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EXISTENCE OF SOLUTIONS AND PERIODIC SOLUTIONS FOR NONLINEAR EVOLUTION INCLUSIONS

NIKOLAOS S. PAPAGEORGIOU
FRANCESCA PAPALINI - FRANCESCA RENZACCI

In this paper we consider nonlinear-dependent systems with multivalued perturbations in the framework of an evolution triple of spaces. First we prove a surjectivity result for generalized pseudomonotone operators and then we establish two existence theorems: the first for a periodic problem and the second for a Cauchy problem. As applications we work out in detail a periodic nonlinear parabolic partial differential equation and an optimal control problem for a system driven by a nonlinear parabolic equation.

1. Introduction.

The purpose of this paper is to prove two existence theorems for evolution inclusions defined in the framework of an evolution triple of spaces. The first existence theorem referes to a periodic problem, while the second referes to a Cauchy problem.

In the past most of the works on the existence of periodic solutions for evolution equation deal with semilinear systems. The assumptions are such so as guarantee uniqueness of the solution, which of course implies single-valuedness of the corrisponding Poincaré map. If we have a singlevalued Poincaré map it is easier to apply one of the classical fixed

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point theorems, while if the Poincaré map is multivalued then most of the known results require convex or in the most general case acyclic values, a condition which in general is difficult to verify. In this respect we should mention the important work of Cellina [6], who illustrated that for finite dimensional differential inclusion, the restrictive requirements of convexity or acyclicity are not necessary and the usual fixed point techniques can work using some approximation arguments (see Cellina [6], part II). The first major result on the periodic problem is that of Browder (cf. [5]), who considered semilinear systems in a Hilbert space driven by a monotone operator and with a single-valued perturbation term $f(t, x)$ which is also monotone in x. Browder's result used the fixed point theorem for nonexpansive maps in a uniformly convex space (in particular a Hilbert space; cf. [4] and [7]). The next major result on the periodic problem can be found in the paper of Pruss (cf. [20]). Pruss drops the monotonicity condition on $f(t, .)$ and instead assumes a Nagumo-type tangential condition. First he proves a result under a uniqueness assumption for the solution of the Cauchy problem and then a result without that uniqueness assumptions. The first theorem is based on Schauder's fixed point theorem applied on an invariant closed convex set whose existence is a result of the assumed tangential condition. The second theorem uses the Leray-Schauder continuation principle applied on a suitably defined family of operators. All his results require that the time-invariant linear unbounded operator governing the equation generates a compact semigroup or alternatively that the single valued perturbation term $f(t, x)$ is compact. Also we should mention that in the context of finite dimensional differential and functional inclusions the Nagumo tangential condition was used by Haddad [9]. Subsequently Becker (cf. [3]), considered semilinear evolution equations driven by a closed densely defined linear operator which generates a compact semigroup. Using a perturbation term of special form and an extra condition amounting to saying that $A - \lambda I$ is m-accretive for some $\lambda > 0$, he proves the existence of a unique periodic solution. More recently we have the works of Vrabie (cf. [25]) and Hirano (cf. [12]) which considered non linear evolution equations. Vrabie's work can be viewed as a nonlinear extension of Becker's result. He also assumes a rather restrictive asymptotic growth condition for his perturbation term which is assumed to be a single valued. Vrabie assumed that the nonlinear operator generates a compact semigroup. Hirano considers an evolution equation defined in a Hilbert space, driven by a subdifferential operator generating a compact semigroup and with a single-valued Caratheodory perturbation term of sublinear growth. We should also mention the recent multivalued work of Hu-Papageorgiou (cf.

[13]), who, considered evolution inclusions in an evolution triple of Hilbert spaces and using a tangential condition and Galerkin approximations proved the existence of a periodic solution. Related is also the work of Lakshmikantham-Papagergiou (cf. [14]) and Papageorgiou (cf. [18]), who extend in a multivalued setting the work of Vrabie.

For the Cauchy problem, related to our work here are the papers of Attouch-Damlamian (cf. [1]), Vrabie (cf. [23] and [24]), Gutman (cf. [8]), Hirano (cf. [11]) and Papageorgiou (cf. [17]). Of these works Attouch-Damlamian consider evolution inclusion driven by a time-invariant subdifferential operator generating a compact semigroup and a multivalued perturbation $F(t,x)$ u.s.c. in both (t,x). Similarly Vrabie considers general maximal monotone operator (not necessarily of the subdifferential type). Gutman assumes that the time-invariant m-accretive operator generates an equicontinuous semigroup of contractions and the perturbation term is single valued and compact. Hirano treats time-invariant systems with single-valued perturbation and finally Papageorgiou assumes that the multivalued perturbation term is defined on all $T \times H$.

Our work here (both the periodic and Cauchy problems) goes beyond the above mentioned papers. We treat time-dependent systems with multivalued perturbations defined within the setting of an evolution triple of spaces (X, H, X^*). We only assume that X embeds compactly in H. This hypothesis does not means that $A(t,.)$ generates a compact semigroup or that the perturbation term (assumed to be mutivalued and defined only on $T \times X$) is compact.

