Quickest descent line during alpine ski racing

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Abstract

Time differences between medalists at Olympic or World Cup alpine ski races are often less than 0.01 s. One factor that could affect these small differences is the line taken between the numerous gates passed through while speeding down the ski slope. The determination of the 'quickest line' is therefore critical to winning races. In this study the quickest lines are calculated by direct optimal control theory which converts an optimal control problem into a parameter optimization problem that is solved using a nonlinear programming method. Specifically, the problem is described in terms of an objective function in which state and control variables are implicitly involved. The objective function is the time between the starting point and finishing gate, while state variables are positions of the ski–skier systems on a ski slope, rotational angles of skis, velocities, and rotational velocity at a discrete time, i.e., a node. The control variable at each node is the skier-controlled edging angle between the ski sole and snow surface. Equations of motion of the ski–skier system on a ski slope are numerically satisfied at the midpoint between neighbouring nodes, and the original problem is converted into a nonlinear programming problem with equality and inequality constraints. The problem is solved by the sequential quadratic programming method in which numerical calculations are carried out using the MATLAB Optimization Toolbox. Numerical calculations are presented to determine the quickest lines of an uphill and a downhill ski turn with a starting point, first gate, and second gate (finish line) having been successfully carried out. The quickest line through four gates could not be calculated due to numerical difficulty. Instead, the descent line was respectively calculated for an uphill and downhill turn and simply added, giving a resultant time that represents an upper bound.

Keywords: alpine skiing, mechanics of skiing, optimal control, optimization, quickest descent line

Introduction

The mechanics of turning alpine skis has been an interesting topic of sports engineering over the past decade. In Japan, for example, much investigation was carried out during the 1950s, with many results published in a book edited by the Society of Ski Science (1971). Renshaw and Mote (1989) subsequently used experimental two-dimensional ice cutting data by Lieu and Mote (1984) to analyze a turning ski in which hypothetical thrust was considered; while Hirano and Tada (1996) numerically simulated alpine ski tracks of a ski–skier system in which the centripetal force necessary for turning motion was considered to be the component transverse to the velocity vector of the impacting force for soft snow, or that of the oblique cutting force for compact snow. The transverse component is obtained

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by placing the ski's longitudinal axis inclined away from the velocity vector and simultaneously edging the ski into the snow. Tada and Hirano (2002) performed oblique snow cutting experiments in a low-temperature room, deriving the cutting force equations by applying multiple regression analysis. The cutting forces were considered functions of the cutting depth which was constant in the experiments. In reality, however, cutting depth is not constant during turning on a ski slope and should be determined considering the equilibrium of forces normal to the snow surface, including the gravitational force, snow reaction force, and the vertical component of the cutting force which is a function of the edging angle between the ski sole and snow surface. The cutting forces were revised considering equilibrium under specific conditions and represented in graphical form (Tada and Hirano, 2002).

Optimal control theory is one method for determining the quickest descent line, i.e., the line in which the least amount of time is expended between the starting and finishing points. Zhang *et al.* (1995) investigated bobsled optimal control by developing minimum-time and smooth-steering algorithms based on the so-called indirect method. This method utilizes the calculus of variations in which the objective function is total racing time for a given sled/track and the control variable is steering angle. Seo *et al.* (2004) utilized the direct method to maximize the flight distance in V-style ski jumping flight, converting their optimal control problem into a parameter optimization problem that is solved using a nonlinear programming method (sequential unconstrained minimization techniques (SUMT)). Flight distance, forward lean, and ski-opening angle were respectively the objective function and control variables.

This study applies optimal control theory in conjunction with empirical snow cutting force equations to determine the quickest descent lines between ski slope gates. The direct method is used to convert the problem into a parameter optimization problem which is then solved by a nonlinear programming method, i.e., sequential quadratic programming (SQP). The objective function and control variable are the time between the starting point and finishing gate and the angle between the ski sole and snow surface, or edging angle, respectively. MATLAB is used for calculations. Results provided here are the quickest descent lines between the starting point, first gate, and second (finishing) gate for an uphill and downhill turn.

