem (P) on a domain Ω with 'rich topology'.

When $n \geqslant 3$, the first program was carried out in Brezis–Nirenberg [10] where they prove existence results for (P) provided the perturbation g is large enough. They also show that if the perturbation is small, the Pohazaev-type non-existence persists. In order to carry out the second program, it is necessary to know precisely the energy levels at which J fails to satisfy the Palais–Smale condition. Such a characterization was obtained by Struwe [21] who showed that Palais–Smale fails only at a discrete (but infinite) set of equally spaced energy levels. Using this characterization and tools from algebraic topology, Bahri–Coron [7] were able to prove that if $n \geqslant 3$ and at least one of the homology groups of Ω (with coefficients in $\mathbb{Z}_2$) is non-trivial, (P) admits a solution even when $f(s) = s^{(n+2)/(n-2)}$. These developments underline the fact that one can exploit the topology of Ω to prove existence results to problem (P) once the energy levels at which Palais–Smale condition fails are known.

In the case $n = 2$ corresponding questions have been considered in [1–5], but the picture remains far from complete. One of the difficulties is that an effective Pohazaev-type identity does not seem to exist when $n = 2$. Hence non-existence results become harder to obtain for general star-shaped domains. Nevertheless, in the radial situation, non-existence results for a large class of convex non-linearities have been obtained in [4] and [5]. We briefly describe these results. The critical non-linearity when $n=2$ is $f(s)=h(s)\exp\{4\pi s^2\}$ with $h\in\mathscr{B}$. In analogy with the results for $n\geqslant 3$, one expects non-existence if the perturbation h is 'small'. This is indeed so if h is 'small' in the sense that

$$\limsup_{s\to\infty} h(s)s<\infty.$$

We refer to [4] and [5] for a precise statement of these non-existence results. Therefore, to reverse this non-existence either one can consider large perturbations h, or one can pose the problem on a domain Ω in $\mathbb{R}^2$ with 'holes'. The former option was considered by Adimurthi in [1] where he proved that if the perturbation is 'large', more precisely, if $\limsup_{s\to\infty} h(s)s=\infty$, then (P) admits a solution on any bounded smooth domain Ω in $\mathbb{R}^2$ (also see remark 1.1 below). Further, he showed in the same paper that Palais–Smale holds in the infinite energy range $(-\infty,\frac{1}{2})$. As we saw earlier for the case $n\geqslant 3$, we need to characterize the energy levels at which J fails to satisfy the Palais–Smale condition if we want to consider the latter option of exploiting the topology of Ω for proving existence results.

1.3 *Contents of the paper*

This paper deals with the two dimensional case. In §2 and §3 we exhibit two Palais–Smale sequences along which J fails to satisfy the Palais–Smale condition at the energy levels $k/2$, k any positive integer. The first Palais–Smale sequence (§2) is the sequence of 'Moser functions' whereas the second sequence (§3) consists of solutions to the problem (ref. [1]),

$$\begin{aligned} -\Delta u &= h(u)\exp\{4\pi u^2\} && \text{in} \quad B(R),\\ u&>0 && \text{in} \quad B(R),\\ u&=0 && \text{on} \quad \partial B(R), \end{aligned}$$

where $B(R)\subset\mathbb{R}^2$ denotes the disc of radius R centered at the origin and $h(s)s\to\infty$ as $s\to\infty$.

A very surprising consequence that comes out of the analysis here is that these two Palais–Smale sequences exhibit different blow-up behaviours. This is in sharp contrast to the situation in higher dimensions where there is essentially only one type of Palais–Smale sequence for J (ref. [21]). The blow-up limit of the first Palais–Smale sequence solves the following equation in $\mathbb{R}^2$, for some compactly supported measure μ:

$$-\Delta u=\mu.$$

The blow-up limit of the second Palais–Smale sequence solves the equation:

$$-\Delta u=\frac{1}{2\pi}\exp\{u\} \quad \text{in} \quad \mathbb{R}^2.$$

Here we recall that any solution u of the above equation corresponds to a conformal change of metric on $\mathbb{R}^2$ from the standard metric to a metric of constant Gaussian curvature $K=1$. Results on these blow-ups are contained in §4(1) and §4(2). We strongly believe that the existence of two distinct types of Palais–Smale sequences has important implications regarding the global compactness properties of J.

For generalizations of theorems A and B to the case of the n-Laplacian in $\mathbb{R}^n$, see §5.

Statement of the theorems. For convenience, we list the statements of all the four theorems we will prove:

Let $0 < l < R$. Define the Moser functions

$$m_{l,R}(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} [\log(R/l)]^{1/2} & 0 \leqslant |x| < l, \\ \dfrac{\log(R/|x|)}{[\log(R/l)]^{1/2}} & l \leqslant |x| \leqslant R, \\ 0 & |x| \geqslant R. \end{cases} \tag{1.2}$$

Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Let $x_0 \in \Omega$ and $R < d(x_0, \partial\Omega)$. Define $m_{l,R,x_0}(x) = m_{l,R}(x - x_0)$.

Theorem A. *Let $f(s) = h(s)\exp\{4\pi s^2\}$, $s \geqslant 0$, $h \in \mathscr{B}$. Assume that h satisfies the hypothesis:*

$$|h'(t)| \leqslant ch(t)t^{1-\eta} \quad \text{and} \quad F(t) \leqslant cf(t)t^{1-\eta}$$

for some $\eta \in (0,1]$, for some $c > 0$ and all large t.

Let R depend on l so that $\lim_{l\to 0}(\log(2/R)/\log(R/l)) = 0$. Then there exists a sequence $\rho_l \to 1$ as $l \to 0$ such that

(a) $\{\rho_l m_{l,R,x_0}\}_{l>0}$ *is a Palais–Smale sequence for J as $l \to 0$,*
(b) *J fails to satisfy the Palais–Smale condition along the above sequence at the energy levels $k/2$, k any positive integer.*

Remark 1.1. Examples of non-linearities satisfying the hypotheses in theorem A are: any $h \in \mathscr{B}$ such that $h(s) = \exp\{\pm s^\beta\}$, for all large s and some $\beta \in (0,2)$; any $h \in \mathscr{B}$ such that $h(s) = s^\theta$ for all large s and some $\theta \in \mathbb{R}$.

Let u_R denote the solution obtained in [1] for the problem (1.1) posed on $B(x_0, R)$. Let $\tilde{u}_R$ denote the extension of u_R obtained by setting $\tilde{u}_R \equiv 0$ in $\Omega \setminus B(x_0, R)$.

Theorem B. *Let $x_0 \in \Omega$ and $R > 0$ be small enough so that $B(x_0, R) = \{x : |x - x_0| < R\} \subset \Omega$. Let $h \in \mathscr{B}$ and $f(s) = h(s)\exp\{4\pi s^2\}$, $s \geqslant 0$, satisfy the following hypotheses: There exists $s_0 > 0$ such that*

(i) $h(s)s \to \infty$ *as* $s \to \infty$,
(ii) $g(s) = \log f(s)$ *is C^3 and convex for all $s \geqslant s_0$,*
(iii) *$f(s)$ is strictly increasing for all $s \geqslant s_0$,*
(iv) $g(s) - \frac{1}{2}sg'(s) + \log(g'(s)/2) \geqslant (1+\theta)\log s$ *for all $s \geqslant s_0$ and some $\theta > -1$,*
(v) $\overline{\lim}\sup_{s\to\infty}(g(s) - \frac{1}{2}sg'(s))s^{-\beta} = O(1)$ *for some $\beta \in [0,2)$,*
(vi) $\overline{\lim}_{s\to\infty}(sg^{(k+1)}(s)/g^{(k)}(s))$ $k = 0, 1$ *exists and is different from 0 (here $g^{(k)}$ denotes the k-th derivative of g).*

Then,

(a) $\{\tilde{u}_R\}_{R>0}$ *as $R \to 0$ are Palais–Smale sequences for J,*
(b) *J does not satisfy Palais-Smale condition at the energy levels $k/2$, $k = 1, 2, 3, \ldots$.*

Remark 1.2. Let $g(s)=\log f(s)$. Then the following are simple consequences of the assumptions (i)–(vi):

$$g'(s)=8\pi s+O\left(\frac{1}{s}\right),$$

$$g''(s)=8\pi+O\left(\frac{1}{s^2}\right),$$

$$g'''(s)=O\left(\frac{1}{s^3}\right).$$

Hence we may choose s_0 large so that g is a convex increasing C^3 function in $[s_0,\infty)$.

Remark 1.3. Examples of non-linearities satisfying the hypotheses in theorem D are: any $h\in\mathscr{B}$ such that $h(s)=\exp\{s^\beta\}$, $0<\beta<2$, for all large s; any $h\in\mathscr{B}$ such that $h(s)=s^\theta$ for all large s and some $\theta>-1$.

Theorem C. *Let $\rho\in(0,1]$ be a parameter depending continuously on l such that $\rho-l\neq 0$ for all $l>0$ and $\rho(0)=0$. We also assume that the following limits (possibly ∞) exist:*

$$a=\lim_{l\to 0}\frac{l}{\rho},$$

$$b=\lim_{l\to 0}\frac{\log\rho}{\log l}.$$

It follows that

$$\log a=\lim_{l\to 0}\log l\left(1-\frac{\log\rho}{\log l}\right). \tag{1.3}$$

Extending each $m_{l,1}$ by zero outside $B(1)$ we may consider $m_{l,1}$ to be members of $H^1(\mathbb{R}^2)$. Let $y\in\partial B(1)$ be arbitrary and $x\in\mathbb{R}^2$. Then,

(i) *If $a=\infty$, uniformly on compact subsets of $\mathbb{R}^2$,*

$$\lim_{l\to 0}m_{l,1}(\rho x)^2-m_{l,1}(\rho y)^2=0.$$

(ii) *If $a<\infty$ and $\rho\leqslant l$ for all $l>0$, uniformly on compact subsets of $\mathbb{R}^2$,*

$$\lim_{l\to 0}\{m_{l,1}(\rho x)^2-m_{l,1}(\rho y)^2\}=\begin{cases}0 & 0\leqslant|x|\leqslant a,\\ \dfrac{b}{\pi}\log\left(\dfrac{a}{|x|}\right) & |x|>a.\end{cases}$$

(iii) *If $a>0$ and $\rho>l$ for all $l>0$, uniformly on compact subsets of $\mathbb{R}^2$,*

$$\lim_{l\to 0}\{m_{l,1}(\rho x)^2-m_{l,1}(\rho y)^2\}=\begin{cases}2\log\left(\dfrac{1}{a}\right) & |x|\leqslant a,\\ 2\log\left(\dfrac{1}{|x|}\right) & |x|>a.\end{cases}$$

(iv) *If $a=0$, for $x\neq 0$, uniformly on compact subsets of $\mathbb{R}^2\setminus\{0\}$,*

$$\lim_{l\to 0}\{m_{l,1}(\rho x)^2-m_{l,1}(\rho y)^2\}=\frac{b}{\pi}\log\left(\frac{1}{|x|}\right),$$

Theorem D. *Let $h \in \mathscr{B}$ and $f(s) = h(s)\exp\{4\pi s^2\}$, $s \geqslant 0$, satisfy the hypotheses listed in the statement of theorem B. Let u_R be any solution of* (P) *posed on $\Omega = B(R)$ with the above choice of f. Then,*

(i) $\|\nabla u_R\|_{L^2(B(R))} \to 1$ *as* $R \to 0$,

(ii) *There exists a parameter ρ depending continuously on R, $\rho \to 0$ as $R \to 0$, such that for any $z \in \partial B(1)$ and $x \in \mathbb{R}^2$,*

$$u_R^2(\rho x) - u_R^2\left(\frac{\rho z}{2}\right) \to 2\log\left(\frac{2}{1+|x|^2}\right)$$

as $\rho \to 0$, uniformly on compact subsets of $\mathbb{R}^2$.

Remark 1.4. Statement (ii) of the theorem implies that the 'limiting equation' to our problem is

$$-\Delta u = \frac{1}{2\pi}\exp\{u\} \quad \text{in } \mathbb{R}^2,$$

$$u = 0 \qquad\qquad \text{on } \partial B(1).$$

Remark 1.5. (1) In fact, for existence of solution to (P) when $n = 2$, it is sufficient if $\limsup_{s\to\infty} h(s)s > c$, where c is a positive constant depending only on Ω and f; refer to remark 4.2 in [1].
(2) In [12], a partial result on non-existence in the case $n = 2$ is obtained, which however, happens to be a special case of theorem B in [5].