2. Mathematical preliminaries.

Let (Ω, Σ) be a measurable space and $(X, \|\cdot\|)$ a separable Banach space. Throughout this paper, we will be using the following notations:

$$P_{f(c)}(X) = \{A \subset X : A \text{ nonempty, closed (convex)}\},$$

$$P_{(w)k(c)}(X) = \{A \subset X : A \text{ nonempty, (weakly)compact(convex)}\}.$$

With X_w we denote the space X equipped with the weak topology.

A multifunction $F : \Omega \to P_f(X)$ is said to be measurable if, for all $x \in X$, the function $\omega \to d(x, F(\omega)) = \inf\{\|x - z\| : z \in F(\omega)\}$ is measurable. A multifunction $F : \Omega \to P_f(X)$ is said to be graph measurable if $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For $P_f(X)$-valued multifunctions, measurability implies graph measurability, while the converse is true, for istance, if there is a σ-finite measure $\mu(.)$ on (Ω, Σ) with respect to which Σ is complete.

We define S_F^p $(1 \leq p \leq \infty)$ to be the set of all $L^p(\Omega, X)$-selectors of $F(.)$; i.e. $S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega)\ \mu\text{-a.e}\}$. Note that for a graph measurable multifunction $F : \Omega \to P_f(X)$, S_F^p is nonempty if and only if the function $\omega \to \inf\{\|z\| : z \in F(\omega)\}$ belongs to $L^p(\Omega, R^+)$.

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z \backslash \{\emptyset\}$ is said to be lower semicontinuous (l.s.c.) (resp. upper semicontinuous (u.s.c.)), if for all $U \subset Z$ open $F^-(U) = \{y \in Y : F(y) \cap U \neq \emptyset\}$ (resp. $F^+(U) = \{y \in Y : F(y) \subset U\}$) is open in Y.

Now let H be a Hilbert space and let X a dense subspace of H carrying the structure of a separable, reflexive Banach space, which embeds into H continuously. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being continuous and dense. Such a triple of spaces is known in the literature as "evolution triple" or "Gelfand triple" (cf. [26]). We will also assume that the embedding of X into H is also compact (in fact, this implies that $H \hookrightarrow X^*$ is compact too). To have a concrete example in mind, let m be a positive integer and $2 \leq p \leq \infty$. Let $Z \subset R^N$ be a bounded domain and set $X = W_0^{m,p}(Z, R)$, $H = L^2(Z, R)$ and $X^* = W^{-m,q}(Z, R)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then from the Sobolev embedding theorem, we know that (X, H, X^*) is an evolution triple and all embeddings are compact. By $|\cdot|$ (resp. $\|\cdot\|_*$) we will denote the norm of H (resp. of X^*). Also by $(\cdot, \cdot)$ we will denote the inner product of H and by $\langle \cdot, \cdot \rangle$ the duality brackets of the pair (X, X^*). The two are compatible in the sense that $\langle \cdot, \cdot \rangle_{X \times H} = (\cdot, \cdot)$.

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $T = [0, b]$; we define:

$$W_{pq}(T) = \{x \in L^p(T, X) : \dot{x} \in L^q(T, X^*)\}.$$

The derivative involved in this definition is understood in the sense of vector valued distributions. Equipped with the norm

$$\|x\|_{W_{pq}} = \left[\|x\|_p^2 + \|\dot{x}\|_q^2\right]^{\frac{1}{2}},$$

the space $W_{pq}(T)$ becomes a separable, reflexive Banach space. It is well known that $W_{pq}(T)$ embeds continuously in $C(T, H)$; i.e. every element in $W_{pq}(T)$ has a unique representative in $C(T, H)$. Since we have assumed that $X \hookrightarrow H$ compactly, we have that $W_{pq}(T) \hookrightarrow L^p(T, H)$ compactly (cf. [26], p. 450).

Now let Y be a reflexive Banach space, $L : D(L) \subset Y \to Y^*$ be a linear densely defined maximal monotone operator and let $T : Y \to P(Y^*)$. $T(.)$ is said to be coercive if $\dfrac{\inf[\langle v, x\rangle : v \in T(x)]}{\|x\|} \to +\infty$, as $\|x\| \to \infty$.

We say that $T(.)$ is a "generalized pseudomonotone operator with respect to $D(L)$" if:

a) $\forall y \in Y,\ T(y) \in P_{wkc}(Y^*)$;

b) $T(.)$ is u.s.c. from every finite dimensional subspace V of $D(L)$ into Y^*_w (with the symbol y^*_n we mean the space y^* equipped with the weak topology);

c) if $\{y_n\}_{n\geq 1} \subset D(L)$, with $y_n \to y$ weakly in $Y, y \in D(L)$, $Ly_n \to Ly$ weakly in Y^*, $y^*_n \in T(y_n)$, $n \geq 1$, $y^*_n \to y^*$ weakly in Y^* and $\limsup\limits_{n\to+\infty}(y^*_n, y_n) \leq (y^*, y)$, then $[y, y^*] \in GrT$ and $(y^*_n, y_n) \to (y^*, y)$ (here $(\cdot, \cdot)$, denotes the duality brackets of the pair (y, y^*)).