Compact snow cutting force and equations of motion

A ski moving down an alpine ski slope (Fig. 1) has snow cutting forces and a component of gravitational force that is parallel to the slope. It is these forces which determine the dynamic motion of the ski–skier system. Oblique snow cutting forces were previously measured by Tada and Hirano (2002) in a low-temperature room. They have a normal, inverse horizontal, and transverse component with respect to ski velocity V, denoted as F_N, F_H, and F_T, respectively (Fig. 2). F_T corresponds to the centripetal force of the system which makes the turning motion possible. Angles γ and α are called the attack and edging angle, respectively. Positive sign conventions for the notation in Fig. 2 are as follows: F_T, $\pi/2$ counter clockwise from velocity vector; F_H, opposite to the velocity vector; R_T, in the y' direction; R_L, opposite to the x' direction; γ, clockwise from the x' axis; and α, clockwise viewed from the tip of the ski. The snow cutting depth is denoted as d_c. Since the measured forces are proportional to cutting width and depth, the forces are divided by cutting width and depth and expressed as pressures P_H, P_N and P_T, i.e.,

$$P_H = e^{5.212}(\sin|\gamma|)^{-0.2371}(\sin|\alpha|)^{0.2366}(\tan|\alpha|)^{-0.0153} \quad (\mathrm{kN\ m^{-2}}) \quad (1)$$

$$P_N = e^{20.84}(\sin|\gamma|)^{-0.8833}|\alpha|^{-3.993}(\sin|\alpha|)^{3.180} \quad (\mathrm{kN\ m^{-2}}) \quad (2)$$

$$P_T = \left(\frac{\gamma}{|\gamma|}\right)e^{5.118}(\tan|\gamma|)^{-0.6279}(\tan|\alpha|)^{0.1610} \quad (\mathrm{kN\ m^{-2}}) \quad (3)$$

where the values of γ and α are in degrees. Equations (1)–(3) were obtained by applying multiple regression analysis. Although cutting forces were previously determined at constant cutting depth (Tada and Hirano, 2002), since this depth is a function of edging

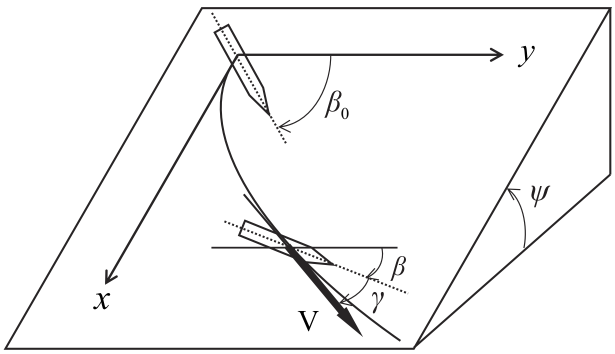

Figure 1 Employed coordinate axes representing a ski slope.

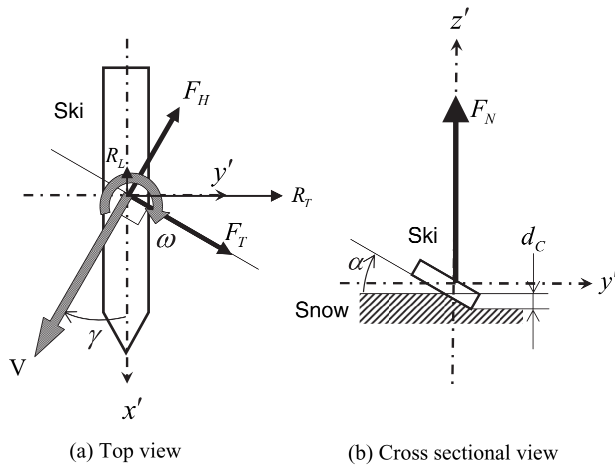

Figure 2 (a) Ski attack angle γ and system velocity V and (b) cross section of a ski with edging angle α on a snow surface.

angle, the forces were revised versus edging angle considering the equilibrium of forces normal to the snow surface under certain conditions; mass of the system, 80 kg; ski slope angle ψ, 15° ; ski length ℓ = 1.8 m; and spring constant of snow per unit area k_G, 3.186×10^7 N m^{-3}. Here we determine transverse and inverse horizontal forces F_T* and F_H* in the ski moving direction per unit ski length using

$$F_T^* = 278\left(\frac{\gamma}{|\gamma|}\right)\sin(2\gamma)\tan\alpha \qquad (\mathrm{N\ m^{-1}}) \tag{4}$$

$$F_H^* = 500\sin\gamma\tan\alpha \qquad (\mathrm{N\ m^{-1}}) \tag{5}$$

Equation (4) indicates that F_T* has a maximum value when attack angle γ = 45°, and that F_T* and F_H* are zero when the edging angle α = 0° and infinite when α = 90°. While these expressions are indeed simplistic, we use them for the sake of convenience because exact expressions are beyond the scope of this study.

