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Stochastic seismic response of structures with added viscoelastic dampers modeled by fractional derivative

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Abstract: Viscoelastic dampers, as supplementary energy dissipation devices, have been used in building structures under seismic excitation or wind loads. Different analytical models have been proposed to describe their dynamic force deformation characteristics. Among these analytical models, the fractional derivative models have attracted more attention as they can capture the frequency dependence of the material stiffness and damping properties observed from tests very well. In this paper, a Fourier-transform-based technique is presented to obtain the fractional unit impulse function and the response of structures with added viscoelastic dampers whose force-deformation relationship is described by a fractional derivative model. Then, a Duhamel integral-type expression is suggested for the response analysis of a fractional damped dynamic system subjected to deterministic or random excitation. Through numerical verification, it is shown that viscoelastic dampers are effective in reducing structural responses over a wide frequency range, and the proposed schemes can be used to accurately predict the stochastic seismic response of structures with added viscoelastic dampers described by a Kelvin model with fractional derivative.

Keywords: fractional derivative; viscoelastic damper; stochastic seismic response

1 Introduction

For years, viscoelastic dampers have been widely used not only for improving residential comfort under strong wind conditions, but also for enhancing structural safety against large earthquake ground motions. There are many examples, such as World Trade Center in New York City and Columbia Center in Seattle, USA, where viscoelastic dampers were applied successfully to improve the structural performance under dynamic loads. For the design of structures with added viscoelastic dampers, it is important to develop or select an accurate force-deformation model and a convenient analytical procedure to calculate the dynamic response for different installation options.

A typical viscoelastic damper consists of thin layers of viscoelastic material bonded between steel plates. It is well-known from previous tests that the force-deformation relationship of a viscoelastic damper dependents upon the frequency of the cyclic loading applied. It has also been shown that fractional derivative models describe the frequency dependence of the viscoelastic damper very well (Bagley and Torvik, 1983a; 1983b; 1985). One of the most frequently used fractional models is the three-parameter Kelvin model with fractional derivative (Bagley and Torvik, 1983) as follows:

$$f_{ve} = k_{ve}u_{ve} + c_{ve}D^{\alpha}\langle u_{ve}\rangle \qquad 0<\alpha<1 \quad (1)$$

In this model, the parameters u_{ve} and c_{ve} are the stiffness and damping coefficients of the damper, respectively, and $D^{\alpha}\langle\cdot\rangle = \mathrm{d}^{\alpha}/\mathrm{d}t^{\alpha}$ denotes the fractional derivative operator. To represent the fractional derivative, two fundamentals have been used. According to the Riemann-Liouville's definition (Miller and Ross, 1993), the fractional derivative can be expressed as:

$$D^{\alpha}\langle f(t)\rangle = \frac{1}{\Gamma(1-\alpha)}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{f(\tau)}{(t-\tau)}\mathrm{d}\tau \quad (2)$$

where $\Gamma(u) = \int_0^{\infty} t^{u-1}\mathrm{e}^{-u}\,\mathrm{d}t$ represents the Gamma function.

Whereas in Caputo's definition (Oldham and Spanier, 1974), the fractional derivative is written in the following form:

$$D^{\alpha}\langle f(t)\rangle = \frac{1}{\Gamma(1-\alpha)}\cdot\int_0^t\frac{f(\tau)}{(t-\tau)}\mathrm{d}\tau \quad (3)$$

Apparently, the equations of motion of the

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structures equipped with supplemental viscoelastic dampers consist of a fractional derivative together with a regular derivative. The appearance of the fractional derivative leads to difficulty in the solution of these equations. It is harder to conduct the dynamic response analysis of these structures, let alone perform a stochastic analysis. However, some numerical schemes have been formulated to solve such equations. For instance, an L1 method based on the Riemann-Liouville definition was proposed by Koh and Kelly (1990) and Suarez and Shokooh (1997). On the other hand, Bagley and Torvik (1985) and Bagley and Calico (1991) adopted the modal decomposition approach to develop a general solution using the Mittag-Leffer functions. For seismic response analysis of multiple-degrees of freedom structures with added viscoelastic dampers modeled by a fractional derivative, Chang and Singh (2002) proposed the modal superposition formulation. On the whole, the above schemes are carried out in the time domain, not the frequency domain. As is well-known, considering the frequency dependence in the time domain is difficult.