The proofs of our existence theorems are based on the following surjectivity result for a particular class of generalized pseudomonotone operators. An analogous result for singlevalued operator can be found in [15] and in [22].

THEOREM 2.1: *If Y is a reflexive, strictly convex Banach space, $L : D(L) \subset Y \to Y^*$ be a linear densely defined maximal monotone operator and $T : Y \to P(Y^*)$ is a multivalued operator, which is bounded, generalized pseudomonotone with respect to $D(L)$ and coercive. Then $R(L + T) = Y^*$.*

Proof. We introduce on $D(L)$ the graph norm, that is

$$\|x\|_{D(L)} = \|x\| + \|Lx\|_*,$$

so $D(L)$ is a reflexive Banach space.

Denoted with $J : Y \to Y^*$ the duality map of Y, for every $m \in N$, we define $M_m : D(L) \to D(L)^*$, as

$$(M_m(u), v) = \frac{1}{m}(J^{-1}Lu, Lv) + (Lu, v), \quad \forall v \in D(L).$$

It is easy to see that M_m is bounded, hemicontinuous and monotone and so pseudomonotone.

Now let $P_m : D(L) \to P(D(L)^*)$ be defined by

$$P_m(u) = M_m(u) + T(u), \quad \forall u \in D(L).$$

We observe that

$$\forall u \in D(L), \ P_m(u) \in P_{wkc}(D(L)^*);$$

and

$P_m(.)$ is u.s.c. from every finite dimensional subspace V of $D(L)$ into $D(L)^*_w$. Now let $[u_n, p_n] \in Gr P_m$, $n \geq 1$, with $u_n \to u$ weakly in $D(L)$, $p_n \to p$ weakly in $D(L)^*$ and $\limsup_{n\to\infty}(p_n, u_n - u) \leq 0$. Let $\rho_n \in T(u_n)$ such that

$$p_n = M_m(u_n) + \rho_n, \quad \forall n \in N.$$

From our conditions, we can assume that $\rho_n \to \rho$ weakly in $D(L)^*$. Moreover we have that

$$\limsup_{n\to\infty}[(M_m(u_n), u_n - u) + (\rho_n, u_n - u)] \leq 0, \tag{2.1}$$

from which we obtain that $\limsup_{n\to\infty}(\rho_n, u_n - u) \leq 0$. Indeed, if $\limsup_{n\to\infty}(\rho_n, u_n - u) = d > 0$, passing to a subsequence, if necessary, we have that $\lim_{n\to\infty}(\rho_n, u_n - u) = d$. But from (2.1) it follows that $\limsup_{n\to\infty}(M_m(u_n), u_n - u) \leq -d$, so, from the pseudomonotonicity of M_m, we obtain that $\liminf_{n\to\infty}(M_m(u_n), u_n - u) \geq 0$, which is a contradiction. Therefore, $\limsup_{n\to\infty}(\rho_n, u_n - u) \leq 0$, then, from our assumptions on $T(.)$, we have $\rho \in T(u)$ and $\lim_{n\to\infty}(\rho_n, u_n) = (\rho, u)$, which implies that $\limsup_{n\to\infty}(M_m(u_n), u_n - u) \leq 0$. By using the pseudomonotonicity of M_m, it follows that $M_m(u_n) \to M_m(u)$ weakly in $D(L)^*$, so $p \in P_m(u)$.

Now, since L is monotone we have that

$$(p, u) \geq \frac{1}{m}\|Lu\|_*^2 + (\rho, u), \quad \forall u \in D(L), \quad \forall p \in P_m(u), \tag{2.2}$$

and so

$$\frac{\inf[(p, u) : p \in P_m(u)]}{\|u\|_{D(L)}} \to +\infty, \quad \text{as } \|u\|_{D(L)} \to \infty.$$

By applying Theorem 5.4 of [19], p. 154, we have that, for every $f \in Y^*$, there exists $u_m \in D(L)$ such that $f \in P_m(u_m)$ and so, it is possible to find $\rho_m \in T(u_m)$ with the property