The attack angle along the ski length ϕ_r is not constant due to the rotation of a ski around the center of the mass of the ski–skier system (Fig. 2), and is expressed as

$$\phi_r = \tan^{-1}\left\{\frac{(V\sin\gamma + \omega r)}{(V\cos\gamma)}\right\} \tag{6}$$

where r is the coordinate along the x' ski axis, and ω is the angular velocity of the system. If the angle between the ski axis and a horizontal line on a ski slope is denoted by β (Fig. 1), then $\omega = d\beta/dt$.

Since rotation should be taken into account in equations (4) and (5), γ should be replaced by ϕ_r. However, the resultant resistance forces acting on a ski are for the sake of convenience simply denoted by the product of the ski length and F_T* and F_H*, respectively. The rotational moment around the centre of the gravity of the system for the front and rear part of the ski can now be calculated, denoted as M_F and M_R. This takes into account the non-uniformity of the attack angle along the ski length, i.e.,

$$M_F = -\int_0^{r_F} r(F_T^*\cos\phi_r + F_H^*\sin\phi_r)dr \tag{7}$$

$$M_R = \int_0^{r_R} r(F_T^*\cos\phi_r + F_H^*\sin\phi_r)dr \tag{8}$$

where r_F and r_R are the front and rear length of the ski. Integration is performed numerically using Simpson's rule.

The equations of motion of the ski–skier system on a ski slope are expressed by taking the steepest and horizontal directions of a ski slope as x and y, respectively (Fig. 1), i.e.,

$$m\left(\frac{d^2x}{dt^2}\right) = -R_T\cos\beta - R_L\sin\beta + mg\sin\psi \tag{9a}$$

$$m\left(\frac{d^2y}{dt^2}\right) = R_T\sin\beta - R_L\cos\beta \tag{9b}$$

$$I\left(\frac{d^2\beta}{dt^2}\right) = M_F + M_R \tag{9c}$$

where m, ψ, and I are the system mass, slope angle, and moment of inertia, respectively. R_T and R_L are components of the resultant force in the direction transverse and parallel to the longitudinal ski axis, and are calculated using equations (4) and (5).

Optimal control problem

The proposed problem is solved by applying optimal control theory. Optimal control methods are roughly separated into two groups; direct and indirect methods. The indirect method is based on variational techniques which are difficult to use when considering various constraints, while the direct method converts the original optimal control problems into a mathematical programming problem. Here, the direct method is used to determine the quickest descent line between gates on a ski slope. The converted problems are then solved by sequential quadratic programming using the MATLAB Optimization Toolbox software (Coleman *et al.*, 1999).

Optimal control problems are characterized by state variables $X(t)$ and control variables $U(t)$ that are vectors in general and continuous functions of time t. System dynamics are expressed as a state equation, i.e.,

$$\frac{dX(\mathrm{t})}{dt} = F[X(t), U(t)] \tag{10}$$

Initial and final times are denoted by $t_0(=0)$ and t_f. The interval is discretized by dividing by N. That is, nodes $t_i(i = 0, \dots ,N)$ are obtained, where $t_0 = 0$ and $t_N = t_\mathrm{f}$. At each node the state and control variables are discretized as follows:

$$X_i = X(t_i) \tag{11}$$

$$U_i = U(t_i) \tag{12}$$

Following Tsuchiya and Suzuki (1998), state variables between two consecutive nodes are approximated by two linear functions with respect to time by taking into account equation (10) as shown in Fig. 3. Since the state variables must be continuous at the midpoint of two consecutive nodes, the following equations must be satisfied at the mid point.

$$\Delta_i = \left\{X_i + F(X_i, U_i)\left(\frac{k_i t_f}{2}\right)\right\} - \left\{X_{i+1} - F(X_{i+1}, U_{i+1})\left(\frac{k_i t_f}{2}\right)\right\} = 0 \tag{13}$$