As for stochastic analysis of the viscoelastic structures with fractional derivatives, some scholars used fractional calculus to model their statistical behavior. Mainardi (1997) presented a fractional calculus approach to model the Brownian motion, and his formulation led to a fractional Langevin, which is solved by using the Laplace transform technique. At the same time, Spanos and Zeldin (1997) presented a frequency-domain approach for random vibration analysis of fractionally damped systems. Recently, Agrawal (1999) presented an analytical scheme for stochastic dynamic systems whose damping behavior is described by a fractional derivative on the order of 1/2. In this approach, the eigenvector expansion method and the properties of Laplace transforms of convolution integrals were used to obtain the desired results. However, the stochastic seismic analysis of structures with added viscoelastic dampers modeled by a fractional derivative has not been reported. Therefore, there is a need to investigate this process, especially to explore the effect of the parametric behavior of the viscoelastic dampers on the seismic response of the structure.

In this paper, an analytical scheme for stochastic seismic analysis of a SDOF structure with added viscoelastic dampers is proposed. A Fourier transform approach is first presented to obtain the fractional unit impulse response function and the Duhamel integral-type closed-form expression for the response of the system. The method is applicable to deterministic as well as random input. Then, these expressions are used to obtain the stochastic seismic response of these types of structures. The responses are achieved through a set of important parametric variables of the viscoelastic dampers modeled by a fractional derivative.

2 Fractional dynamic model and general solution

Under earthquake excitation, the differential equation of a SDOF structure with added viscoelastic dampers whose force-deformation relationship is described by a fractional derivative can be written as:

$$m\ddot{x}(t) + c\dot{x}(t) + c_{ve}D^{\alpha}\langle x(t)\rangle + (k + k_{ve})x(t) = -ma(t) \tag{4}$$

where m, c and k represent the mass, damping and stiffness coefficient, respectively, k_{ve} and c_{ve} are, respectively, the stiffness and damping coefficients of the damper, and $a(t)$ is the ground acceleration. Note that the dimension of c_{ve} is not the same as that of the general damping coefficient, c.

Eq. (4) can also be written as:

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega^2x(t) + 2\xi_{ve}\omega_{ve}D^{\alpha}\langle x(t)\rangle + \omega_{ve}^2x(t) = -a(t) \tag{5}$$

where $\omega_n = \sqrt{k/m}$, $\xi = c/2m\omega_n$

$$\omega_{ve} = \sqrt{k_{ve}/m},\ \eta = c_{ve}/2m\omega_{ve}$$

For the Riemann-Liouville derivative (i. e., when $D^{\alpha} = D_R^{\alpha}$), the application of the Fourier transform to Eq. (5) leads to:

$$H(\omega)X(\omega) = -A(\omega) + (\mathrm{i}\omega)\cdot P_1 + P_2 \tag{6}$$

where $X(\omega)$ and $A(\omega)$ are the Fourier transform of $x(t)$ and $a(t)$, respectively, and $H(\omega)$ is an indicial polynomial defined as:

$$H(\omega) = \omega_n^2 + \omega_{ve}^2 - \omega^2 + \mathrm{i}2\xi\omega_n\omega + 2\xi_{ve}\omega_{ve}(\mathrm{i}\omega)^{\alpha} \tag{7}$$

the constants P_1 and P_2 are given as:

$$P_1 = x(0)$$

$$P_2 = \dot{x}(0) + 2\xi_{ve}\omega_{ve}D_R^{\alpha-1}$$

and the fractional derivative of $x(t)$ at $t=0$ can be obtained using an extension of the initial value theorem for the Fourier transform. Taking the inverse Fourier transform of Eq. (6), one obtains:

$$x(t) = x_R(t; x(0); \dot{x}(0)) - \int_0^t h(t-\xi)a(t)\mathrm{d}\xi \tag{8}$$

where $x_R(t;\ x(0);\ \dot{x}(0))$ is the response of the system corresponding to the initial state only, and

$$h(t) = F^{-1}\left[\frac{1}{H(\omega)}\right]$$

is the fractional unit impulse response function. The subscript R in the variables above is referred to as the Riemann-Liouville derivative. Differentiating Eq. (8) with respect to time and using the properties of the fractional unit impulse response function, the following is obtained:

$$\dot{x}(t) = \dot{x}_R(t;\, x(0);\, \dot{x}(0)) - \int_0^t \dot{h}(t-\xi)a(t)\mathrm{d}\xi \tag{9}$$