(2.3) $$f = M_m(u_m) + \rho_m.$$

Therefore we obtain that

(2.4) $$\frac{1}{m}(J^{-1}Lu_m, Lv) = (f - \rho_m - Lu_m, v), \quad \forall v \in D(L),$$

so $J^{-1}Lu_m \in D(L^*)$ (L^* is the adjoint map of L) and

(2.5) $$\frac{1}{m}L^*J^{-1}Lu_m + Lu_m + \rho_m = f, \quad \forall m \in N.$$

Now, from (2.2) and (2.3) we have that $(u_m)_m$ is bounded in Y; moreover from (2.5) it follows that

$$\frac{1}{m}(L^*J^{-1}Lu_m J^{-1}Lu_m) + (Lu_m, J^{-1}Lu_m) + (\rho_m, J^{-1}Lu_m) = (f, J^{-1}Lu_m),$$

and, since L^* is monotone, we obtain

$$\|Lu_m\|_*^2 \leq (f, J^{-1}Lu_m) - (\rho_m, J^{-1}Lu_m).$$

But $T(.)$ is bounded, so we can find a positive number K such that

$$\|Lu_m\|_*^2 \leq K\|Lu_m\|_*,$$

then $(Lu_m)_m$ is bounded in Y^*. By passing to subsequences we get

$u_m \to u$ weakly in $Y, u \in D(L)$,

$Lu_m \to Lu$ weakly in Y^*,

$\rho_m \to \rho$ weakly in Y^*.

By (2.3) we have that

$$(\rho_m, u_m - u) = (f - M_m(u_m),$$

$$u_m - u) \leq (f - Lu, u_m - u) - \frac{1}{m}(J^{-1}Lu_m, Lu_m - Lu),$$

which implies that

$$\limsup_{m \to +\infty}(\rho_m, u_m - u) \leq 0.$$

Therefore $\rho \in T(u)$ and, from (2.4), we get, finally that $f \in Lu + T(u)$.

3. Periodic solutions.

Let $T = [0, b]$, (X, H, X^*) be an evolution triple of spaces with X embedding compactly into $H, 2 \leq p < \infty$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We consider the following evolution inclusion

$$\text{(1)} \qquad \begin{cases} \dot{x}(t) + A(t, x(t)) + F(t, x(t)) \ni h(t), & \text{a.e. on } T \\ x(0) = x(b). \end{cases}$$

The hypotheses on the data of (1) are the following:

$H(A)$: $A : T \times X \to X^*$ is an operator such that

i) $\forall x \in X, t \mapsto A(t, x)$ is measurable;

ii) $\forall t \in T, x \mapsto A(t, x)$ is monotone and hemicontinuous;

iii) $\exists a_1 \in L^q(T, R_0^+)$ and $\exists c_1 > 0$:

$$\|A(t, x)\|_* \leq a_1(t) + c_1 \|x\|^{p-1}, \quad \text{a.e. on } \; T, \forall x \in X;$$

iv) $\exists c > 0 : \langle A(t, x), x \rangle \geq c\|x\|^p$, a.e.on $T, \forall x \in X$.

H(F): $F : T \times X \to P_{fc}(H)$ is a multifunction such that

j) $\forall x \in X,\ t \mapsto F(t, x)$ is measurable;

jj) $\forall t \in T,\ x \mapsto F(t, x)$ is sequentially closed in $X_w \times H_w$;

jjj) $\exists a_2 \in L^q(T, R_0^+)$ and $c_2 > 0$:

$$|F(t, x)| \leq a_2(t) + c_2\|x\|^{p-1}, \quad \text{a.e. on } \; T, \forall x \in X;$$

jv) $\exists c_3, c_4 > 0$ with $2^{2p} c_3 < c$:
$\forall (t, x) \in T \times X,\ \forall y \in F(t, x)$ we have that $(y, x) \geq -c_3\|x\|^p - c_4$.

THEOREM 3.1. *If hypotheses $H(A)$, $H(F)$ hold, and $h \in L^q(T, X^*)$ then problem (1) has a solution.*

Proof. Let $L : D(L) \subset L^p(T, X) \to L^q(T, X^*)$ be a linear operator defined by $Lx = \dot{x}$ with $D(L) = \{x \in W_{pq}(T) : x(0) = x(b)\}$.

Claim 1. L is a densely defined, maximal monotone operator.

That $D(L)$ is dense in $L^p(T, X)$ follows immediately from the fact that the set of all functions $C^\infty(T, X)$ with $x(0) = x(b)$ is dense in $L^p(T, X)$.

From integration by parts formula for functions in $W_{pq}(T)$ (cf. [26], Proposition 23.23, p. 422), we have

$$((Lx, x)) = \frac{1}{2}(|x(b)|^2 - |x(0)|^2) = 0$$

Here $((\cdot,\cdot))$ denotes the duality brackets for the pair $(L^p(T, X), L^q(T, X^*))$. So L is a monotone operator.

To prove the maximality of L, we need to show that if $[u, u^*] \in L^p(T, X) \times L^q(T, X^*)$ and $0 \leq ((u^* - Lx, u - x))$ for all $x \in D(L)$, then $u \in D(L)$ and $u^* = Lu$ (i.e. $u^* = \dot{u}$). To this end let $x = \varphi z$ with $\varphi \in C_0^\infty(T)$ and $z \in X$.