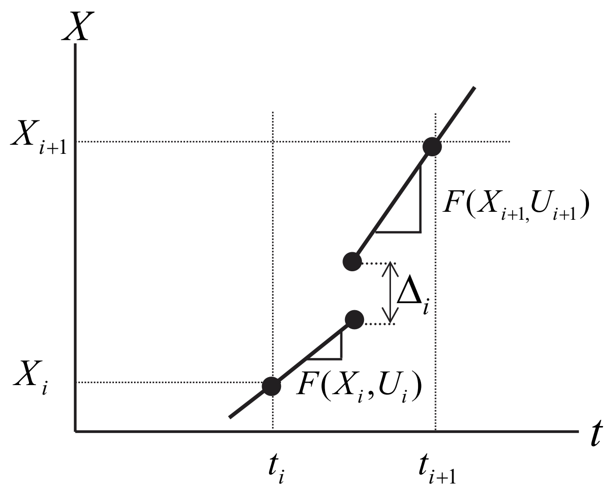

Figure 3 Discrete state variables.

where

$$k_i = \frac{(t_{i+1} - t_i)}{t_f} \tag{14}$$

In the present problem, the final time t_f is the objective function to be minimized under the equality constraints given by equation (13). The original problem is now converted into

Minimize t_f

$$\text{subject to } \Delta_i = 0 (i = 0, 1, \dots , N-1) \tag{15}$$

Quickest line between gates

The second-order differential equations in equation (9) can be rewritten using six first-order differential equations, thereby allowing use of the present optimal control method, i.e.,

$$\frac{dx}{dt} = u \tag{16a}$$

$$\frac{dy}{dt} = v \tag{16b}$$

$$\frac{d\beta}{dt} = w \tag{16c}$$

$$\frac{du}{dt} = \frac{(-R_T \cos\beta - R_L \sin\beta + mg \sin\psi)}{m} = f_4 \tag{16d}$$

$$\frac{dv}{dt} = \frac{(R_T \sin\beta - R_L \cos\beta)}{m} = f_5 \qquad (16e)$$

$$\frac{d\omega}{dt} = \frac{(M_F + M_R)}{I} = f_6 \qquad (16f)$$

State variables are x, y, β, u, v, and ω while the control variable is α. As mentioned, the starting point and first and second gates are considered. The objective function is the terminal time t_f. Time is expressed as

$$t_f = t_{01} + t_{12} \qquad (17)$$

where t_{01} and t_{12} are times between the starting point and the first gate, and the first and second gate, respectively. Both t_{01} and t_{12} are divided by $N = 10$ and the state and control variables are discretized. The state variables are made to be continuous at the midpoint of two adjacent nodes. The following are 60 continuity conditions between the starting point and the first gate.

$$\{x_0 + u_0(0.05t_{01})\} - \{x_1 - u_1(0.05t_{01})\} = 0$$

$$\{x_1 + u_1(0.05t_{01})\} - \{x_2 - u_2(0.05t_{01})\} = 0$$

$$\vdots$$

$$\{\beta_9 + \omega_9(0.05t_{01})\} - \{\beta_{10} - \omega_{10}(0.05t_{01})\} = 0$$

$$\{u_0 + (f_4)_0(0.05t_{01})\} - \{u_1 - (f_4)_1(0.05t_{01})\} = 0$$

$$\{u_1 + (f_4)_1(0.05t_{01})\} - \{u_2 - (f_4)_2(0.05t_{01})\} = 0$$

$$\vdots$$

$$\{\omega_9 + (f_6)_9(0.05t_{01})\} - \{\omega_{10} - (f_6)_{10}(0.05t_{01})\} = 0 \qquad (18)$$

The subscripts of the state variables and the functions f_4, f_5 and f_6 denote the node number. Between the first and second gates there are 60 more equality constraints.

The conditions that a ski racer must pass between gates on a ski slope can be expressed as follows for the case of an uphill turn (turning away from the fall-line and decreasing the angle of descent) when the first gate is located at $x = 20$ m, $y = 20$ m and the second at $x = 40$ m, $y = 60$ m:

$$y_{10} = -x_{10} + 40 \qquad (16 \le x_{10} \le 24) \qquad (19a)$$

$$y_{20} = -x_{20} + 100 \qquad (36 \le x_{20} \le 44) \qquad (19b)$$

Equations (19) indicate that the state variables (x_{10}, y_{10}) and (x_{20}, y_{20}) are on the lines that connect the two poles of each gate. Both gates are open in the direction of 225° clockwise from the y-axis and are about 11.3 m wide. To perform a regular turning motion, other constraints are also imposed on the attack angle γ and edging angle α. These constraints are $0 < \alpha < \pi/2$ and $\gamma > 0$ for the uphill turn.