2. Proof of Theorem A: Palais–Smale fails along 'Moser functions'

We first prove a few preliminary lemmas from which the theorem will follow readily. In what follows we assume without loss of generality that $0 \in \Omega$, $x_0 = 0$ and $B(0,3) \subset \Omega$. also we equip $H_0^1(\Omega)$ with gradient norm $\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$.

Lemma 2.1. *Let $u \in H_0^1(B(0,R_1)) \subset H_0^1(\Omega)$ be a function radial about the origin. Then there exists a positive constant ω depending only on Ω such that for $R_0 \in (R_1, d(0,\partial\Omega))$,*

$$\|J'(u)\|^2 \leqslant \frac{\omega}{(R_0 - R_1)^2}\left(\int_0^{R_0}\left|\frac{\partial u}{\partial r} + \frac{1}{r}\int_0^r f(u)t\,\mathrm{d}t\right|^2 r\,\mathrm{d}r\right). \tag{2.1}$$

Proof. For $\phi \in H_0^1(\Omega)$, define the θ-average $\tilde{\phi}$ of ϕ to be

$$\tilde{\phi}(r) = \frac{1}{2\pi}\int_0^{2\pi}\phi(r\cos\theta,\ r\sin\theta)\,\mathrm{d}\theta.$$

Then,

$$\langle J'(u),\phi\rangle = \int_\Omega \nabla u\cdot\nabla\phi\,\mathrm{d}x - \int_\Omega f(u)\phi\,\mathrm{d}x$$
$$= 2\pi\left(\int_0^{R_1}\frac{\partial u}{\partial r}\frac{\partial\tilde{\phi}}{\partial r}r\,\mathrm{d}r - \int_0^{R_1} f(u)\tilde{\phi}r\,\mathrm{d}r\right).$$

Let $\eta \in C_0^\infty(B(0,R_0))$ be such that η is radial about the origin, $\eta \equiv 1$ on $B(0,R_1)$,

$|\partial\eta/\partial r| \leqslant 4/(R_0 - R_1)$. Let $\phi_1 = \eta\,\tilde{\phi}$. Then,

$$-\int_0^{R_1} f(u)\,\tilde{\phi} r\,\mathrm{d}r = -\int_0^{R_0} f(u)\,\phi_1 r\,\mathrm{d}r$$

$$= \int_0^{R_0} f(u)\left(\int_r^{R_0} \frac{\partial\phi_1}{\partial s}\,\mathrm{d}s\right) r\,\mathrm{d}r$$

$$= \int_0^{R_0} \left(\frac{1}{r}\int_0^r f(u) t\,\mathrm{d}t\right) \frac{\partial\phi_1}{\partial r} r\,\mathrm{d}r.$$

Therefore,

$$|\langle J'(u), \phi\rangle|^2 = 2\pi \left| \int_0^{R_0} \left(\frac{\partial u}{\partial r} + \frac{1}{r}\int_0^r f(u) t\,\mathrm{d}t\right) \frac{\partial\phi_1}{\partial r} r\,\mathrm{d}r \right|^2$$

$$\leqslant 2\pi \left(\int_0^{R_0} \left| \frac{\partial u}{\partial r} + \frac{1}{r}\int_0^r f(u) t\,\mathrm{d}t \right|^2 r\,\mathrm{d}r \right) \left(\int_0^{R_0} \left| \frac{\partial\phi_1}{\partial r} \right|^2 r\,\mathrm{d}r \right).$$

Since $\partial\phi_1/\partial r = \eta\,(\partial\tilde{\phi}/\partial r) + (\partial\eta/\partial r)\,\tilde{\phi}$, we get, for some positive constants, ω_1, ω depending only on Ω,

$$\int_0^{R_0} \left(\frac{\partial\phi_1}{\partial r}\right)^2 r\,\mathrm{d}r \leqslant \omega_1 \left\{ \int_\Omega |\nabla\phi|^2 \mathrm{d}x + \frac{1}{(R_0 - R_1)^2} \int_\Omega |\phi|^2 \mathrm{d}x \right\}$$

$$\leqslant \frac{\omega}{(R_0 - R_1)^2} \int_\Omega |\nabla\phi|^2 \mathrm{d}x.$$

Therefore, the last two inequalities imply that

$$||J'(u)||^2 \leqslant \frac{\omega}{(R_0 - R_1)^2} \int_0^{R_0} \left| \frac{\partial u}{\partial r} + \frac{1}{r}\int_0^r f(u) t\,\mathrm{d}t \right|^2 r\,\mathrm{d}r.$$

This proves the lemma. ∎

For simplicity of notation, we let $m_{l,R,0}(x) = m_{l,R}(x)$. For $\rho \in \mathbb{R}$, using the facts $\| m_{l,R} \| = 1$ and f has exponential growth, we have

$$J(\rho m_{l,R}) = \frac{\rho^2}{2} - \int_\Omega F(\rho m_{l,R})\,\mathrm{d}x$$

$$\to -\infty \text{ as } |\rho| \to \infty.$$

Hence, we can find $\rho_{l,R} \geqslant 0$ such that

$$J(\rho_{l,R} m_{l,R}) = \sup_{\rho \in \mathbb{R}} J(\rho m_{l,R}).$$

Therefore, the derivative of the function $\rho \mapsto J(\rho m_{l,R})$ vanishes at $\rho = \rho_{l,R}$ which gives, denoting $u_{l,R} = \rho_{l,R} m_{l,R}$,

$$\| \nabla u_{l,R} \|^2_{L^2(\Omega)} = \int_\Omega f(u_{l,R})\,u_{l,R}\,\mathrm{d}x. \tag{2.2}$$

That is, since $\|\nabla u_{l,R}\|_{L^2(\Omega)} = \rho_{l,R}$,

$$\rho_{l,R}^2 = \int_\Omega f(u_{l,R})\, u_{l,R}\, \mathrm{d}x. \tag{2.3}$$

We note that the integrands in equations (2.2), (2.3) and (2.4) are radial functions supported in $B(0,2)$. Therefore, we can transform the integrals in these equations to integrals on $(0,\infty)$ via the transformation

$$|x| = r = 2\exp\{-t/2\}. \tag{2.4}$$

Let $\tau(t) = 2\exp\{-t/2\}$. For any $u \in H_0^1(\Omega)$, u radial about the origin denote $v = u \circ \tau$. Let $w_{l,R} = m_{l,R} \circ \tau$ and $v_{l,R} = \rho_{l,R} w_{l,R}$. Then it can be checked that

$$w_{l,R}(t) = \frac{1}{\sqrt{4\pi}} \begin{cases} 0 & 0 \leqslant t < t_R, \\ (t - t_R)(t_l - t_R)^{-1/2} & t_R \leqslant t < t_l, \\ (t_l - t_R)^{1/2} & t \geqslant t_l, \end{cases} \tag{2.5}$$

where $t_R = 2\log(2/R)$ and $t_l = 2\log(2/l)$.

Now, under the transformation (2.5), the equations (2.2), (2.3) and (2.4) will become respectively

$$\|J'(u)\|^2 \leqslant \frac{4\pi\omega}{R_0 - R_1} \left\| \frac{\mathrm{d}v}{\mathrm{d}t} - \int_t^\infty f(v)\exp\{-s\}\,\mathrm{d}s \right\|_{L^2(0,\infty)}^2, \tag{2.6}$$

$$\int_0^\infty \left|\frac{\mathrm{d}v_{l,R}}{\mathrm{d}t}\right|^2 \mathrm{d}t = \int_0^\infty f(v_{l,R})\, v_{l,R} \exp\{-t\}\,\mathrm{d}t, \tag{2.7}$$

and

$$\frac{\rho_{l,R}^2}{4\pi} = \int_0^\infty f(v_{l,R})\, v_{l,R}\exp\{-t\}\,\mathrm{d}t. \tag{2.8}$$

Also, if $u \in H_0^1(\Omega)$ is radial about the origin, $J(u)$ is transformed under (2.5) as

$$J(u) = 2\pi \int_0^\infty \left|\frac{\mathrm{d}v}{\mathrm{d}t}\right|^2 \mathrm{d}t - 4\pi \int_0^\infty F(v)\exp\{-t\}\,\mathrm{d}t. \tag{2.9}$$

Let $\{l_n\}$ and $\{R_n\}$ be two sequences of positive numbers with $l_n \leqslant R_n$ for all n,

$$\lim_{n\to\infty} l_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\log(2/R_n)}{\log(R_n/l_n)} = 0.$$

By abuse of notation, let $u_n = u_{l_n,R_n}$, $v_n = v_{l_n,R_n}$, $m_n = m_{l_n,R_n}$, $t_n = t_{l_n}$, $\bar{t}_n = t_{R_n}$, $\rho_n = \rho_{l_n R_n}$. We prove that $\{u_n\}$ is a Palais–Smale sequence for J, and also that J does not satisfy the Palais-Smale condition at the energy level $\frac{1}{2}$. This we do by showing that

$$J(u_n) \to \tfrac{1}{2} \quad \text{as } n \to \infty \quad \text{and,}$$

$$\|J'(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

Hereafter, we work only with the transformed equations (2.6)–(2.10)

Lemma 2.2.

$$\rho_n \to 1 \quad as\ n \to \infty.$$

Proof. We prove the lemma by showing

$$\limsup \rho_n \leqslant 1 \quad \text{and} \quad \liminf \rho_n \geqslant 1.$$

$\limsup \rho_n \leqslant 1$: We argue by contradiction. Suppose there exists a subsequence of ρ_n, which we again denote by ρ_n, and $\varepsilon > 0$, such that $\rho_n^2 \geqslant 1 + \varepsilon$ for all n. Then (2.8) gives, denoting $s_n = (\sqrt{4\pi})^{-1}(t_n - \bar{t}_n)^{1/2}$,

$$\begin{aligned}
\frac{\rho_n^2}{4\pi} &\geqslant \int_{t_n}^{\infty} f(v_n)\, v_n \exp\{-t\}\, dt \\
&= \int_{t_n}^{\infty} h(v_n)\, v_n \exp\{4\pi v_n^2 - t\}\, dt \\
&= \int_{t_n}^{\infty} h(\rho_n s_n)\, \rho_n s_n \exp\{\rho_n^2(t_n - \bar{t}_n) - t\}\, dt \\
&= h(\rho_n s_n)\, \rho_n s_n \exp\{(\rho_n^2 - 1)\, t_n - \rho_n^2 \bar{t}_n\}. \qquad (2.10)
\end{aligned}$$

By the growth assumption on h, for any $\delta \in (0,1)$ we may find $c_\delta > 0$ such that for all large t,

$$h(t)\, t^{-1} \geqslant c_\delta \exp\{-4\pi\delta t^2\}.$$

Therefore, the inequality (2.10) gives, for all large n,

$$\begin{aligned}
\frac{1}{4\pi} &\geqslant h(\rho_n s_n)(\rho_n s_n)^{-1} s_n^2 \exp\{(\rho_n^2 - 1)\, t_n - \rho_n^2 \bar{t}_n\} \\
&\geqslant c_\delta s_n^2 \exp\{-\delta \rho_n^2(t_n - \bar{t}_n) + (\rho_n^2 - 1) t_n - \rho_n^2 \bar{t}_n\} \\
&= c_\delta s_n^2 \exp\{\rho_n^2(1-\delta)(t_n - \bar{t}_n) - t_n\}. \qquad (2.11)
\end{aligned}$$

Now, the assumption

$$\lim_{n\to\infty} \frac{\log(2/R_n)}{\log(R_n/l_n)} = 0$$

implies that

$$\lim_{n\to\infty} \frac{\bar{t}_n}{t_n - \bar{t}_n} = 0$$

and hence in particular that $\bar{t}_n = o(t_n)$. Therefore we may choose positive numbers δ, v and η small enough so that $\bar{t}_n \leqslant \eta t_n$ for all large n and $(1+\varepsilon)(1-\delta)(1-\eta) > 1 + v$. Therefore, the inequality (2.11) together with the assumption $\rho_n^2 \geqslant 1 + \varepsilon$ for all n implies that for all large n,

$$\frac{1}{4\pi} \geqslant c_\delta s_n^2 \exp\{v t_n\}$$

which is a contradiction. This contradiction shows that $\limsup_n \rho_n^2 \leqslant 1$.