For Caputo derivative (i. e., when $D^\alpha = D_C^\alpha$), the application of the Fourier transform to Eq. (5) leads to:

$$H(\omega)X(\omega) = -A(\omega) + C_1(\omega)x(0) + C_2(\omega)\dot{x}(0) \tag{10}$$

where

$$C_1(\omega) = \mathrm{i}\omega + a(\mathrm{i}\omega)^{(\alpha-1)}$$

$$C_2(\omega) = 1$$

taking the inverse Fourier transform of Eq. (10), gives

$$x(t) = x_c(t;\, x(0);\dot{x}(0)) - \int_0^t h(t-\xi)a(t)\mathrm{d}\xi \tag{11}$$

where x_C represents the response of the system corresponding to the initial state only when the Caputo fractional derivative is considered. Differentiating Eq. (11) with respect to time and using the properties of the fractional unit impulse response function, the following is obtained:

$$\dot{x}(t) = \dot{x}_C(t;\, x(0);\, \dot{x}(0)) - \int_0^t \dot{h}(t-\xi)a(t)\mathrm{d}\xi \tag{12}$$

Eqs. (8) [or Eq. (9)] and (11) [or Eq. (12)] represent general closed-form solutions for the displacement and the velocity for Eq. (4) or (5) corresponding to the Riemann-Liouville (the Caputo) fractional derivative. Note that these equations contain two parts, the force and initial condition, each on the right-hand side.. The force part represents the zero-state response, and the initial condition part represents the zero-input response. These equations are similar to the Duhamel integral solution for a linear system. Therefore, they can be considered as the Duhamel integral formula for the dynamic system described by Eq. (4). Also note that the fractional unit impulse response functions for the two fractional operators are the same.

3 Stochastic response analysis

Eqs. (8), (9), (11), and (12) are applicable for an arbitrary forcing function, and therefore, they are also applicable for a random input. For stochastic analysis, $a(t)$ is considered as a Gaussian random process with a zero mean function and a specific correlation function $R(t,u)$, i. e.,

$$E[a] = 0, \tag{13}$$

$$R(t,u) = E[a(t)a(u)] \tag{14}$$

where E is the expectation operator. The process $a(t)$ need not have a zero mean function, however, this assumption is made for simplicity. Applying E to Eqs. (8), (9), (11) and (12), and using Eq. (13), the mean function for displacement and velocity process is obtained as

$$\bar{x}(t) = E[x(t)] = x_R(t;x(0);\dot{x}(0)) \tag{15}$$

$$\bar{\dot{x}}(t) = E[\dot{x}(t)] = \dot{x}_R(t;x(0);\dot{x}(0)) \tag{16}$$

for the Riemann-Liouville fractional derivative, and

$$\bar{x}(t) = E[x(t)] = x_C(t;x(0);\dot{x}(0)) \tag{17}$$

$$\bar{\dot{x}}(t) = E[\dot{x}(t)] = \dot{x}_C(t;x(0);\dot{x}(0)) \tag{18}$$

for the Caputo fractional derivative. Using Eqs. (8), (9), (11), and (12) and Eqs. (15) through (18), it follows that for both derivatives

$$x(t) - E[x(t)] = -\int_0^t h(t-\xi)a(\xi)\mathrm{d}\xi \tag{19}$$

$$\dot{x}(t) - E[[\dot{x}(t)] = -\int_0^t \dot{h}(t-\xi)a(\xi)\mathrm{d}\xi \tag{20}$$

Using Eqs. (13), (14), (19) and (20) gives the variance and covariance function as follows

$$E[(x(t) - E[x(t)])^2] = \int_0^t\int_0^t h(t-\xi_1)h(t-\xi_2) \times R(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \tag{21}$$

$$E[(\dot{x}(t) - E[\dot{x}(t)])^2] = \int_0^t\int_0^t \dot{h}(t-\xi_1)\dot{h}(t-\xi_2) \times$$

$$R(\xi_1,\xi_2)\,d\xi_1 d\xi_2 \quad (22)$$

$$E[x(t) - E[x(t)]]\,E[(\dot{x}(t) - E[\dot{x}(t)]] = \int_0^t\int_0^t h(t-\xi_1)\dot{h}(t-\xi_2)R(\xi_1,\xi_2)\,d\xi_1 d\xi_2 \quad (23)$$

Note that in the above derivative, the initial conditions are assumed to be deterministic. For random initial conditions, the above equations can be modified using an approach similar to that for a stochastic damped system of order 1.