Then $\dot{x} = \dot{\varphi} z$ and $x \in D(L)$. So we have $((Lx, x)) = 0$. Hence we get

$$0 \leq ((u^*, u)) - \int_0^b \langle \dot{\varphi}(t)u(t) + \varphi(t)u^*(t), z\rangle dt,$$

for all $\varphi \in C_0^\infty(T)$ and all $z \in X$.

$$\Rightarrow \int_0^b (\dot{\varphi}(t)u(t) + \varphi(t)u^*(t))dt = 0, \text{ for all } \varphi \in C_0^\infty(T)$$

$$\Rightarrow \dot{u} = u^*,\ u \in W_{pq}(T).$$

It remains to show that $u \in D(L)$. Using once again the integration by parts formula for functions in $W_{pq}(T)$ and the fact that $x(0) = x(b)$, we get

$$0 \leq ((\dot{u} - \dot{x}, u - x)) = \frac{1}{2}|u(b)|^2 - \frac{1}{2}|u(0)|^2 - (u(b) - u(0), x(0)),$$

for all $x \in D(L)$.

Choose $x(t) = v$ for arbitrary $v \in X$. Exploiting the density of X in H we deduce at once that $u(b) = u(0)$, thus $u \in D(L)$ and so we have proved the maximal monotonicity of L.

Next let $\hat{A} : L^p(T, X) \to L^q(T, X^*)$ be the Nemitsky (superposition) operator corresponding to $A(t, x)$ (i.e. $\hat{A}(x)(.) = A(., x(.))$. By virtue of hypothesis $H(A)$, $\hat{A}(.)$ is monotone, hemicontinuous, bounded and coercive.

Also let $G : L^p(T, X) \to P_{fc}(L^q(T, X^*))$ be defined by $G(x) = S^q_{F(.,x(.))}$. Then let $T : L^p(T, X) \to P_{fc}(L^q(T, X^*))$, defined as $T(x) = \hat{A}(x) + G(x)$, for all $x \in L^p(T, X)$.

Claim 2. $T(.)$ is a generalized pseudomonotone operator with respect to $D(L)$. It is easy to see that the values of T are convex and weakly compact in $L^q(T, X^*)$ and that T is u.s.c. from $L^p(T, X)$ into $L^q(T, X^*)_w$.

Now let $\{x_n\}_{n\geq 1} \subset D(L)$, $x_n \to x$ weakly in $L^p(T, X)$, $x \in D(L)$, $Lx_n \to Lx$ weakly in $L^q(T, X^*)$, $v_n \in T(x_n)$, $n \geq 1$, with $v_n \to v$ weakly in $L^q(T, X^*)$ and assume that $\limsup_{n\to\infty}((v_n, x_n - x)) \leq 0$. Note that $v_n = \hat{A}(x_n) + g_n$ with $g_n \in G(x_n)$, $n \geq 1$.

Observe that we actually have $g_n \in L^q(T, H)$, $n \geq 1$, and because of hypothesis $H(F)$-jjj), by passing to a subsequence, if necessary, we may assume that $g_n \to g$ weakly in $L^q(T, H)$. Also since $x_n \to x$ weakly in $W_{pq}(T)$ and the latter embeds compactly in $L^p(T, H)$, we have $x_n \to x$ in $L^p(T, H)$. So we get

$$\limsup_{n\to\infty}((\hat{A}(x_n), x_n - x)) + \lim_{n\to\infty}((g_n, x_n - x)) =$$

$$= \limsup_{n\to\infty}((v_n), x_n - x)) \leq 0.$$

Because $\lim_{n\to\infty}(g_n, x_n - x) = 0$, we have that

$$\limsup_{n\to\infty}((\hat{A}(x_n), x_n - x)) \leq 0.$$

But $\hat{A}(.)$ being monotone, hemicontinuous, bounded is pseudomonotone (cf. [26], Proposition 27.6, p. 586) and so $\hat{A}(x_n) \to A(x)$ weakly in $L^q(T, X^*)$ and

$$((\hat{A}(x_n), x_n)) \to ((\hat{A}(x), x)). \tag{3.1}$$

Now let $\xi_n(t) = \langle A(t, x_n(t)) - A(t, x(t)), x_n(t) - x(t)\rangle \geq 0$ a. e. on T. From (3.1) we have $\int_0^b \xi_n(t)dt \to 0$ and so $\xi_n \to 0$ in $L^1(T, R)$. Moreover by passing to a subsequence we can also assume that $\xi_n(t) \to$

0 a.e. on T. Also by using hypotheses $H(A)$-iii) and $H(A)$-iv) we have

$$\xi_n(t) \geq c\|\text{sen}(t)\|^p - c_1\|x(t)\|\|\text{sen}(t)\|^{p-1} \cdot \|\text{sen}(t)\|(a_1(t)+ \tag{3.2}$$
$$+c_1\|x(t)\|^{p-1}) - a_1(t)\|x(t) + c\|x(t)\|^p, \qquad \text{a.e. on } T.$$