$\liminf \rho_n \geqslant 1$: Again, suppose there is a subsequence of ρ_n, denoted still by ρ_n, and $\varepsilon > 0$ such that $\rho_n^2 \leqslant 1 - \varepsilon$ for all n. From (2.8),

$$\frac{1}{4\pi} = \int_0^{t_n} \rho_n^{-2} f(v_n) v_n \exp\{-t\}\, dt + \int_{t_n}^{\infty} \rho_n^{-2} f(v_n) v_n \exp\{-t\}\, dt$$
$$= I_1^n + I_2^n. \tag{2.12}$$

We have,

$$\begin{aligned} I_1^n &= \int_{\bar{t}_n}^{t_n} \rho_n^{-2} h(v_n) v_n \exp\left\{\rho_n^2 \frac{(t-\bar{t}_n)^2}{(t_n - \bar{t}_n)} - t\right\} dt \\ &= \frac{1}{4\pi(t_n - \bar{t}_n)} \int_{\bar{t}_n}^{t_n} h(v_n) v_n^{-1} (t - \bar{t}_n)^2 \exp\left\{\rho_n^2 \frac{(t-\bar{t}_n)^2}{(t_n - \bar{t}_n)} - t\right\} dt \\ &\leqslant \frac{1}{4\pi(t_n - \bar{t}_n)} \int_{\bar{t}_n}^{t_n} h(v_n) v_n^{-1} (t - \bar{t}_n)^2 \exp\{(\rho_n^2 - 1)t - \rho_n^2 \bar{t}_n\}\, dt \\ &\leqslant \frac{1}{4\pi(t_n - \bar{t}_n)} \int_{\bar{t}_n}^{t_n} h(v_n) v_n^{-1} (t - \bar{t}_n)^2 \exp\{-\varepsilon t - \rho_n^2 \bar{t}_n\}\, dt. \end{aligned}$$

By the growth assumption on h and the assumption $h(0) = 0$, there exists for any $\delta > 0$ a positive constant $C(\delta)$ such that,

$$h(t)t^{-1} \leqslant C(\delta) \exp\{4\pi\delta t^2\} \quad \text{for all } t \geqslant 0.$$

We now choose $\delta < \min\{\varepsilon, (\varepsilon(1-\varepsilon)^{-1})/2\}$. For all large n, since $\delta < \varepsilon$,

$$\begin{aligned} &\frac{1}{4\pi(t_n - \bar{t}_n)} \int_{\bar{t}_n}^{t_n} h(v_n) v_n^{-1} (t - \bar{t}_n)^2 \exp\{-\varepsilon t - \rho_n^2 \bar{t}_n\}\, dt \\ &\qquad \leqslant \frac{C(\delta)}{4\pi(t_n - \bar{t}_n)} \int_{\bar{t}_n}^{t_n} (t - \bar{t}_n)^2 \exp\{\delta \rho_n^2 (t - \bar{t}_n) - \varepsilon t - \rho_n^2 \bar{t}_n\} dt \\ &\qquad \leqslant \frac{C(\delta)}{4\pi(t_n - \bar{t}_n)} \int_{\bar{t}_n}^{t_n} (t - \bar{t}_n)^2 \exp\{(\delta - \varepsilon)t\}\, dt \\ &\qquad = O(t_n^{-1}). \end{aligned}$$

Hence, $I_1^n = O(t_n^{-1})$. Similarly, for all n large

$$\begin{aligned} I_2^n &= \frac{1}{4\pi} \int_{t_n}^{\infty} h(v_n) v_n^{-1} (t_n - \bar{t}_n) \exp\{\rho_n^2 (t_n - \bar{t}_n) - t\}\, dt \\ &\leqslant \frac{C(\delta)}{4\pi} (t_n - \bar{t}_n) \exp\{(1+\delta)\rho_n^2 t_n\} \int_{t_n}^{\infty} \exp\{-t\}\, dt \\ &= \frac{C(\delta)}{4\pi} (t_n - \bar{t}_n) \exp\{((1+\delta)\rho_n^2 - 1)t_n\}. \end{aligned}$$

Since $\delta < (\varepsilon(1-\varepsilon)^{-1})/2$, the above inequality gives that $I_2^n = O(\exp\{-(\varepsilon/2)t_n\})$. Hence $I_1^n + I_2^n = O(t_n^{-1})$ which contradicts (2.12). This contradiction shows that $\liminf \rho_n \geqslant 1$ and this proves the lemma. ■

Let $\bar{v}_n = v_n(t_n)$. From (2.8) one gets

$$\frac{\rho_n^2}{4\pi} = \int_0^{t_n} f(v_n)v_n \exp\{-t\}\,dt + f(\bar{v}_n)\bar{v}_n \exp\{-t_n\}. \tag{2.13}$$

In the following lemma we compute the limit, as $n \to \infty$, of the two terms on the right of equation (2.13). These limits will enable us to show that $||J'(u_n)|| \to 0$ as $n \to \infty$.

Lemma 2.3.

$$\lim_{n\to\infty} \int_0^{t_n} f(v_n)v_n \exp\{-t\}\,dt = \frac{1}{8\pi},$$

$$\lim_{n\to\infty} f(\bar{v}_n)\bar{v}_n \exp\{-t_n\} = \frac{1}{8\pi}.$$

Since the proof is technical, we prove Lemma 2.3 in the appendix (§8).

Lemma 2.4.

$$J(u_n) \to \tfrac{1}{2} \quad \textit{as} \quad n \to \infty,$$

Proof. Since

$$2\pi \int_0^\infty \left|\frac{dv_n}{dt}\right| dt = \frac{\rho_n^2}{2} \to \frac{1}{2} \quad \text{as} \quad n \to \infty,$$

(2.9) implies that the lemma is proved if we show that

$$\int_0^\infty F(v_n)\exp\{-t\}\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

Let N be a large fixed positive number. Noting that $F(t) \leqslant Mf(t)t^{1-\eta}$ for some M positive, $\eta \in (0, 1]$ and all large t, we have

$$\begin{aligned}
&\int_0^\infty F(v_n)\exp\{-t\}\,dt \\
&\quad = \int_{\{v_n \leqslant N\}} F(v_n)\exp\{-t\}\,dt + \int_{\{v_n > N\}} F(v_n)\exp\{-t\}\,dt \\
&\quad \leqslant \int_{\{v_n \leqslant N\}} F(v_n)\exp\{-t\}\,dt + M\int_{\{v_n > N\}} f(v_n)v_n^{1-\eta}\exp\{-t\}\,dt.
\end{aligned}$$

Using the dominated convergence theorem, the first integral on the right tends to zero as $n \to \infty$. Also, since $\int_0^\infty f(v_n)v_n \exp\{-t\}\,dt$ is bounded,

$$\begin{aligned}
\int_{\{v_n > N\}} f(v_n)v_n^{1-\eta}\exp\{-t\}\,dt &\leqslant \frac{1}{N^\eta}\int_{\{v_n > N\}} f(v_n)v_n \exp\{-t\}\,dt \\
&= O(N^{-\eta}).
\end{aligned}$$

Since N was an arbitrarily large positive number, it follows that

$$\int_0^\infty F(v_n)\exp\{-t\}\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

This proves the lemma. ■

Lemma 2.5.

$$\|J'(u_n)\| \to 0 \quad as \quad n \to \infty.$$

Since the proof is again technical, we prove Lemma 2.5 in the appendix.

Proof of Theorem A. Lemmas 2.4 and 2.5 show that $\{u_n\}$ is a Palais–Smale sequence for J thereby yielding (a) of the theorem. Since $u_n \to 0$ in $H_0^1(\Omega)$ (up to a subsequence), and $J(u_n) \to \frac{1}{2}$, it follows that $\{u_n\}$ does not contain any subsequence that converges strongly in $H_0^1(\Omega)$. Hence J does not satisfy the Palais–Smale condition at the energy level $\frac{1}{2}$. By a standard procedure, for any positive integer k, we can construct a sequence of functions ξ_n, each of them being sum of k Moser functions with mutually disjoint supports, to get as $n \to \infty$, $J(\xi_n) \to k/2$ and

$$\|J'(\xi_n)\| \to 0.$$

This shows that J does not satisfy Palais–Smale condition at the energy levels $k/2$, k any positive integer. This proves (b) and hence the theorem. ■

3. Proof of theorem B: Palais–Smale fails along solutions to (P) on a disc

We prove a few preliminary lemmas from which the theorem will follow easily. But first we transform the problem to an ODE (ordinary differential equation). We are considering solutions u_R, obtained in [1] and [6], of the problem

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in} \quad B(R),\\ u &> 0 \quad \text{in} \quad B(R),\\ u &= 0 \quad \text{on} \quad \partial B(R). \end{aligned} \tag{P$_R$}$$

By regularity (see Remark 4.1 in [1]), u is in $C^{2,\alpha}(\overline{B(R)})$ for some $\alpha > 0$. We note that, by Gidas–Ni–Nirenberg [14], u is radial about the origin. Defining $y(t) = u(x)$ for any x such that $|x| = 2\exp\{-t/2\}$, we see that y solves the problem, with $\gamma = u(0)$,

$$\begin{aligned} -y'' &= \exp\{-t\}f(y),\\ y(\infty) &= \gamma > 0,\\ y'(\infty) &= 0. \end{aligned} \tag{P$_\gamma$}$$

Let $T_0(\gamma)$ denote the first zero of y. Let $g(\gamma) = \log f(\gamma)$.

Lemma 3.1.

$$y'(T_0(\gamma))T_0^{1/2}(\gamma) \to 0 \quad as \quad \gamma \to \infty.$$

Proof. Let $T_2(\gamma)$ be the point at which $y(T_2(\gamma)) = s_0$ for all $\gamma > s_0$ (s_0 as in the hypotheses of theorem). We will require the following asymptotics:

$$T_0(\gamma) \geqslant \left(g(\gamma) - \frac{\gamma g'(\gamma)}{2}\right) + \log\left(\frac{g'(\gamma)}{2}\right) + O(1),$$

$$T_2(\gamma) = \frac{s_0 g'(\gamma)}{2} + \left(g(\gamma) - \frac{\gamma g'(\gamma)}{2}\right) + \log\left(\frac{g'(\gamma)}{2}\right) + O(1),$$

$$y'(T_2(\gamma)) = \frac{2}{g'(\gamma)}\left[1 + O\left(\frac{\log^2(\gamma)}{g(\gamma)}\right)\right].$$

(For the proof of the first asymptotic we refer to lemma 2 in [6]. The second and third asymptotics have been proved in lemma 4.4 in the next section.) Define $\delta(\gamma)=g(\gamma)-\frac{1}{2}\gamma g'(\gamma)+\log(\frac{1}{2}g'(\gamma))$. Let $\gamma\to\infty$ be a given sequence. To prove the lemma we distinguish two cases:

Case (i). *For some subsequence* $\gamma_k\to\infty$, $\delta(\gamma_k)\geqslant\log\gamma_k$: In this case $T_0(\gamma_k)\geqslant\log\gamma_k+O(1)$. Hence,

$$\begin{aligned}y'(T_0(\gamma_k)) &= y'(T_2(\gamma_k))+\int_{T_0(\gamma_k)}^{T_2(\gamma_k)} f(y)\exp\{-t\}\,dt\\ &= O\left(\frac{1}{\gamma_k}\right)+O(\exp\{-T_0(\gamma_k)\})\\ &= O\left(\frac{1}{\gamma_k}\right).\end{aligned}$$

Case (ii). $\delta(\gamma)\leqslant\log\gamma$ *for all large* γ: Define $T_3(\gamma)=4\log\gamma$. Then,

$$\begin{aligned}y(T_3(\gamma)) &= y(T_2(\gamma))-y'(T_2(\gamma))(T_2(\gamma)-T_3(\gamma))+O(\exp\{-T_3(\gamma)\})\\ &= s_0-\frac{2}{g'(\gamma)}\left[1+O\left(\frac{\log^2(\gamma)}{g(\gamma)}\right)\right]\left[\frac{s_0g'(\gamma)}{2}+\delta(\gamma)-T_3(\gamma)\right]+O\left(\frac{1}{\gamma^4}\right)\\ &= O\left(\frac{\log\gamma}{\gamma}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}y'(T_3(\gamma)) &= y'(T_2(\gamma))+\int_{T_3(\gamma)}^{T_2(\gamma)} f(y)\exp\{-t\}\,dt\\ &= O\left(\frac{1}{\gamma}\right)+O(\exp\{-T_3(\gamma)\})\\ &= O\left(\frac{1}{\gamma}\right).\end{aligned}$$