For the case of a downhill turn (turning toward the fall-line and increasing the angle of descent), the starting point is taken at the calculated terminal point of the uphill turn. The locations of the first and second gate are $x = 80$ m, $y = 60$ m, and $x = 100$ m, $y = 30$ m, respectively. The conditions that a ski racer must pass through the gates are as follows.

$$y_{10} = x_{10} - 20 \qquad (76 \le x_{10} \le 84) \qquad (20a)$$

$$y_{20} = x_{20} - 70 \qquad (96 \le x_{20} \le 104) \qquad (20b)$$

The conditions for regular turning motion are $-\pi/2 < \alpha < 0$ and $\gamma < 0$.

Numerical results

Uphill turn

Calculations for the quickest line were made under the following conditions: starting point at $x = 0$ m and $y = 0$ m, initial velocity $u_0 = 6$ m s^{-1}, $v_0 = 6$ m s^{-1}; initial angle between ski's longitudinal axis x' and y axis $\beta_0 = 30°$, initial angular velocity $\omega_0 = -0.01$ rad s^{-1}, system mass m = 80 kg, moment of inertia $I = 2.5$ kg m^2, ski length $\ell = 1.8$ m, boot location $\eta = (r_F/r_R) = 0.53$, and slope angle $\psi = 15°$. To perform the numerical calculation, the initial values of state variables and the control variable (edging angle) at every node must be guessed, with results being very sensitive to guessed values. Also times t_{01} and t_{12} must be initially guessed. Accordingly, insight is needed regarding the occurring phenomena, with inappropriate values yielding either late convergence or divergence.

Figure 4a shows the initial discrete values for the state variables x and y (small circles). The other four state variables and control variable are also initially guessed. These values are those that successfully converged for the case of η = 0.505. For a boot location of η = 0.505, convergence was obtained for the uphill case, but not for the downhill case. For this reason only the results for η = 0.53 are presented (convergence in both cases). The indicated short straight lines are the first and second gates. Figure 4b shows the resultant quickest descent line, where the difference between the initial and resultant variables for x and y are small because of good initially guessed values. Time t_{01} between the starting point and the first gate is calculated as 3.37 s, while time t_{12} between the first and second gates is 6.34 s (total time t_f = 9.71 s). Please note the time difference between two adjacent dots before the first gate is 0.337 s and that after the gate is 0.634 s. Also note that the quickest line obtained does not look like a brachistochrone trajectory because of the snow cutting forces.

Downhill turn

The starting point for this case is taken at the terminal point of the previous uphill turn case, i.e., x = 44 m, y = 56 m. Figure 5a shows the considered gates. The initial conditions used are the terminal state variables (u_0 = 1.42 m s^{-1}, v_0 = 3.41 m s^{-1}) of the uphill turn except the angle β and ω. The initial values of β and ω were taken as 48° and 0.2 rad s^{-1}. The conditions for the ski itself and the ski–slope are the same as for the uphill turn. The initial state values guessed at every node for the iterative calculation were obtained by trial and error methods, and the initially assumed values for x and y are indicated (small circles).

For this case convergence occurred, with the resultant quickest line being shown in Fig. 5b. Time t_{01} between starting point and first gate is calculated as 5.02 s, while time t_{12} between the first and second gate is 4.12 s (total time t_f = 9.15 s).