Our claim is that (3.2) above implies that for almost $t \in T$, $\sup_{n\in N}\|\text{sen}(t)\| < \infty$. Let $N \subset T$, $mN = 0$, such that

$$\begin{aligned} \xi_n(t) \geq\ & c\|x_n(t)\|^p - c_1\|x(t)\| \cdot \|x_n(t)\|^{p-1} \\ & - \|x_n(t)\|(a_1(t) + c_1\|x(t)\|^{p-1})+ \\ & - a_1(t)\|x(t)\| + c\|x(t)\|^p, \quad \forall t \in T\backslash N, \end{aligned} \tag{3.3}$$

and

$$\xi_n(t) \to 0, \quad \forall t \in T\backslash N. \tag{3.4}$$

Suppose that there exists $t \in T\backslash N$ such that $\sup_{n\in\mathbb{N}}\|x_n(t)\| = \infty$. Then it is possible to find a subsequence of $\{x_n(t)\}_{n\geq 1}$, denoted also with $\{x_n(t)\}_{n\geq 1}$ diverging to $+\infty$. But from (3.3) we obtain that $\xi_n(t) \to +\infty$, which contradicts (3.4).

Hence if we fix $t \in T\backslash N$ and pass to an appropriate subsequence (depending in general on t), we can have $x_n(t) \to x(t)$ weakly in X as $n \to \infty$. Using Theorem 3.1 of [16] and the fact that $GrF(t,.)$ is sequentially closed in $X_w \times H_w$ we get $g(t) \in \overline{conv}\, w-\limsup\{g_n(t)\}_{n\geq 1} \subset \overline{conv}\, w - \limsup F(t, x_n(t)) \subset F(t, x(t))$.

Therefore $g \in G(x)$.

So finally we have that $v_n \to \hat{A}(x) + g = v$ weakly in $L^q(T, X^*)$ and so $v \in T(x)$.

Moreover, since $g_n \to g$ weakly in $L^q(T, H)$, from (3.1), we obtain that

$$\lim_{n\to\infty}((v_n, x_n)) = ((v, x)).$$

Therefore $T(.)$ is a generalized pseudomonotone operator with respect to $D(L)$.

Claim 3. $T(.)$ is coercive;

i.e. $$\frac{\inf[((v,x)) : v \in T(x)]}{\|x\|_p} \to +\infty, \quad \text{as } \|x\|_p \to \infty,\ x \in D(L).$$

This is an immediate consequence of the coercivity property of $\hat{A}$ and of hypothesis $H(F)$-iv) toke away.

Now rewrite the problem (1) as the following equivalent abstract operator inclusion:

$$Lx + T(x) \ni h, \ x \in D(L). \tag{3.5}$$

Apply theorem 2.1 to get a solution for (3.5). Evidently this is the desired periodic solution of problem (1).

4. The Cauchy problem.

Now we pass to the Cauchy problem. So our object of investigation is the following Cauchy problem

$$\begin{cases} \dot{x}(t) + A(t, x(t)) + F(t, x(t)) \ni h(t), & \text{a.e. on } T, \\ x(0) = x_0. \end{cases} \tag{2}$$

THEOREM 4.1. *If hypotheses $H(A)$, $H(F)$ hold and $x_0 \in H$, then problem (2) has a solution $x \in W_{pq}(T)$.*

Proof. First assume that $x_0 \in X$. Let $\hat{A}_1 : L^p(T, X) \to L^q(T, X^*)$ be defined by $\hat{A}_1(x)(.) = A(., x(.) + x_0)$ and $G_1 : L^p(T, X) \to P_{fc}(L^q(T, X^*))$ by $G_1(x) = S_F^q(., x(.) + x_0)$. Because of hypothesis $H(A)$, we have that $\hat{A}_1(.)$ is monotone, hemicontinuous and bounded. Moreover it is easy to see that

$$\exists \beta, \gamma > 0 : ((\hat{A}_1(x), x)) \geq \frac{c}{2^p} \|x\|_p^p - \gamma - \beta \|x\|_p^{p-1}, \tag{4.1}$$

$$\exists\, \beta_1, \gamma_1 > 0 : ((g, x)) \geq -c_3 2^p \|x\|_p^p - \gamma_1 - \beta_1 \|x\|_1^{p-1}, \quad \text{for all } g \in G_1(x) \tag{4.2}$$

Now let $L_1 : D(L_1) \subset L^p(T, X) \to L^q(T, X^*)$ be defined by $L_1 x = \dot{x}$ with $D(L_1) = \{x \in W_{pq}(T) : x(0) = 0\}$.

Claim 1: L_1 is a densely defined, maximal monotone operator.

It is easy to see that L_1 is a densely defined operator. So it remains to show the maximal monotonicity of L_1. First note that by the integration by parts formula for functions in $W_{pq}(T)$, we have

$$((L_1 x, x)) = ((\dot{x}, x)) = \frac{1}{2} |x(b)|^2 \geq 0$$

$\Rightarrow L_1$ is monotone.