Hence, with θ as in the hypothesis of the theorem,

$$\begin{aligned}y'(T_0(\gamma)) &= y'(T_3(\gamma))+\int_{T_0(\gamma)}^{T_3(\gamma)} f(y)\exp\{-t\}\,dt\\ &= O\left(\frac{1}{\gamma}\right)+O\left(\int_{T_0(\gamma)}^{T_3(\gamma)} y(t)\exp\{-t\}\,dt\right.\\ &= O\left(\frac{1}{\gamma}\right)+O\left(\frac{\log\gamma\exp\{-T_0(\gamma)\}}{\gamma}\right)\\ &= O\left(\frac{1}{\gamma}\right)+O\left(\frac{\log\gamma\exp\{-(1+\theta)\log(\gamma)\}}{\gamma}\right)\\ &= O\left(\frac{1}{\gamma}\right).\end{aligned}$$

Thus in both the cases $y'(T_0(\gamma)) = O(1/\gamma)$. Hence, noting that $g(\gamma) - \frac{1}{2}\gamma g'(\gamma) = O(\gamma^\beta)$ for some $\beta \in [0, 2)$, and $T_0(\gamma) \leqslant T_2(\gamma)$ we get

$$y'(T_0(\gamma)) T_0^{1/2}(\gamma) = O\left(\frac{1}{\gamma}[\max\{\gamma^{1/2}, \gamma^{\beta/2}\}]\right)$$
$$\to 0 \quad \text{as} \quad \gamma \to \infty.$$

This proves the lemma. ∎

Lemma 3.2.

$$\| J'(\tilde{u}_R) \| \to 0 \quad \text{as} \quad R \to 0.$$

Proof. We assume without loss of generality that $x_0 = 0$ and $\overline{B(0,2)} \subset \Omega$. Also we consider R to be much smaller than 1. Let $\chi \in C_0^\infty(\Omega)$ be such that

$$\text{Support}(\chi) \subset B(0,2),$$
$$\chi \equiv 1 \quad \text{on} \quad B(0,1),$$
$$0 \leqslant \chi \leqslant 1.$$

For $\phi \in H_0^1(\Omega)$ set

$$\phi_1 = \chi\phi,$$
$$\psi(r) = \frac{1}{2\pi}\int_0^{2\pi} \phi_1(r\cos\theta, r\sin\theta)\,d\theta.$$

We note that for some $c > 0$ (independent of ϕ), it holds that

$$\|\phi_1\| \leqslant c\|\phi\|.$$

Now,

$$|\psi(R)| = \left|\int_R^2 \psi'(r)\,dr\right|$$
$$\leqslant \left(\int_R^2 |\psi'(r)|^2 r\,dr\right)^{1/2}\left(\int_R^2 \frac{dr}{r}\right)^{1/2}$$
$$\leqslant \frac{1}{\sqrt{2\pi}}\|\phi_1\|\left(\log\left(\frac{2}{R}\right)\right)^{1/2}$$
$$\leqslant \frac{c}{\sqrt{2\pi}}\|\phi\|\left(\log\left(\frac{2}{R}\right)\right)^{1/2}.$$

Letting $d\sigma$ denote the surface measure on $\partial B(R)$, we have

$$\langle J'(\tilde{u}_R), \phi\rangle = \langle J'(\tilde{u}_R), \phi_1\rangle$$
$$= \int_\Omega \nabla\tilde{u}_R \cdot \nabla\phi_1 dx - \int_\Omega f(\tilde{u}_R)\phi_1 dx$$
$$= \int_{B(R)} \nabla u_R \cdot \nabla\phi_1 dx - \int_{B(R)} f(u_R)\phi_1 dx$$
$$= \int_{\partial B(R)} \frac{\partial u_R}{\partial r}\phi_1 d\sigma - \int_{B(R)} [\Delta u_R + f(u_R)]\phi_1\,dx$$

$$
\begin{aligned}
&= \int_{\partial B(R)} \frac{\partial u_R}{\partial r} \phi_1 \, d\sigma \\
&= u_R'(R) \int_{\partial B(R)} \phi_1 \, d\sigma \\
&= R u_R'(R) \int_0^{2\pi} \phi_1(R\cos\theta, R\sin\theta) \, d\theta \\
&= 2\pi R u_R'(R) \psi(R).
\end{aligned}
$$

Hence, for some $c > 0$,

$$|\langle J'(\tilde{u}_R), \phi \rangle| \leqslant c \|\phi\| R |u_R'(R)| \left(\log\left(\frac{2}{R}\right) \right)^{1/2}.$$

Therefore $\|J'(\tilde{u}_R)\|$ will tend to zero as $R \to 0$ if we can show that $R|u_R'(R)| (\log(2/R))^{1/2} \to 0$ as $R \to 0$. Since $u_R(R) = 0$, if we define T_0 by $R = 2\exp\{-T_0/2\}$ and set $y(t) = u_R(2\exp\{-t/2\})$, we see that y solves (P_γ) with $\gamma = u_R(0)$ and T_0 is the first zero of y. Hence by lemma 3.1,

$$
\begin{aligned}
|u_R'(R)| R \left(\log\left(\frac{2}{R}\right) \right)^{1/2} &= \sqrt{2} y'(T_0) T_0^{1/2} \\
&\to 0
\end{aligned}
$$

as $u_R(0) = \gamma \to \infty$. Since $u_R(0) \to \infty$ if and only if $R \to 0$, it follows that

$$\|J'(\tilde{u}_R)\| \to 0 \quad \text{as} \quad R \to 0.$$

This proves the lemma. ■

Lemma 3.3.

$$J(\tilde{u}_R) \to \tfrac{1}{2} \quad \textit{as} \quad R \to 0.$$

Proof. Since

$$\int_{B(R)} |\nabla u_R|^2 \, dx \to 1 \quad \text{as} \quad R \to 0,$$

by (i) of theorem D in §4 it is sufficient to show that

$$\int_\Omega F(\tilde{u}_R) \, dx \to 0 \quad \text{as} \quad R \to 0.$$

Let N be a large positive number. We have, using the fact $F(t) \leqslant M f(t) t^{1-\eta}$ for some $M > 0$, $\eta \in (0, 1]$ and all large t,

$$
\begin{aligned}
\int_\Omega F(\tilde{u}_R) \, dx &= \int_{B(R)} F(u_R) \, dx \\
&= \int_{u_R \leqslant N} F(u_R) \, dx + \int_{u_R > N} F(u_R) \, dx. \\
&\leqslant \int_{u_R \leqslant N} F(u_R) \, dx + \frac{M}{N^\eta} \int_{u_R > N} f(u_R) u_R \, dx.
\end{aligned}
$$

Now, since u_R solves (P_R), we have

$$\int_{B(R)} f(u_R)u_R\,\mathrm{d}x = \int_{B(R)} |\nabla u_R|^2\,\mathrm{d}x.$$

Therefore, the second integral on the right in the last inequality is of the order $1/N^\eta$. By the dominated convergence theorem, it follows that the first integral in the same inequality tends to zero as R tends to 0. Therefore,

$$\int_\Omega F(\tilde{u}_R)\,\mathrm{d}x \to 0 \quad \text{as} \quad R \to 0.$$

Hence $J(u_R) \to 1/2$ as $R \to 0$ which proves the lemma. ■

Proof of theorem B. Putting lemmas 3.2 and 3.3 together, we get that $\{\tilde{u}_R\}$ as $R \to 0$ is a Palais–Smale sequence for J. Since $\tilde{u}_R \to 0$ in $H_0^1(\Omega)$ and $J(\tilde{u}_R) \to \frac{1}{2}$ as $R \to 0$, it follows that $\{\tilde{u}_R\}$ has no strongly convergent subsequence. Thus J does not satisfy the Palais–Smale condition at the energy level $\frac{1}{2}$. By a standard construction, for each positive integer k, instead of u_R we may consider sum of k such solutions with mutually disjoint supports. Then, we will have that J does not satisfy the Palais–Smale condition at the energy levels $k/2$, $k = 1, 2, 3, \ldots$. This proves the theorem. ■

4(1). Proof of theorem C: Blow-up behaviour of the Moser functions $m_{l,1}(x)$

Define $A_{l,\rho}(x) = 2\pi(m_{l,1}(\rho x)^2 - m_{l,1}(\rho y)^2)$.

Case (i). $\rho \leqslant l$ *for all* $l \geqslant 0$: In this case $|\rho y| \leqslant l$. Therefore

$$m_{l,1}(\rho y)^2 = -\frac{1}{2\pi}\log l,$$

and hence

$$A_{l,\rho}(x) = \begin{cases} 0 & 0 \leqslant |x| \leqslant \dfrac{l}{\rho}, \\[2ex] \dfrac{-(\log\rho + \log|x|)^2 + (\log l)^2}{\log l} & \dfrac{l}{\rho} \leqslant |x| \leqslant \dfrac{1}{\rho}. \end{cases}$$

$$= \begin{cases} 0 & 0 \leqslant |x| \leqslant \dfrac{l}{\rho}, \\[2ex] \left(\dfrac{\log\rho}{\log l} + 1\right)\log\left(\dfrac{l}{\rho}\right), & \\[2ex] -2\left(\dfrac{\log\rho}{\log l}\right)\log|x| - \dfrac{(\log|x|)^2}{\log l} & \dfrac{l}{\rho} \leqslant |x| \leqslant \dfrac{1}{\rho}. \end{cases} \tag{4.1}$$

Also, since $\rho \leqslant l$,

$$a = \lim_{l \to 0} \frac{l}{\rho} \geqslant 1.$$

If $a=\infty$, it easily follows from (4.1) that $A_{l,\rho}(x)\to 0$ as $l\to 0$, the convergence being uniform on compact subsets of $\mathbb{R}^2$.

Suppose $a<\infty$. Then, (4.1) immediately implies (since $b=\lim_{l\to 0}(\log\rho/\log l)$) that $b=1$. Therefore,

$$\frac{\log\rho}{\log l}=1+o(l)\quad\text{as}\quad l\to 0.$$

Hence,

$$\left(\frac{\log\rho}{\log l}+1\right)\log\left(\frac{l}{\rho}\right)\to 2\log a\quad\text{as}\quad l\to 0.$$

Therefore, from (4.1), as $l\to 0$,

$$\lim_{l\to 0}A_{l,\rho}(x)=\begin{cases}0 & 0\leqslant|x|\leqslant a,\\ 2\log\left(\dfrac{a}{|x|}\right) & a\leqslant|x|,\end{cases}$$

uniformly on compact subsets of $\mathbb{R}^2$.

Case (ii). $l<\rho<1$ *for all* $l>0$: In this case, $|\rho y|>l$. Therefore

$$m_{l,1}(\rho y)^2=-\frac{1}{2\pi}\frac{(\log\rho)^2}{\log l},$$

and hence,

$$A_{l,\rho}(x)=\begin{cases}\dfrac{(\log\rho)^2-(\log l)^2}{\log l} & 0\leqslant|x|\leqslant\dfrac{l}{\rho},\\ \dfrac{(\log\rho)^2-(\log\rho+\log|x|)^2}{\log l} & \dfrac{l}{\rho}\leqslant|x|\leqslant\dfrac{1}{\rho}.\end{cases}\tag{4.2}$$

Also, we have

$$a\leqslant 1\quad\text{and}\quad b\leqslant 1.$$

If $b<1$, by (4.1) we get $a=0$. Hence $l/\rho\to 0$ as $l\to 0$ and (4.2) gives

$$\lim_{l\to 0}A_{l,\rho}(x)=2b\log\left(\frac{1}{|x|}\right),$$

uniformly on compact subsets of $\mathbb{R}^2\backslash\{0\}$.

If $b=1$, there are two cases: $a=0$ and $a>0$. In the former case, again $l/\rho\to 0$ as $l\to 0$ and as before

$$\lim_{l\to 0}A_{l,\rho}(x)=2\log\left(\frac{1}{|x|}\right),$$

uniformly on compact subsets of $\mathbb{R}^2\backslash\{0\}$.