Eqs. (19) through (23) provide the stochastic response of the system for a general class of random ground motion. For the case of white noise with spectral density function, $S(\omega) = S_0$, and the corresponding correlation function $R(t,u) = 2\pi S_0\delta(t-u)$, Eqs. (21) through (23) reduce, after some algebraic manipulation, to

$$E[(x(t) - E[x(t)])^2] = 2\pi S_0\int_0^t h^2(\xi)\,d\xi \quad (24)$$

$$E[(\dot{x}(t) - E[\dot{x}(t)])^2] = 2\pi S_0\int_0^t \dot{h}^2(\xi)\,d\xi \quad (25)$$

$$E[(x(t) - E[x(t)])\,(\dot{x}(t) - E[\dot{x}(t)])] = 2\pi S_0 \times \int_0^t h(\xi)\dot{h}(\xi)\,d\xi \quad (26)$$

However, the random behavior of the ground is generally presented by the power spectral density function. The power spectral density of ground acceleration is modeled as filtered white noise, such as the Kanai-Tajimi Spectrum:

$$S(\omega) = \frac{1 + 4\xi_g^2(\omega/\omega_g)^2}{[1-(\omega/\omega_g)^2]^2 + 4\xi_g^2(\omega/\omega_g)^2}S_0 \quad (27)$$

where ω_g and ξ_g are the filter parameters; and S_0 is the spectral density of the input white noise. The corresponding correlation function can be expressed in the following form:

$$R(t,u) = \frac{\pi S_0}{2\xi_g}e^{-\xi_g\omega_g(t-u)}\Big\{(1+4\xi_g^2)\cos[\bar{\omega}_g(t-u)] + \frac{\xi_g\omega_g}{\bar{\omega}_g}(1-4\xi_g^2)\sin[\bar{\omega}_g(t-u)]\Big\} \quad (28)$$

where $\bar{\omega}_g = \omega_g\sqrt{1-\xi_g^2}$.

Closed form or numerical computation requires knowledge of $h(t)$ and $\dot{h}(t)$. In Appendix I, the MATLAB computer program for the fractional unit impulse response function $h(t)$ using the fast Fourier transform method is presented. Its derivative $\dot{h}(t)$ can be obtained in a similar way. Note that an eigenvector expansion approach for obtaining equations with $\alpha = 1/2$ was presented by Agrawal (1999). The scheme presented is general and applicable to all fractional derivatives of positive rational order.

Eqs. (21) through (23) (or Eqs. (24) through (26)) can be used to compute the stochastic response of dynamic systems. The approach is similar to the impulse function approach to find the stochastic response of damped systems of order 1. The above formulations present expressions of covariance functions for an SDOF system only. For MDOF systems, the formulation can be developed in a similar manner.

4 Numerical study

In this section, numerical results of parametric studies on the stochastic behavior of a structure with added viscoelastic dampers modeled by fractionally derivative are given and compared with those obtained for a structure without added viscoelastic dampers. In the numerical calculation, it is assumed that $\omega_n = 6.28\text{rad}\cdot\text{s}^{-1}$, $\xi = 0.02$, and $\omega_{ve} = 1.57\text{rad}\cdot\text{s}^{-1}$. Moreover, the filtered white noise is used as the random earthquake excitation with $S_0 = 8.83\times10^{-3}\times \text{m}^3/(\text{rad}\cdot\text{s}^{-3})$, $\omega_g = 12.56\text{rad}\cdot\text{s}^{-1}$ and $\xi_g = 0.5$.

Figs. 1 through 5 show, respectively, the obtained fractional unit impulse response functions $h(t)$ and $\dot{h}(t)$, the time variation of the variance function $E[x^2]$ and $E[v^2]$, and the covariance function $E[xv]$ with $\alpha = 0.6$ for $\eta = 0.1, 0.5, 1.0$ and 5.0. Here $v = \dot{x}$. These numerical results confirm that viscoelastic dampers can be effectively used to reduce the structural responses over the entire frequency range. It is also not surprising that as η increases, the values of these functions decrease. Note that both fractional unit impulse response functions $h(t)$ and $\dot{h}(t)$ exhibit oscillation characteristics of an over-damped system when $\eta = 5.0$.