Working as in claim 1 in the proof of theorem 3.1 let $[u, u^*] \in L^p(T, X) \times L^q(T, X^*)$ and $0 \leq ((u^* - L_1 x, u - x))$ for all $x \in D(L_1)$. That $u^* = \dot{u}$ follows exactly as in the proof of theorem 3.1 (cf. claim 1). Now to show that $u \in D(L_1)$, we remark that:

$$0 \leq ((\dot{u} - \dot{x}, u - x)) = \frac{1}{2}[|u(b) - x(b)|^2 - |u(0) - x(0)|^2].$$

Let $v_n \in X$, $n \geq 1$ such that $bv_n \to u(b)$ in H, as $n \to \infty$. Set $x_n(t) = tv_n$. Then $x_n \in D(L_1)$ and for every $n \geq 1$ we have $|u(0)|^2 \leq |bv_n - u(b)|^2 \to 0$, as $n \to \infty$. Hence $u(0) = 0$ and so $u \in D(L_1)$.

Let $T_1 : L^p(T, X) \to P_{fc}(L^q(T, X^*))$, defined as $T_1(x) = \hat{A}_1(x) + G_1(x)$, for all $x \in L^p(T, X)$. Observe that $T_1(x) = T(x + x_0)$, where T is the operator defined in the proof of theorem 3.1, so, also $T_1(.)$ is a generalized pseudomonotone operator with respect to $D(L_1)$.

Claim 2. $T_1(.)$ is coercive;

$$\text{i.e. } \frac{\inf[((v, x)) : v \in T_1(x)]}{\|x\|_p} \to +\infty, \quad \text{as } x \in D(L_1).$$

This claim follows immediately from the coercivity properties (4.1) and (4.2).

Then consider the operator equation

$$L_1 x + T_1(x) \ni h, \ x \in D(L_1).$$

By theorem 2.1 this has a solution $\hat{x}$. Let $x(t) = \hat{x}(t) + x_0$. Evidently $x \in W_{pq}(T)$ is a solution of the Cauchy problem (2) but under the extra hypothesis that we have a regular initial condition; i.e. $x_0 \in X$.

Now for the general case let $x_0 \in H$. Let $\{x_0^n\}_{n \geq 1} \subset X$ such that $x_0^n \to x_0$ in H, as $n \to \infty$ (recall that X is dense in H). Then from the first part of the proof, we know that the multivalued Cauchy problem

$$\begin{cases} \dot{x}_n(t) + A(t, x_n(t)) + F(t, x_n(t)) \ni h(t), & \text{a.e. on } T, \\ x_n(0) = x_0^n, \end{cases}$$

has a solution $x_n \in W_{pq}(T)$, $n \geq 1$. So $\dot{x}_n(t) + A(t, x_n(t)) + g_n(t) = h(t)$, a.e. on T, $x_n(0) = x_0^n$, with $g_n \in G(x_n) = S^q_{F(.,x_n(.))}$. From standard a priori estimation (cf. for example [26] or [17]) we can have that $\{x_n\}_{n \geq 1}$ is bounded in $W_{pq}(T)$.

So by passing to a subsequence, if necessary, we may assume that $x_n \to x$ weakly in $W_{pq}(T)$ and $g_n \to g$ in $L^q(T, H)$ (cf. $H(F) - jjj$).

Then we have:

$$\limsup_{n\to+\infty}((\hat{A}(x_n) + g_n, x_n - x)) = \limsup_{n\to+\infty}((\dot{x}_n, x - x_n)).$$

But from the integration by parts formula for functions in $W_{pq}(T)$ we have

$$((\dot{x}_n, x - x_n)) = -\frac{1}{2}|x(b) - x_n(b)|^2 + \frac{1}{2}|x_0^n - x_0|^2 + ((\dot{x}, x - x_n))$$

$$\Rightarrow \limsup_{n\to+\infty}((\dot{x}_n, x - x_n)) \leq \lim_{n\to+\infty} \frac{1}{2}|x_0^n - x_0|^2 + \lim_{n\to+\infty}((\dot{x}, x - x_n)) = 0$$

$\Rightarrow \limsup_{n\to+\infty}((\hat{A}(x_n), x_n - x)) \leq 0$ (because $((g_n, x_n - x)) = (g_n, x_n - x) \to 0$ since $x_n \to x$ in $L^p(T, H)$ by the compact embedding of $W_{pq}(T)$ into $L^p(T, H)$).

Since $\hat{A}(.)$ is pseudomonotone we have $\hat{A}(x_n) \to \hat{A}(x)$ weakly in $L^q(T, X^*)$ and

$$(\hat{A}(x_n), x_n)) \to ((\hat{A}(x), x)).$$

Also working as in the proof of theorem 3.1, via the use of $\xi_n(t)$, we can show that $g \in G(x)$. So in the limit as $n \to \infty$ we get $\dot{x} + \hat{A}(x) + g = h$ with $x(0) = x_0$. So $x \in W_{pq}(T)$ solves problem (2) with $x_0 \in H$.