In the latter case, for $|x| > a$, from (4.2) it follows that uniformly on compact subsets of $\{x: |x| > a\}$,

$$\lim_{l\to 0} A_{l,\rho}(x) = 2\log\left(\frac{1}{|x|}\right).$$

For $|x| \leqslant a$, we have from (4.2)

$$\begin{aligned} A_{l,\rho}(x) &= \left(\frac{\log\rho}{\log l}+1\right)\left(\log\left(\frac{\rho}{l}\right)\right) \\ &\to 2\log\left(\frac{1}{a}\right) \quad \text{as } l\to 0 \end{aligned} \tag{4.3}$$

uniformly on $|x| \leqslant a$. That is

$$\lim_{l\to 0} A_{l,\rho}(x) = \begin{cases} 2\log\left(\dfrac{1}{a}\right) & 0\leqslant |x| \leqslant a, \\ 2\log\left(\dfrac{1}{|x|}\right) & |x| > a, \end{cases}$$

uniformly on compact subsets of $\mathbb{R}^2$. This proves the theorem. ∎

4(2). Proof of theorem D: Blow-up behaviour of the solutions u_R

Here we consider a more general situation where u_R is any solution of the problem (P) posed on $\Omega = B(R)$ and with $f(s) = h(s)\exp\{4\pi s^2\}$, $h \in \mathscr{B}$, satisfying the hypotheses listed in the statement of theorem B. The theorem shows that u_R concentrates to the Dirac mass at the origin as $R \to 0$. Also, the blow-up behaviour of u_R helps us to find the 'limiting equation' associated to the problem under consideration. The asymptotics developed in [6] will be crucial to the proof of the theorem.

We will find it convenient to prove the ODE version of theorem D. This ODE version is stated as theorem D′ below. First we indicate how the statements in theorem D transform to corresponding statements in theorem D′. We are considering the problem

$$\begin{aligned} -\Delta u &= h(u)\exp\{4\pi u^2\} && \text{in} \quad B(R), \\ u &> 0 && \text{in} \quad B(R), \\ u &= 0 && \text{on} \quad \partial B(R). \end{aligned} \tag{P_R}$$

By Gidas–Ni–Nirenberg [14], every solution u_R of (P_R) is radial about the origin. Define for $r \in [0, R]$, and any $x \in \partial B(r)$,

$$w_R(r) = u_R(x).$$

Clearly, w_R solves the following ODE:

$$\begin{aligned} -w'' - \frac{1}{r}w' &= h(w)\exp\{4\pi w^2\} && \text{in} \quad (0,R), \\ w &> 0 && \text{in} \quad (0,R), \\ w'(0) &= w(R) = 0. \end{aligned} \tag{$\tilde{\mathrm{P}}_R$}$$

Let $\gamma = w_R(0)$. Define $y_\gamma(t) = w_R(2\exp\{-t/2\})$, $t \in \mathbb{R}$. Then, $T_0(\gamma) = 2\log(2/R)$ will be the first zero of y_γ. Also, y_γ solves the following ODE:

$$\begin{aligned} -y'' &= \exp\{-t\}h(y)\exp\{4\pi y^2\} \quad \text{in } (T_0(\gamma), \infty),\\ y &> 0 \quad \text{in } (T_0(\gamma), \infty),\\ y(\infty) &= \gamma, y'(\infty) = 0. \end{aligned} \tag{P_γ}$$

Assuming that a parameter ρ as in statement (ii) of the theorem exists, we let $\mu = 2\log(2/\rho)$. Now, $\gamma \to \infty$ if and only if $R \to 0$ and we have assumed that $\rho \to 0$ as $R \to 0$. Therefore, we get that $\mu \to \infty$ as $\gamma \to \infty$. Henceforth, we shall denote simply by y the solution y_γ of (P_γ). Due to the above reduction arguments, it is clear that statements (i) and (ii) in the theorem concerning the behaviour of u_R can be rewritten as the following statements concerning the behaviour of y.

Theorem D′.

(a)
$$\int_{T_0(\gamma)}^{\infty} y'(t)^2 \,\mathrm{d}t \to \frac{1}{4\pi} \quad \textit{as} \quad \gamma \to \infty,$$

(b) *There exists a parameter μ depending continuously on γ, $\mu \to \infty$ as $\gamma \to \infty$, such that,*

$$y^2(\mu + t) - y^2(\mu) \to \frac{1}{2\pi}\log\left(\frac{2}{1 + \exp\{-t\}}\right)$$

uniformly (with respect to t) on compact subsets of $\mathbb{R}$.

Therefore, it is sufficient to prove theorem D′ in order to prove the theorem D. We prove few simple lemmas from which theorem D′ will follow readily. Before doing so we make some preliminary observations.

Let $T_0(\gamma)$ denote the first zero of the solution y of (P_γ). Let k be large, but fixed, positive integer. Define

$$\delta = k\log\gamma,$$

$$T_1(\gamma) = g(\gamma) + \log\left(\frac{g'(\gamma)}{2}\right) - \delta.$$

For easy notation, in the sequel we let $g = g(\gamma)$, $g' = g'(\gamma)$, $g'' = g''(\gamma)$, $g''' = g'''(\gamma)$, $T_0 = T_0(\gamma)$, $T_1 = T_1(\gamma)$, etc. Define

$$z(t) = \gamma - \frac{2}{g'}\log\left(1 + \frac{g'}{2}\exp\{g - t\}\right).$$

The analysis in [6] gives the following relations:

$$y(t) \leqslant z(t), t \geqslant T_1, \tag{4.4}$$

$$y'(t) \geqslant z'(t), t \geqslant T_1, \tag{4.5}$$

$$y(T_1) = \gamma - \frac{2\delta}{g'} + O\left(\frac{\delta^2}{g}\right), \tag{4.6}$$

$$g(y(T_1)) = g - 2\delta + O\left(\frac{\delta^2}{g}\right), \tag{4.7}$$

$$y'(T_1) = \frac{2}{g'}\left[1 + O\left(\frac{\delta^2}{g}\right)\right]. \tag{4.8}$$

Equations (4.4) to (4.8) appear respectively as equations (4.4), (5.1), (7.8), (7.9) *and* (7.17) *in* [6].

Lemma 4.1.

$$\sup_{t\in[T_1,\infty)} |z(t)-y(t)|\to 0 \quad as \quad \gamma\to\infty.$$

Proof. Let

$$l(t)=\frac{2}{g'}\log\left(1+\frac{g'}{2}\exp\{g-t\}\right).$$

Then, from lemma 1 in [6], for $t>T_1$,

$$g(y(t))\geqslant g-g'l(t).$$

Applying g^{-1} on either side of the above inequality and using Taylor's expansion of order two, we get, for some $\xi\in[g-g'l(t),g]$,

$$y(t)\geqslant g^{-1}(g)-(g^{-1})'(g)g'l(t)+\frac{(g^{-1})''(\xi)}{2}(g')^2l^2(t). \tag{4.9}$$

Differentiating the identity $g^{-1}(g(s))=s$ successively twice, we get

$$(g^{-1})'(g(s))=\frac{1}{g'(s)},$$

$$(g^{-1})''(g(s))=\frac{-g''(s)}{(g'(s))^2}.$$

The above equations together with the facts $g'\sim\gamma$ and $g''=O(1)$ imply that

$$\frac{(g^{-1})''(\xi)(g')^2l^2(t)}{2}=O\left(\frac{\log^2\gamma}{\gamma^3}\right).$$

Hence, from (4.9) we get

$$\begin{aligned} y(t)&\geqslant\gamma-\frac{2}{g'}\log\left(1+\left(\frac{g'}{2}\right)\exp\{g-t\}\right)+O\left(\frac{\log^2\gamma}{\gamma^3}\right)\\ &=z(t)+O\left(\frac{\log^2\gamma}{\gamma^3}\right)\end{aligned}$$

uniformly with respect to $t\geqslant T_1$. Thus the lemma is proved. ∎

Lemma 4.2.

$$\sup_{t\in[T_1,\infty)} |(z')^2(t)-(y')^2(t)|\to 0 \quad as \quad \gamma\to\infty.$$

Proof. Clearly $z(\infty)=\gamma$. Also, it can be checked easily that $z(t)$ satisfies the differential equation

$$z''(t)=-\exp\{g+g'(z(t)-\gamma)-t\}. \tag{4.10}$$

Now we have

$$z'(t) = \frac{\exp\{g-t\}}{(1+(g'/2)\exp\{g-t\})}$$

which implies

$$z'(T_1) = \frac{2}{g'}\left[1 + O\left(\frac{1}{\gamma^k}\right)\right]. \tag{4.11}$$

From (4.4) we have, for all $t \geqslant T_1$,

$$y(t) \leqslant z(t).$$

Let

$$l(t) = \frac{2}{g'}\log\left(1 + \frac{g'}{2}\exp\{g-t\}\right). \tag{4.12}$$

Then for all large γ and $t \geqslant T_1$ we have

$$\begin{aligned} g(z(t)) &= g(\gamma - l(t)) \\ &= g(\gamma) - g'(\gamma)l(t) + \frac{g''(\xi)}{2} l^2(t), \end{aligned}$$

for some $\xi \in [\gamma - l(t), \gamma]$. Since g'' is bounded for large γ,

$$\begin{aligned} g(z(t)) = g - g'\left[\frac{2}{g'}\log\left(1 + \frac{g'}{2}\exp\{g-t\}\right)\right] \\ + O\left(\frac{\log^2(1+(g'/2)\exp\{g-t\})}{(g')^2}\right). \end{aligned} \tag{4.13}$$

Since $g' \sim \gamma$ and for $t \geqslant T_1$, $\log(1+(g'/2)\exp\{g-t\}) = O(\log\gamma)$, the above equation becomes

$$g(z(t)) = g + g'(z(t) - \gamma) + O\left(\frac{\log^2\gamma}{\gamma^2}\right). \tag{4.14}$$

Since $y(t) \leqslant z(t)$ for $t \geqslant T_1$, and g is increasing in $[T_1, \infty)$, it follows from (4.14) and (4.10)

$$\begin{aligned} y'(t) &= \int_t^\infty \exp\{g(y) - s\}\,\mathrm{d}s \\ &\leqslant \int_t^\infty \exp\{g(z) - s\}\,\mathrm{d}s \\ &= \exp\left\{O\left(\frac{\log^2\gamma}{\gamma^2}\right)\right\}\int_t^\infty \exp\{g + g'(z(s) - \gamma) - s\}\,\mathrm{d}s \\ &= z'(t)\left[1 + O\left(\frac{\log^2\gamma}{\gamma^2}\right)\right]. \end{aligned}$$

Since from (4.5), $y'(t) \geqslant z'(t)$ for all $t \geqslant T_1$, we get

$$z'(t) \leqslant y'(t) \leqslant z'(t)\left[1 + O\left(\frac{\log^2\gamma}{\gamma^2}\right)\right]$$

and hence for all $t \geqslant T_1$,

$$(z')^2(t) \leqslant (y')^2(t) \leqslant (z')^2(t)\left[1 + O\left(\frac{\log^2 \gamma}{\gamma^2}\right)\right]$$

which proves the lemma. ■

Lemma 4.3.

$$\int_{T_1}^{\infty} (y')^2(t)\,dt \to 0 \quad as \quad \gamma \to \infty.$$

Proof. By the previous lemma, it is enough to show

$$\int_{T_1}^{\infty} (z')^2(t)\,dt \to 0 \quad \text{as} \quad \gamma \to \infty.$$

We have

$$z'(t) = \frac{\exp\{g - t\}}{1 + (g'/2)\exp\{g - t\}}$$

which implies $z'(t) \leqslant 1$ for all $t \in [T_1, \infty)$, for all large γ. Hence

$$\begin{aligned}\int_{T_1}^{\infty} (z')^2(t)\,dt &\leqslant \int_{T_1}^{\infty} z'(t)\,dt \\ &= z(\infty) - z(T_1) \\ &= \frac{2}{g'}\log(1 + \gamma^k) \\ &\to 0 \text{ as } \gamma \to \infty.\end{aligned}$$

■

Define $T_2 < T_1$ by $y(T_2) = s_0$, where s_0 as in the hypotheses of the theorem is chosen as large as required.