Combination of uphill and downhill turn

Four gates are considered having the same locations as the uphill and downhill cases as shown in Fig. 6. Although various initial values were guessed and tried for this gate placement, numerical convergence was not obtained. The upper bound for this problem was

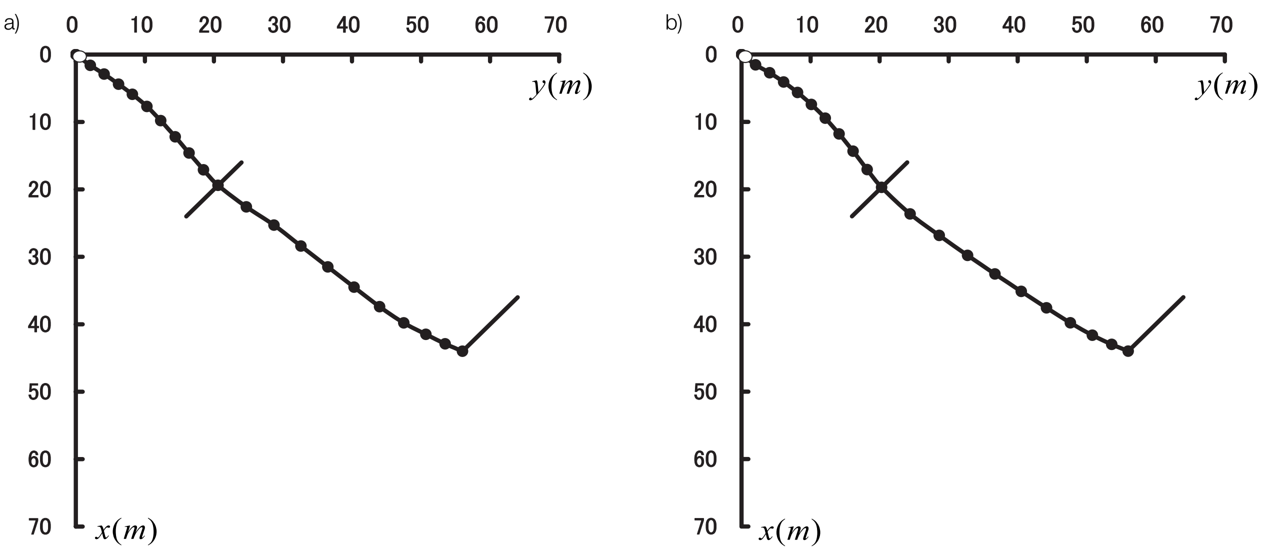

Figure 4 (a) Initial values for state variables x and y on a ski slope for uphill turn (b) Resultant quickest descent line for uphill turn.

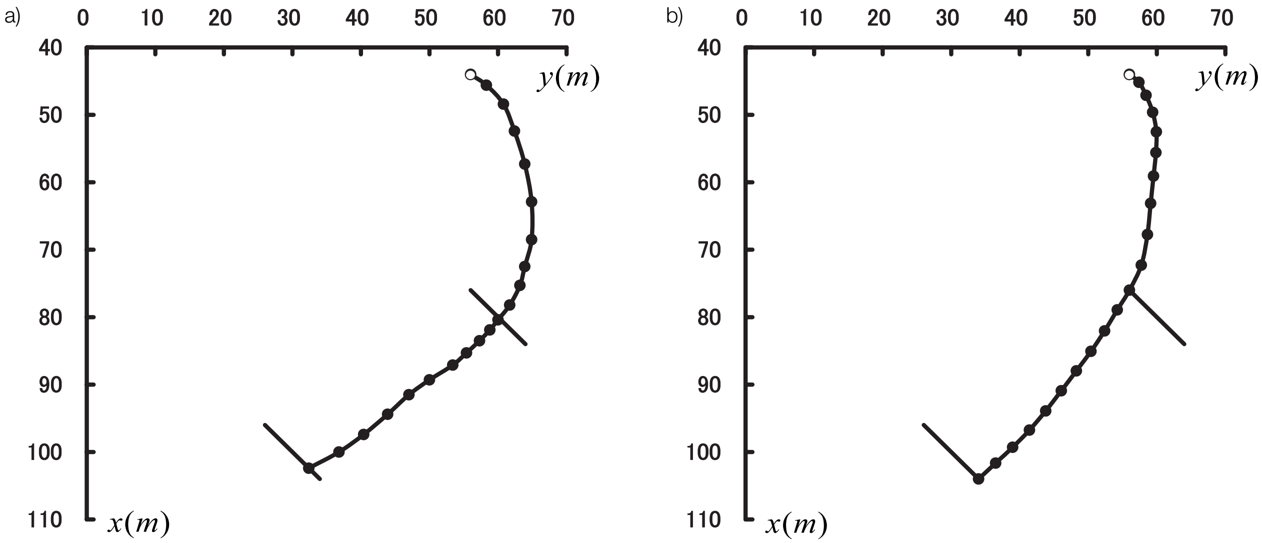

Figure 5 (a) Initial values for state variables *x* and *y* on a ski slope for downhill turn. (b) Resultant quickest descent line for downhill turn.