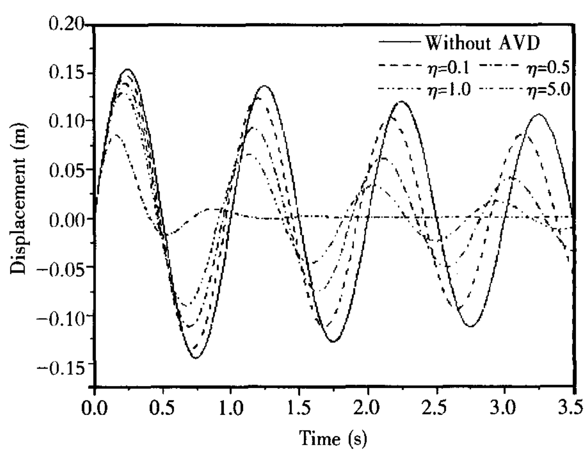

Fig. 1 Fractional unit impulse response function $h(t)$ (α = 0.6)

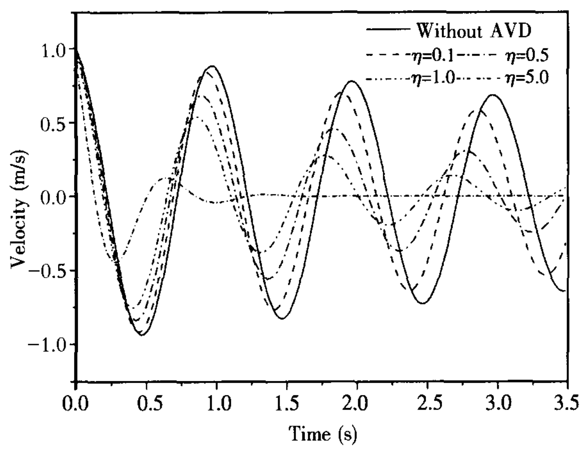

Fig. 2 Fractional unit impulse response function $\dot{h}(t)(\alpha=06)$

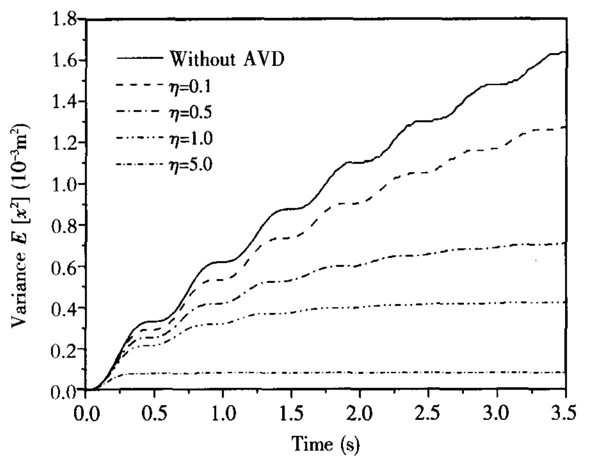

Fig. 3 Variance function $E[x^2]$ as function of time(α = 0.6)

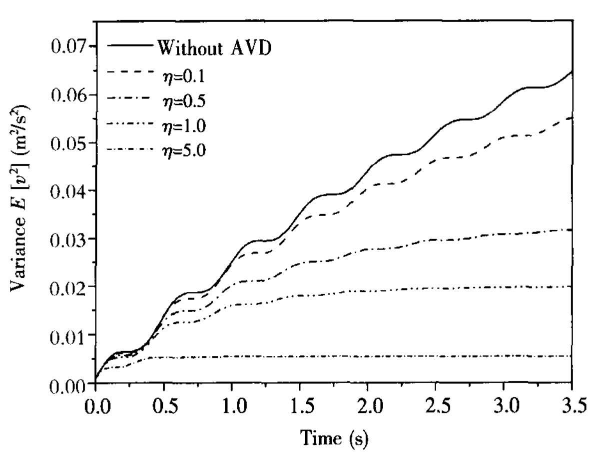

Fig. 4 Variance function $E[v^2]$ as function of time(α = 0.6)

Similar results are plotted in Figs. 6 through 10 with $\eta = 0.5$ for $\alpha = 0.2, 0.4, 0.6, 0.8$. From these figures, it is seen that when α increases, all related functions decrease. However, compared to the effect of η shown in Figs. 1 through 5, α has less of an effect on the impulse functions$h(t)$ and $\dot{h}(t)$, and a comparable effect on variance function $E[x^2]$ and $E[v^2]$, and covariance function $E[xv]$. These conditions are true for the variation range of η and η values that were assumed in the numerical calculation.