Lemma 4.4.

$$\int_{T_2}^{T_1} (y')^2(t)\,dt \to \frac{1}{4\pi} \quad as \quad \gamma \to \infty.$$

Proof. From lemma 2 in [6], we have for all $t \in [T_2, T_1]$,

$$g(y(t)) - t \leqslant g(y(t)) - g + \frac{g'}{2}(\gamma - y(t)) - \log\left(\frac{g'}{2}\right) \equiv \psi(y).$$

Then $\psi''(s) = g''(s) > 0$ for all $s \in [y(T_2), y(T_1)]$. Thus, ψ is convex in $[y(T_2), y(T_1)]$ and hence for all $t \in [T_2, T_1]$,

$$g(y(t)) - t \leqslant \max\{\psi(y(T_1)), \psi(y(T_2))\}.$$

We have by the hypothesis (iv), for some $\eta > 0$ and all large γ,

$$\begin{aligned}\psi(y(T_2)) &= g(s_0) - g + \frac{1}{2}\gamma g' - \frac{1}{2}s_0 g' - \log\left(\frac{g'}{2}\right) \\ &\leqslant -\eta\gamma.\end{aligned}$$

Also, using the asymptotics (4.7) and (4.8) we get

$$\psi(y(T_1)) = g(y(T_1)) - g + \frac{1}{2}(\gamma - y(T_1))g' - \log\left(\frac{g'}{2}\right)$$
$$\leqslant -\delta + O(1). \tag{4.15}$$

Therefore, for all $t \in [T_2, T_1]$,

$$g(y(t)) - t \leqslant \max\{-\eta\gamma, -\delta + O(1)\} = -\delta + O(1).$$

Hence, for all $t \in [T_2, T_1]$ we have, if k is large enough

$$\begin{aligned} y'(t) &= y'(T_1) + \int_t^{T_1} \exp\{g(y(s)) - s\}\,\mathrm{d}s \\ &\leqslant y'(T_1) + O\left(\frac{T_1 - t}{\gamma^k}\right) \\ &= y'(T_1) + O\left(\frac{1}{\gamma^{k-2}}\right) \\ &= \frac{2}{g'}\left[1 + O\left(\frac{\delta^2}{g}\right)\right]. \end{aligned}$$

Since $y'(t) \geqslant y'(T_1)$ for any $t \in [T_2, T_1]$, we get, for any t in this range,

$$y'(t) = \frac{2}{g'}\left[1 + O\left(\frac{\delta^2}{g}\right)\right].$$

Therefore, for some $\xi \in [T_2, T_1]$,

$$\begin{aligned} T_2 &= T_1 - \frac{1}{y'(\xi)}(y(T_2) - y(T_1)) \\ &= g + \log\left(\frac{g'}{2}\right) - \delta \\ &\quad + \left[\frac{2}{g'}\left(1 + O\left(\frac{\delta^2}{g}\right)\right)\right]^{-1}\left(s_0 - \gamma + \frac{2\delta}{g'} + O\left(\frac{\delta^2}{g}\right)\right) \\ &= g - \frac{\gamma g'}{2} + \frac{s_0 g'}{2} + \log\left(\frac{g'}{2}\right) + O\left(\frac{\delta^2}{g'}\right). \end{aligned} \tag{4.16}$$

Thus,

$$\begin{aligned} \int_{T_2}^{T_1} (y')^2(t)\,\mathrm{d}t &= \frac{4}{(g')^2}\left[1 + O\left(\frac{\delta^2}{g}\right)\right](T_1 - T_2) \\ &= \frac{2\gamma}{g'} + O\left(\frac{1}{\gamma}\right) \\ &\to \frac{1}{4\pi} \quad \text{as} \quad \gamma \to \infty. \end{aligned}$$

This proves the lemma. ∎

Lemma 4.5.

$$\int_{T_0}^{T_2} (y')^2(t)\,\mathrm{d}t \to 0 \quad as \quad \gamma \to \infty.$$

Proof. Let $T_3 = \max\{T_0, 4\log\gamma\}$. Note that T_3 depends on γ. Also, since

$$\sup\{f(y): 0 \leqslant y \leqslant y(T_2) = s_0\} = O(1),$$

we get, upon integrating the ODE $-y'' = \exp\{-t\} f(y)$ successively twice between the limits $t = T_3$ and $t = T_2$ and using the concavity of y together with the fact $T_3 \geqslant 4\log\gamma$,

$$y(T_3) = y(T_2) - y'(T_2)(T_2 - T_3) + O\left(\frac{1}{\gamma^3}\right), \tag{4.17}$$

$$y'(T_3) = y'(T_2) + O\left(\frac{1}{\gamma^4}\right). \tag{4.18}$$

We now distinguish two cases regarding the behaviour of $T_0(\gamma)$ as $\gamma \to \infty$.

Case (i). *For some subsequence* $\gamma_k \to \infty$, $T_0(\gamma_k) > 4\log\gamma_k$: Clearly in this case, $T_3 = T_0$ and hence from (4.18) and the fact that y' is decreasing on $[T_0, \infty)$ we get, $y'(t) = (2/g')[1 + O(\delta^2/g)]$ for all $t \in [T_0, T_2]$. Hence

$$\int_{T_0}^{T_2} (y')^2(t)\,\mathrm{d}t = \frac{4}{(g')^2}\left[1 + O\left(\frac{\delta^2}{g}\right)\right](T_2 - T_0)$$
$$= O\left(\frac{T_2}{\gamma^2}\right).$$

Since $g - (\gamma g'/2) = O(\gamma^\beta)$ for some $\beta \in [0, 2)$, (4.16) implies that $T_2 = O(\gamma^\beta)$. Therefore, the last inequality implies that

$$\int_{T_0}^{T_2} (y')^2(t)\,\mathrm{d}t \to 0 \quad \text{as} \quad \gamma_k \to \infty$$

and so the lemma is true in this case.

Case (ii). $T_0(\gamma) \leqslant 4\log\gamma$ *for all large* γ: In this case, $T_3 = 4\log\gamma$ for all large γ. Hence (4.17) gives

$$y(T_3) = s_0 - \frac{2}{g'}\left[1 + O\left(\frac{\delta^2}{g}\right)\right]\left(g - \frac{\gamma g'}{2} + \frac{s_0 g'}{2} + \log\left(\frac{g'}{2}\right)\right.$$
$$\left. - 4\log\gamma + O\left(\frac{\delta^2}{g}\right)\right)$$
$$= -\frac{2}{g'}\left(g - \frac{g'\gamma}{2} + \log\left(\frac{g'}{2}\right)\right) + O\left(\frac{\log\gamma}{\gamma}\right).$$

Since by assumption (iv) in theorem B, $g - (g'\gamma/2) + \log(g'/2) \geqslant (1 + \theta)\log\gamma$ for some $\theta > -1$ and all large γ, the above equation gives

$$y(t) = O\left(\frac{\log\gamma}{\gamma}\right) \quad \text{for all} \quad t \in [T_0, T_3].$$

Now, for any $t \in [T_0, T_3]$,

$$y'(t) = y'(T_3) + \int_t^{T_3} f(y) \exp\{-s\} \, ds.$$

Since $f(0) = 0$, for any $s \in [T_0, T_3]$, $f(y(s)) = O(y(s)) = O(\log \gamma/\gamma)$. Therefore, the last equation implies that for all $t \in [T_0, T_3]$,

$$\begin{aligned} y'(t) &= y'(T_3) + O\left(\frac{\log \gamma}{\gamma}\right) \\ &= O\left(\frac{\log \gamma}{\gamma}\right). \end{aligned}$$

Therefore, since $T_0 \geqslant 0$ for all large γ (in fact, $T_0 \to \infty$ as $\gamma \to \infty$),

$$\int_{T_0}^{T_2} (y')^2(t) \, dt \leqslant O\left(\frac{\log^2 \gamma}{\gamma^2}\right) T_2.$$

Since $T_2 = O(\gamma^\beta)$ for some $\beta \in [0, 2)$, the above inequaliy implies

$$\int_{T_0}^{T_2} (y')^2(t) \, dt \to 0 \quad \text{as} \quad \gamma \to \infty$$

and so the lemma is true in this case also. This proves the lemma. ■

Proof of theorem D′. (a) follows from lemmas 4.3–4.5. In view of lemma 4.1, (b) is proved if we show that there exists a parameter μ depending continuously on $\gamma, \mu \to \infty$ as $\gamma \to \infty$, such that as $\mu \to \infty$

$$z^2(\mu + t) - z^2(\mu) \to \frac{1}{2\pi} \log\left(\frac{2}{1 + \exp\{-t\}}\right)$$

uniformly (with respect to t) on compact subsets of $\mathbb{R}$.

We take $\mu = g + \log(g'/2)$. Then,

$$\begin{aligned} z^2(\mu + t) - z^2(\mu) &= \left(\gamma - \frac{2}{g'} \log(1 + \exp\{-t\})\right)^2 - \left(\gamma - \frac{2}{g'} \log 2\right)^2 \\ &= \frac{4\gamma}{g'} \log\left(\frac{2}{1 + \exp\{-t\}}\right) + O\left(\frac{1}{\gamma^2}\right) \\ &\to \frac{1}{2\pi} \log\left(\frac{2}{1 + \exp\{-t\}}\right) \end{aligned}$$

uniformly (with respect to t) on compact subsets of $\mathbb{R}$ as $\mu \to \infty$. This proves (b) and hence the theorem. ■

5. Generalizations

In this section we generalize theorems A and B by replacing $-\Delta$ in problem (P) by the n-Laplacian $-\Delta_n$ in $\mathbb{R}^n$, $n \geqslant 2$. Let $\mathscr{B}_n$ denote the subset of $C^1(\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_+)$ consisting of functions $h(s)$ which vanish only at $s=0$ and which satisfy the following growth conditions:

For every $\delta > 0$,

$$\lim_{s\to\infty} h(s)\exp\{\delta s^{n/(n-1)}\} = \infty,$$

$$\lim_{s\to\infty} h(s)\exp\{-\delta s^{n/(n-1)}\} = 0.$$

We call a function $f \in C^1(\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_+)$ a function of critical growth in $\mathbb{R}^n$ if $f(s) = h(s)s^{n-2}\exp\{\alpha_n s^{(n/(n-1))}\}$ with $h \in \mathscr{B}_n$ and $\alpha_n = n\omega_n^{1(n-1)}$, $\omega_n = \text{volume}(S^{n-1})$. Consider the following problem with f having critical growth in $\mathbb{R}^n$ and Ω a smooth bounded domain in $\mathbb{R}^n$:

$$\begin{aligned} -\Delta_n u &= f(u) \quad \text{in} \quad \Omega,\\ u &> 0 \quad \text{in} \quad \Omega,\\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{P_n}$$

The associated energy functional J_n: $W_0^{1,n}(\Omega) \to \mathbb{R}$ is given by

$$J_n(u) = \frac{1}{n}\int_\Omega |\nabla u|^n \mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x,$$

where F denotes the primitive of f.

5.1 *Generalization of theorem A*

Let $0 < l < R$. Define the Moser functions

$$m_{l,R}(x) = \frac{1}{\omega_n^{1/n}}\begin{cases} [\log(R/l)]^{1-1/n} & 0 \leqslant |x| \leqslant l,\\ \dfrac{\log(R/|x|)}{[\log(R/l)]^{1/n}} & l \leqslant |x| \leqslant R,\\ 0 & |x| \geqslant R. \end{cases}$$

Let $x_0 \in \Omega$ and $R < d(x_0, \partial\Omega)$. Define $m_{l,R,x_0}(x) = m_{l,R}(x - x_0)$. We can now state the following.

Theorem A′. *Let* $f(s) = h(s)s^{n-2}\exp\{\alpha_n s^{(n/(n-1))}\}$, $s \geqslant 0$, $h \in \mathscr{B}_n$ *and* h *satisfies the following hypothesis:* $|h'(t)| \leqslant ch(t)t^{(((1/(n-1))-\eta)}$ and $F(t) \leqslant cf(t)t^{(((1/(n-1))-\eta)}$ *for* $\eta \in (0, 1/(n-1))]$, *for some* $c > 0$ *and all large* t.

Let R *depend on* l *so that* $\lim_{l\to 0}(\log(2/R)/\log(R/l)) = 0$. *Then there exists a sequence* $\rho_l \to 1$ *as* $l \to 0$ *such that*

(a) $\{\rho_l m_{l,R,x_0}\}_{l>0}$ *is a Palais–Smale sequence for* J_n *as* $l \to 0$,

(b) J_n *fails to satisfy the Palais–Smale condition along the above sequence at the energy levels* k/n, k *any positive integer.*

Proof. Except for some technical modifications, same as that of theorem A.