then taken by the addition of the uphill and downhill results. That is, at the connecting point between the uphill and downhill descent lines, discontinuity of β (the angle between the ski axis x' and y axis) and ω (angular velocity) occurs. To successfully carry out numerical calculations the author had to abruptly change the attack angle (γ) at the second gate. Ski racers can manage such discontinuity by stepping outer-ski outwards or jumping sideways in order to change the descent direction. Figure 6 shows the line obtained by simple addition of the quickest lines for the uphill and downhill turn, where the upper bound of the descent time between the starting point and fourth (finishing) gate is 18.86 s (9.71 s + 9.15 s). The line shown in the figure represents the one for the upper bound for the problem of passing through four gates.

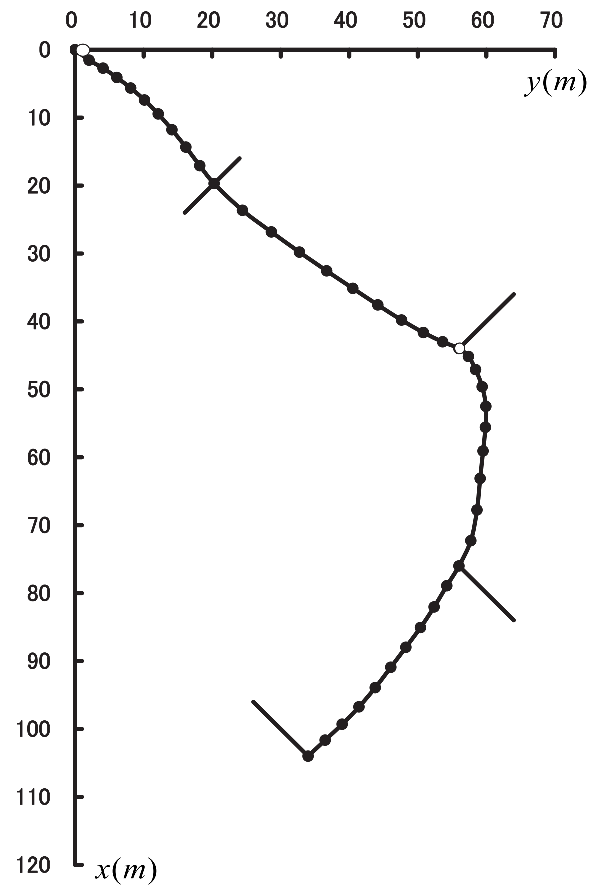

Figure 6 Addition of uphill and sequential downhill result.

Conclusions

This study provides a method to determine the quickest descent line between gates on a ski slope. A direct optimal control method is used to convert the original optimal control problem into a parameter optimization problem which is solved by a mathematical programming method. The objective function of the present problem is the descent time between the starting point and the finishing gate. The control variable is the edging angle which can be adjusted by ski racers. Six position and velocity state variables of the ski–skier system are considered. The control variable and state variables are discretized at several nodes between the initial time and finishing time. The state variables must satisfy the equations of motion of the system on a ski slope. This is numerically accomplished by considering continuity at the midpoint between adjacent nodes, a condition that yields many equality constraints. Since inequality constraints are also taken into account to make regular turning motion possible, the problem is to minimize the descent time under equality and inequality constraints. The results of numerical calculations are presented for both an uphill and downhill turn with the starting point, first gate, and second (finish) gate. If the quickest lines between gates have been calculated, ski racers can follow them by intuitively adjusting the edging angle. The values of the control and state variables at the nodes must be initially guessed, and the numerical calculations are very sensitive to these initially guessed values, such that many trials are necessary for convergence. Such results were obtained for an uphill and a downhill turn, although they are not guaranteed to be a global minimum. When an uphill and a downhill case were combined, however, successful results were not obtained. This indicates that determining the quickest descent line during actual alpine ski racing with many gates most likely requires other methods or software. Future work will be directed at taking into account bending deformation, side cut of the ski, and carving turns. Assuming that new descent lines will be found, alpine ski racers can attempt to follow the quickest line by instinctively adjusting the edging angle.

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