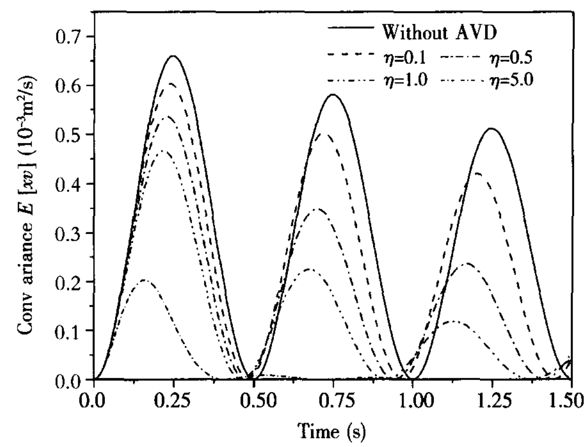

Fig. 5 Convariance function $E[xv]$ as function of time (α = 0.6)

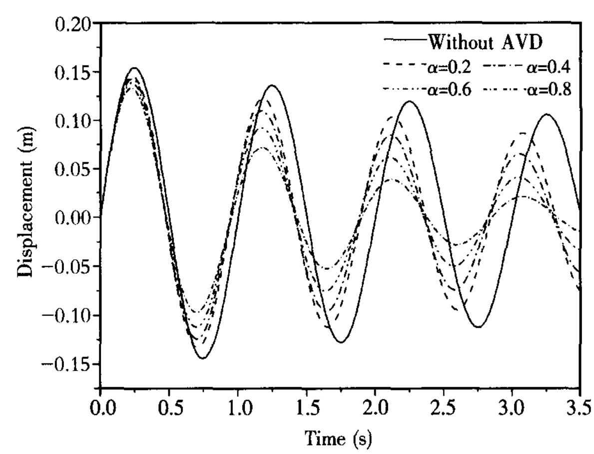

Fig. 6 Comparison of $h(t)$ for fractional models(η = 0.5)

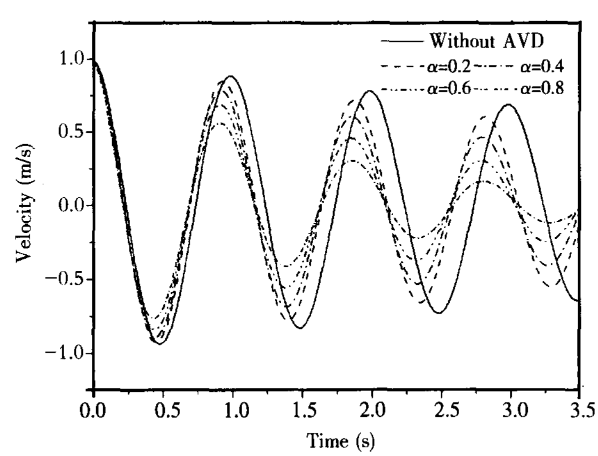

Fig. 7 Comparison of $\dot{h}(t)$ for fractional models(η = 0.5)

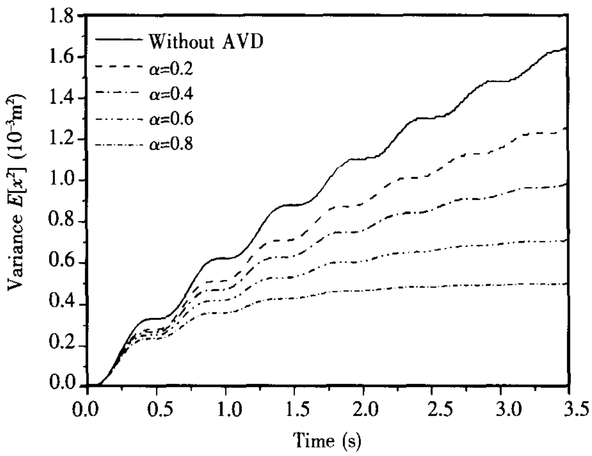

Fig. 8 Comparison of $E[x^2]$/for fractional models (η = 0.5)