5.2 *Generalization of Theorem B*

Let $x_0 \in \Omega$ and $R > 0$ be small enough so that $B(x_0, R) = \{x : |x - x_0| < R\} \subset \Omega$. We now state the following generalization of theorem B.

Theorem B′. *Let* $h \in \mathscr{B}_n$ *and* $f(s) = h(s)s^{n-2}\exp\{\alpha_n s^{(n/(n-1))}\}$, $s \geqslant 0$, *satisfies the following hypothesis: There exists* $s_0 > 0$ *such that*

(i) $h(s)s^{n-1} \to \infty$ *as* $s \to \infty$,
(ii) $g(s) = \log f(s)$ *is* C^3 *and convex for* $s \geqslant s_0$,
(iii) $f(s)$ *is strictly increasing for* $s \geqslant s_0$,
(iv) $\left(g(s) - \left(\frac{n-1}{n}\right)sg'(s)\right) + (n-1)\log\left(\frac{(n-1)}{n}g'(s)\right) \geqslant (n-1+\theta)\log s$ *for all* $s \geqslant s_0$ *and some* $\theta > 1-n$,
(v) $\limsup_{s\to\infty}\left(g(s) - \left(\frac{n-1}{n}\right)sg'(s)\right)s^{-\beta} = O(1)$ *for some* $\beta \in [0, (n/(n-1)))$,
(vi) $\limsup_{s\to\infty} \frac{sg^{(k+1)}(s)}{g^{(k)}(s)}$, $k = 0, 1$, *exist and are different from* 0 (*here* $g^{(k)}$ *denotes the k-th derivative of g*).

Let u_R *denote the solution to problem* (P_n) *with the above choice of* f *which is obtained in* [1]. *Let* $\tilde{u}_R$ *denote the extension of* u_R *obtained by setting* $\tilde{u}_R \equiv 0$ *in* $\Omega \backslash B(x_0, R)$. *Then*

(a) $\{\tilde{u}_R\}_{R>0}$ *as* $R \to 0$ *are Palais–Smale sequences for* J_n,
(b) J_n *does not satisfy the Palais–Smale condition at the energy levels* k/n, *k any positive integer.*

Proof. Except for some technical modifications, same as that of theorem B.

6. The Neumann case

If we consider the problem (P_n) with the Dirichlet boundary condition replaced by the homogeneous Neumann condition, the associated energy functional J_n: $W^{1,n}(\Omega) \to \mathbb{R}$ will be given again by

$$J_n(u) = \frac{1}{n}\int_\Omega |\nabla u|^n \,\mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x.$$

If f satisfies the assumptions of either theorem A′ or theorem B′, it can be shown that J_n fails to satisfy the Palais–Smale condition at the energy levels $k/2n$, k any positive integer. The proof is done first for the case of half-space $\mathbb{R}_n^+$, where one uses the results from Dirichlet case to show that the associated energy functional fails to satisfy the Palais–Smale condition at the energy levels $k/2n$, k any positive integer. The proof for a general domain Ω with smooth boundary is accompolished by the standard localization argument involving partition of unity.

7. Concluding remarks

1. Analogues of theorems A and B can be shown to hold even when the Palais–Smale sequences considered in §2 and §3 approach a boundary point of Ω and/or there are multiple concentrations at a single point in Ω or boundary of Ω. In each of these cases Palais–Smale still fails at the energy levels $k/2$, k any positive integer.
2. If the non-linearity f grows like $\exp\{bs^2\}$ for some $b > 0$ as $s \to \infty$ similar arguments as in theorems A and B will show that the corresponding energy functional

will fail to satisfy the Palais–Smale condition at the energy levels $2k\pi/b$, k any positive integer.
3. Blow-up result similar to the one presented in part (ii) of theorem D were first obtained in [20] for the maximizing sequence for the Trudinger imbedding.
4. The blow-up result in part (ii) of theorem D extends the results of [17, 18], where h is assumed to have atmost polynomial growth. Also, we have greatly simplified the proof given in these references.
5. In fact, the blow-up analysis given here can be easily seen to hold even when the non-linearity f grows like $\exp\{s^\alpha\}$ as $s \to \infty$ for any $\alpha \geqslant 1$.
6. Lemmas 4.1 and 4.2 are generalizations of the corresponding results in Volkmer [23].

8. Appendix

Proof of Lemma 2.3. Let $\varepsilon \in (0,1)$ be fixed. For $\delta > 0$ denote $\hat{\delta} = \delta/4\pi$. Define $T(\varepsilon, n) = (1-\varepsilon)t_n + \varepsilon \bar{t}_n$. Then, for any $\delta > 0$, and all large n,

$$\begin{aligned}
&\int_0^{T(\varepsilon,n)} f(v_n)v_n \exp\{-t\}\,\mathrm{d}t \\
&= \int_0^{T(\varepsilon,n)} h(v_n)v_n \exp\left\{\rho_n^2 \frac{(t-\bar{t}_n)^2}{t_n - \bar{t}_n} - t\right\}\mathrm{d}t \\
&\leqslant \int_0^{T(\varepsilon,n)} h(v_n)v_n \exp\{\rho_n^2(1-\varepsilon)(t-\bar{t}_n) - t\}\,\mathrm{d}t \\
&= \int_0^{T(\varepsilon,n)} h(v_n)\exp\{\rho_n^2(1-\varepsilon)(t-\bar{t}_n) - t - \delta v_n^2 + \delta v_n^2\}\,\mathrm{d}t \\
&\leqslant \int_0^{T(\varepsilon,n)} h(v_n)\exp\{-\delta v_n^2\} v_n \exp\{\rho_n^2(1-\varepsilon)(1+\hat{\delta})(t-\bar{t}_n) - t\}\,\mathrm{d}t \\
&\leqslant \int_0^{T(\varepsilon,n)} h(v_n)\exp\{-\delta v_n^2\} v_n \exp\{\rho_n^2(1-\varepsilon)(1+\hat{\delta}) - 1)t\}\,\mathrm{d}t.
\end{aligned} \tag{8.1}$$

Let M be an upper bound for $h(v_n)\exp\{-\delta v_n^2\}$.

We choose $\delta > 0$ small enough so that for some $\eta > 0$ and all large n,

$$(1-\varepsilon)(1+\hat{\delta})\rho_n^2 - 1 \leqslant -\eta.$$

Now (8.1) implies, with η and M chosen as above,

$$\begin{aligned}
&\int_0^{T(\varepsilon,n)} f(v_n)v_n \exp\{-t\}\,\mathrm{d}t \\
&\leqslant \frac{\rho_n M}{\sqrt{4\pi}(t_n - \bar{t}_n)^{1/2}} \int_{\bar{t}_n}^{T(\varepsilon,n)} t\exp\{-\eta t\}\,\mathrm{d}t \\
&= O(t_n^{-1/2}).
\end{aligned} \tag{8.2}$$

Integrating by parts, we obtain for some $\nu > 0$,

$$\int_{T(\varepsilon,n)}^{t_n} h(v_n)v_n \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt$$

$$= \int_{T(\varepsilon,n)}^{t_n} h(v_n)v_n \left(\frac{2\rho_n^2(t-\bar{t}_n)}{t_n-\bar{t}_n} - 1\right)^{-1} \frac{d}{dt}\left(\exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\}\right) dt$$

$$= \left[h(v_n)v_n\left(\frac{2\rho_n^2(t-\bar{t}_n)}{t_n-\bar{t}_n} - 1\right)^{-1} \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\}\right]_{t=T(\varepsilon,n)}^{t=t_n}$$

$$- \int_{T(\varepsilon,n)}^{t_n} \left\{- h(v_n)v_n\left(\frac{2\rho_n^2(t-\bar{t}_n)}{t_n-\bar{t}_n} - 1\right)^{-2} \frac{2\rho_n^2}{(t_n-\bar{t}_n)}\right.$$

$$\left. + (h(v_n)v_n)'\left(\frac{2\rho_n^2(t-\bar{t}_n)}{t_n-\bar{t}_n} - 1\right)^{-1}\right\} \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt$$

$$= \frac{h(\bar{v}_n)\bar{v}_n}{2\rho_n^2-1} \exp\left\{\rho_n^2(t_n-\bar{t}_n) - t_n\right\} + O(\exp\{-\nu t_n\})$$

$$+ O\left(\frac{1}{t_n}\int_{T(\varepsilon,n)}^{t_n} h(v_n)v_n \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt\right)$$

$$+ O\left(\int_{T(\varepsilon,n)}^{t_n} (h(v_n)v_n)' \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt\right). \tag{8.3}$$

By the hypothesis on h we get, for some $\eta \in (0,1]$, for some $c > 0$ and all large t,

$$|h'(v_n)|v_n \leqslant c h(v_n) v_n^{1-\eta} v_n$$

$$\leqslant c\left(\frac{t_n-\bar{t}_n}{4\pi}\right)^{(1-\eta)/2} h(v_n)v_n.$$

Therefore, for some positive numbers c_1, c_2 and c_3,

$$\int_{T(\varepsilon,n)}^{t_n} |h(v_n)v_n)'| \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt$$

$$\leqslant \int_{T(\varepsilon,n)}^{t_n} \{|h'(v_n)v_n'|v_n + h(v_n)|v_n'|\} \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt$$

$$= \frac{c_1}{(t_n-\bar{t}_n)^{1/2}} \int_{T(\varepsilon,n)}^{t_n} \{|h'(v_n)|v_n + h(v_n)\} \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt$$

$$\leqslant \frac{c_2}{(t_n-\bar{t}_n)^{\eta/2}} \int_{T(\varepsilon,n)}^{t_n} h(v_n)v_n \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt$$

$$+ \frac{c_3}{t_n-\bar{t}_n} \int_{T(\varepsilon,n)}^{t_n} h(v_n)v_n \exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n} - t\right\} dt. \tag{8.4}$$

Since by lemma 2.2, $\rho_n^2 \to 1$ as $n \to \infty$, (2.13) implies

$$\int_0^{t_n} f(v_n) \exp\{-t\}\, dt = O(1).$$

Therefore (8.4) gives

$$\int_{T(\varepsilon,n)}^{t_n} |(h(v_n)v_n)'|\exp\left\{\frac{\rho_n^2(t-\bar{t}_n)^2}{t_n-\bar{t}_n}-t\right\}\mathrm{d}t = O(t_n^{-\eta/2}).$$

Hence, (8.3) implies

$$\int_{T(\varepsilon,n)}^{t_n} f(v_n)v_n\exp\{-t\}\,\mathrm{d}t = \frac{f(\bar{v}_n)\bar{v}_n\exp\{-t_n\}}{2\rho_n^2-1}+O(t_n^{-\eta/2}). \tag{8.5}$$

Combining (8.2) with the last equation, we get

$$\int_0^{t_n} f(v_n)v_n\exp\{-t\}\,\mathrm{d}t = \frac{f(\bar{v}_n)\bar{v}_n\exp\{-t_n\}}{2\rho_n^2-1}+O(t_n^{-\eta/2}). \tag{8.6}$$

The lemma now follows from (2.13) and the fact that $\rho_n^2\to 1$ as $n\to\infty$. ■

Proof of Lemma 2.5. From (2.6), it is sufficient to show that

$$I_n = \int_0^\infty\left|\int_t^\infty f(v_n)\exp\{-s\}\,\mathrm{d}s-\frac{\mathrm{d}v_n}{\mathrm{d}t}\right|^2\mathrm{d}t\to 0$$

as $n\to\infty$.

Using the explicit form of v_n, we find that

$$\frac{\mathrm{d}v_n}{\mathrm{d}t}=\frac{\rho_n}{\sqrt{4\pi}}\begin{cases}0 & 0\leqslant t<\bar{t}_n,\\ (t_n-\bar{t}_n)^{-1/2} & \bar{t}_n<t<t_n,\\ 0 & t_n<t.\end{cases}$$

We split

$$I_n = A_n+B_n+C_n$$

with

$$A_n=\int_0^{\bar{t}_n}\left|\int_t^\infty f(v_n)\exp\{-s\}\,\mathrm{d}s\right|^2\mathrm{d}t,$$

$$B_n=\int_{\bar{t}_n}^{t_n}\left|\int_t^\infty f(v_n)\exp\{-s\}\,\mathrm{d}s-\frac{\rho}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}\right|^2\mathrm{d}t,$$

$$C_n=\int_{t_n}^\infty\left|\int_t^\infty f(v_n)\exp\{-s\}\,\mathrm{d}s\right|^2\mathrm{d}t.$$

We show that A_n, B_n and C_n tend to zero as $n\to\infty$. This we do in the following series of claims. Let $\varepsilon>0$ be arbitrary. Define $T(\varepsilon,n)=(1-\varepsilon)t_n+\varepsilon\bar{t}_n$.

Claim 1. There exists an $\eta>0$ such that for any $t\in[\bar{t}_n, T(\varepsilon,n)]$, as $n\to\infty$,

$$\int_t^{T(\varepsilon,n)} f(v_n)\exp\{-s\}\,\mathrm{d}s = O((t_n-\bar{t}_n)^{-1/2}(1+t)\exp\{-t\}) + O(\exp\{-\eta(t_n-\bar{t}_n)^{1/2}\}). \tag{8.7}$$

Proof of claim.

Since $h(0)=0$ and $v_n(s)\leqslant 1$ in $[0,(t_n-\bar{t}_n)^{1/2}]$, we have $h(v_n(s))=O(v_n(s))$ as $n\to\infty$, uniformly for $s\in[0,(t_n-\bar{t}_n)^{1/2}]$. Let $\delta>0$ and $c_\delta>0$ be such that $h(t)\leqslant c_\delta\exp\{4\pi\delta t^2\}$ for all $t\geqslant 0$. Also, we may choose δ small enough so that the following inequality holds for all $s\leqslant T(\varepsilon,n)$ and some $\eta>0$:

$$(1+\delta)\rho_n^2(s-\bar{t}_n)^2(t_n-\bar{t}_n)^{-1}-s\leqslant-\eta s.$$

Therefore, for $t\in[\bar{t}_n,T(\varepsilon,n)]$, all large n and some positive constant c,

$$\begin{aligned}\int_t^{T(\varepsilon,n)}f(v_n)\exp\{-s\}\,\mathrm{d}s&=\int_t^{(t_n-\bar{t}_n)^{1/2}}f(v_n)\exp\{-s\}\,\mathrm{d}s\\&\quad+\int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)}f(v_n)\exp\{-s\}\,\mathrm{d}s\\&\leqslant c\int_t^{(t_n-\bar{t}_n)^{1/2}}v_n(s)\exp\{-s\}\,\mathrm{d}s\\&\quad+c_\delta\int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)}\exp\left\{(1+\delta)\rho_n^2\,\frac{(s-\bar{t}_n)^2}{(t_n-\bar{t}_n)}-s\right\}\mathrm{d}s\\&\leqslant\frac{c\rho_n}{\sqrt{4\pi}(t_n-\bar{t}_n)^{1/2}}\int_t^{(t_n-\bar{t}_n)^{1/2}}s\exp\{-s\}\,\mathrm{d}s\\&\quad+c_\delta\int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)}\exp\{-\eta s\}\,\mathrm{d}s\\&=O((t_n-\bar{t}_n)^{-1/2}(1+t)\exp\{-t\})\\&\quad+O(\exp\{-\eta(t_n-\bar{t}_n)^{1/2}\}).\end{aligned}$$

This proves the claim.

Claim 2. For all large enough n,

$$\int_{T(\varepsilon,n)}^{t_n}f(v_n)\exp\{-s\}\,\mathrm{d}s\in\left[\frac{(1-\varepsilon)(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}\rho_n},\frac{(1+\varepsilon)(t_n-\bar{t}_n)^{-1/2}}{(1-\varepsilon)4\sqrt{\pi}\rho_n}\right].$$

Proof of claim. From lemma 2.3, we may obtain the following two relations for all large n:

$$\int_0^{t_n}f(v_n)v_n\exp\{-s\}\,\mathrm{d}s\in\left[\frac{1-\varepsilon}{8\pi},\frac{1+\varepsilon}{8\pi}\right],\tag{8.8}$$

$$f(\bar{v}_n)\bar{v}_n\exp\{-t_n\}\in\left[\frac{1-\varepsilon}{8\pi},\frac{1+\varepsilon}{8\pi}\right].\tag{8.9}$$

We have, for all large n,

$$\begin{aligned}\int_{T(\varepsilon,n)}^{t_n}f(v_n)\exp\{-s\}\,\mathrm{d}s&\leqslant\frac{1}{v_n(T(\varepsilon,n))}\int_{T(\varepsilon,n)}^{t_n}f(v_n)v_n\,exp\{-s\}\,\mathrm{d}s\\&\leqslant\frac{1+\varepsilon}{8\pi v_n(T(\varepsilon,n))}\\&=\frac{(1+\varepsilon)(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}(1-\varepsilon)\rho_n}.\end{aligned}\tag{8.10}$$

Now,

$$\int_{\bar{t}_n}^{T(\varepsilon,n)} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s = \int_{\bar{t}_n}^{(t_n-\bar{t}_n)^{1/2}} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s + \int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s.$$

Since by (2.13) $f(v_n)v_n \exp\{-t\} \in L^1((0,\infty))$ and $f(v_n)v_n \to 0$ as $n \to \infty$ pointwise in $[\bar{t}_n, (t_n - \bar{t}_n)^{1/2}]$, by dominated convergence theorem

$$\int_{\bar{t}_n}^{(t_n-\bar{t}_n)^{1/2}} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s \to 0 \quad \text{as} \quad n \to \infty.$$

By the reasoning that led to the second 'order term' in claim 1, we can find an $\eta > 0$ such that

$$\int_{(t_n-\bar{t}_n)^{1/2}}^{T(\varepsilon,n)} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s = O(\exp\{-\eta(t_n - \bar{t}_n)^{1/2}\}).$$

Therefore,

$$\int_{\bar{t}_n}^{T(\varepsilon,n)} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s \to 0 \ \text{ as } n \to \infty.$$

Now, since

$$\int_{\bar{t}_n}^{t_n} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s \to \frac{1}{8\pi} \ \text{ as } n \to \infty,$$

we obtain for all large n,

$$\int_{T(\varepsilon,n)}^{t_n} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s \geqslant \frac{1-\varepsilon}{8\pi}.$$

Therefore,

$$\int_{T(\varepsilon,n)}^{t_n} f(v_n) \exp\{-s\}\,\mathrm{d}s \geqslant \frac{\sqrt{4\pi}(t_n - \bar{t}_n)^{-1/2}}{\rho_n} \int_{T(\varepsilon,n)}^{t_n} f(v_n)v_n \exp\{-s\}\,\mathrm{d}s \geqslant \frac{(1-\varepsilon)(t_n - \bar{t}_n)^{-1/2}}{4\sqrt{\pi}\rho_n}. \tag{8.11}$$

Combining (8.10) and (8.11) we prove the claim. ■

Claim 3.

$$\lim_{n\to\infty} A_n = 0.$$

Proof of Claim. Since $f(0) = 0$, it follows that

$$A_n \leqslant \bar{t}_n \left| \int_{\bar{t}_n}^{\infty} f(v_n) \exp\{-s\}\,\mathrm{d}s \right|^2.$$

Claims 1 and 2 imply that

$$\int_{\bar{t}_n}^{t_n} f(v_n)\exp\{-s\}\,ds = O((t_n-\bar{t}_n)^{-1/2}).$$

Since by lemma 2.3,

$$\int_{t_n}^{\infty} f(v_n)\exp\{-s\}\,ds = f(\bar{v}_n)\exp\{-t_n\} = O((t_n-\bar{t}_n)^{-1/2}),$$

it follows that

$$A_n \leqslant O\left(\frac{\bar{t}_n}{t_n-\bar{t}_n}\right).$$

Since $(\bar{t}_n/(t_n-\bar{t}_n)) \to 0$ by assumption, the claim follows. ■

Claim 4.

$$\lim_{n\to\infty} B_n = 0.$$

Proof of claim. We split

$$B_n = B_n^1 + B_n^2$$

with

$$B_n^1 = \int_{\bar{t}_n}^{T(\varepsilon,n)} \left| \int_t^{\infty} f(v_n)\exp\{-s\}\,ds - \frac{\rho}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2} \right|^2 dt.$$

Now,

$$\int_t^{\infty} f(v_n)\exp\{-s\}\,ds - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}$$

$$= \int_t^{t_n} f(v_n)\exp\{-s\}\,ds + f(\bar{v}_n)\exp\{-t_n\} - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}. \tag{8.12}$$

From (8.9) we obtain,

$$f(\bar{v}_n)\exp\{-t_n\} - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}$$

$$\in \left[\frac{(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}\rho_n}(1-\varepsilon-2\rho_n^2), \frac{(t_n-\bar{t}_n)^{-1/2}}{4\sqrt{\pi}\rho_n}(1+\varepsilon-2\rho_n^2)\right]. \tag{8.13}$$

Therefore, using claims 1 and 2 we get from (8.12)

$$\int_t^{\infty} f(v_n)\exp\{-s\}\,ds - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}$$

$$\in \left[\frac{(t_n-\bar{t}_n)^{-1/2}}{\sqrt{4\pi}\rho_n}(1-\varepsilon-\rho_n^2), \frac{(t_n-\bar{t}_n)^{-1/2}}{\sqrt{4\pi}\rho_n}\left(\frac{1+\varepsilon}{1-\varepsilon}-\rho_n^2\right)\right]$$

$$+ O((t_n-\bar{t}_n)^{-1/2}(1+t)\exp\{-t\})$$

$$+ O(\exp\{-\eta(t_n-\bar{t}_n)^{-1/2}\}).$$

Therefore,

$$B_n^1 \leqslant 2\int_{\bar{t}_n}^{T(\varepsilon,n)} \frac{(t_n-\bar{t}_n)^{-1}}{4\sqrt{\pi}\rho_n^2}\max\left\{(1-\varepsilon-\rho_n^2)^2, \left(\frac{1+\varepsilon}{1-\varepsilon}-\rho_n^2\right)^2\right\}dt$$
$$+ O\left(\int_{\bar{t}_n}^{T(\varepsilon,n)} (t_n-\bar{t}_n)^{-1}(1+t)^2\exp\{-2t\}\,dt\right)$$
$$+ O\left(\int_{\bar{t}_n}^{T(\varepsilon,n)} \exp\{-2\eta(t_n-\bar{t}_n)^{1/2}\}\,dt\right)$$
$$= \frac{(1-\varepsilon)}{2\pi\rho_n^2}\max\left\{(1-\varepsilon-\rho_n^2)^2, \left(\frac{1+\varepsilon}{1-\varepsilon}-\rho_n^2\right)^2\right\} + O((t_n-\bar{t}_n)^{-1}).$$

Hence,

$$\limsup B_n^1 \leqslant \frac{(1-\varepsilon)}{2\pi}\max\left\{\varepsilon^2, \left(\frac{2\varepsilon}{1-\varepsilon}\right)^2\right\}$$
$$= \frac{2\varepsilon^2}{\pi(1-\varepsilon)}. \tag{8.14}$$

From claim 2 and the fact $f(\bar{v}_n)\exp\{-t_n\} = O((t_n-\bar{t}_n)^{-1/2})$, it follows that for $t\in[T(\varepsilon,n), t_n]$,

$$\left|\int_t^\infty f(v_n)\exp\{-s\}\,ds - \frac{\rho_n}{\sqrt{4\pi}}(t_n-\bar{t}_n)^{-1/2}\right|^2 = O((t_n-\bar{t}_n)^{-1}).$$

Therefore,

$$B_n^2 = O\left(\int_{T(\varepsilon,n)}^{t_n} (t_n-\bar{t}_n)^{-1}\,dt\right)$$
$$= O(\varepsilon). \tag{8.15}$$

Since $\varepsilon > 0$ was arbitrary, combining (8.14) and (8.15) we obtain the claim. ∎

Claim 5.

$$\lim_{n\to\infty} C_n = 0.$$

Proof of claim. We have

$$C_n = f(\bar{v}_n)^2\int_{t_n}^\infty \exp\{-2t\}\,dt$$
$$= \frac{1}{2}(f(\bar{v}_n)\exp\{-t_n\})^2$$
$$= O((t_n-\bar{t}_n)^{-1})$$

which proves the claim. ∎

Claims 3, 4 and 5 imply that $\lim_{n\to\infty} I_n = 0$, which proves the lemma. ∎

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