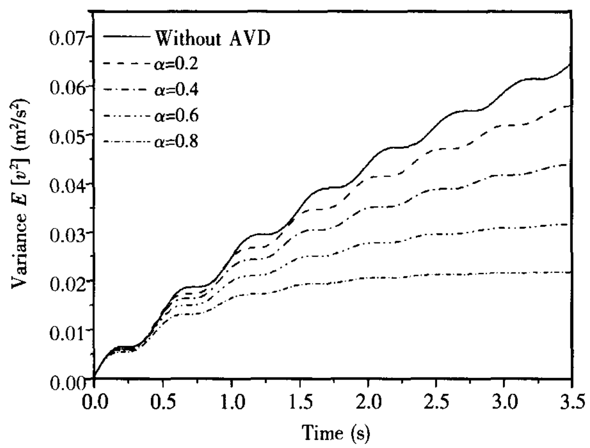

Fig. 9 Comparison of $E[v^2]$/for fractional models (η = 0.5)

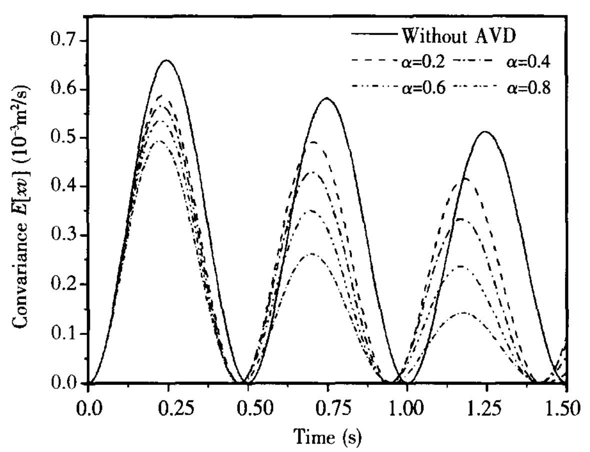

Fig. 10 Comparison of $E[xv]$/for fractional models (η =0.5)

5 Conclusions

A Fourier-transform-based technique has been presented to obtain the fractional unit impulse response function and the response of a single-degree-of-freedom structure with added viscoelastic dampers whose force-deformation relationship is described by a fractional derivative model. A Duhamel integral-type expression has been presented for the response of fractional damped dynamic systems that may be subjected to deterministic or random input. These expressions are used to obtain the stochastic response of the structure subjected to random ground motion. Through numerical verification, it is shown that viscoelastic dampers can be effectively used to reduce structural responses over the entire frequency range, and the proposed schemes can be used to accurately predict the stochastic seismic response of structures with added viscoelastic dampers modeled by fractional derivative.

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Appendix Ⅰ: MATLAB computer program for the fractional unit impulse response function $h(t)$.

This program is developed for computing the fractional unit impulse response function $h(t)$ illustrated in Fig. 1 or Fig. 6. The constants wn = 2 * pi, wve = pi/2, zn = 0.02, zve = 0.1 and alfa = 0.6 , which are defined near the beginning of the code, stand for $\omega_n = 6.28 rad/s$, $\omega_{ve} = 1.57 rad/s$, $\xi = 0.02$, $\eta = 0.1$ and $\alpha = 0.6$, respectively. The meaning of the related variables and functions is as follows:

1. dur: the duration time
2. dt: time interval
3. dw: radius frequency interval
4. wmax: maximum radius frequency
5. om1: positive radius frequency range
6. om2: negative radius frequency range
7. om: radius frequency range
8. ifft: inverse discrete Fourier transform function in MATLAB
9. hiw: fractional frequency response function in discrete form
10. ht: fractional unit impulse response function in discrete form

Program

```
%
clear;
wn = 2 * pi;
wve = pi/2;
zn = 0.02;
zve = 0.1;
alfa = 0.6;
dur = 100.0;
dt = 0.01;
t = 0:dt:dur;
wmax = pi/dt;
dw = 2 * pi/dur;
om1 = 0:dw:-wmax;
om2 = - wmax:dw:-dw;
om = [om1 om2]
hiw = 1./(wn^2 + wve^2 - om.^2 + i * 2 * zn * wn * pi * om + (i * om).^alfa * 2 * zve * wve);
ht = real(ifft(hiw/dt));
```