A careful reading of the above proof can convince the reader that the following is true:

COROLLARY 4.2. *If hypotheses $H(A), H(F)$ hold, $h \in L^q(T, X^*)$ and $x_0 \in H$, then the solution set of problem (2) is nonempty and weakly compact in $W_{pq}(T)$ and so is compact in $L^p(T, H)$.*

5. Examples.

We will conclude this paper with two examples.

EXAMPLE 1. A periodic nonlinear Parabolic partial differential equation.

Let $T = [0, b]$ and $Z \subset R^N$ a bounded domain with a C^1-boundary Γ. Let $D_k = \frac{\partial}{\partial z_k}$, $k = 1, ..., N$, $D = \text{grad} = (D_k)_{k=1}^N$ and $\Delta = \sum_{k=1}^N D_k^2$ (the Laplacian). We consider the following multivalued periodic parabolic problem:

$$(3)\quad \begin{cases} \frac{\partial x}{\partial t} - \Delta x + r\sum_{k=1}^N (\sin x) D_k x + u(t,z) = h(t,z) & \text{a.e. on } T \times Z \\ u(t,z) \in [f_1(t,z,x(t,z)), f_2(t,z,x(t,z))] & \text{a.e. on } T \times Z \\ x(0,z) = x(b,z) & \text{a.e. on } Z,\ x|_{T\times\Gamma} = 0. \end{cases}$$

Our hypotheses on f_1, f_2 are the following:

$H(f)$: $f_1, f_2 : T \times Z \times R \to R$ are functions such that:

i) $f_1(t,z,x) \le f_2(t,z,x)$, $\forall (t,z,x) \in T \times Z \times R$;

ii) for every $x : Z \to R$ measurable, $(t,z) \mapsto f_i(t,z,x(z))$ is measurable, $i = 1, 2$;

iii) $\forall (t,z) \in T \times Z$, $x \mapsto f_1(t,z,x)$ and $x \mapsto -f_2(t,z,x)$ are l.s.c.;

iv) $\exists a_2 \in L^2(T, L^2(Z, R_0^+))$ and $c_2 \in L^\infty(T, R_0^+)$:
$|f_i(t,z,x)| \le a_2(t,z) + c_2(t)|x|$, a.e. on $T \times Z, \forall x \in R, i = 1, 2$;

iv) $\exists \beta > 0$: $f_i(t,z,x)x \ge -\beta$, a.e. on $Z, \forall (t,x) \in T \times R$, $i = 1, 2$.

Remark 1. Problems like (3) arise when we deal with partial differential equations whose perturbation term $f(t,z,x)$ is discontinuous in x. Then in order to obtain solutions, we need replace the original single-valued problem by a multivalued one in which we have filled the gaps at the discontinuity points. Namely we introduce $\underline{f}(t,z,x) = \liminf_{y\to x} f(t,z,y)$ and $\bar{f}(t,z,x) = \limsup_{y\to x} f(t,z,y)$ and replace $f(t,z,x)$ by the interval $[\underline{f}(t,z,x), \bar{f}(t,z,x)]$. Under suitable assumptions on f, we are within the framework of problem (3) with conditions $H(f)$ (cf. [21]).

THEOREM 5.1. *If hypothesis $H(f)$ holds, $N \le 3$, $16|r| < \frac{\lambda_1}{1+\lambda_1}$, where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(Z, R))$ and $h \in L^2(T \times$*

Z, R), then problem (3) has a solution $x \in L^2(T, H_0^1(Z, R)) \cap C(T, L^2(Z, R))$, with $\frac{\partial x}{\partial t} \in L^2(T, H^{-1}(Z, R))$.

Proof. In this case the evolution triple is $X = H_0^1(Z, R)$, $H = L^2(Z, R)$ and $X^* = H^{-1}(Z, R)$.

From the Sobolev embedding theorem we know that X embeds compactly in H. Also note that, since we have assumed $N \leq 3$, we actually have that X embeds compactly in $L^4(Z, R)$. Let $A : X \to X^*$ be defined by

$$\langle A(x), y \rangle = \int_Z (Dx(z), D(y(z)))dz.$$

Evidently $A(.)$ is linear, continuous, monotone and

$$\langle A(x), x \rangle = \|Dx\|^2_{L^2(Z,R^N)} \geq \frac{\lambda_1}{1 + \lambda_1} \|x\|^2.$$

Next let $v : X \to H$ be defined by

$$v(x)(z) = r \sum_{k=1}^{N} (\sin x(z)) D_k x(z).$$

We claim that $v(.)$ is sequentially continuous from X_w into H_w. To this end view $v(.)$ as an X^*-valued function. We will show that $v(.)$ is strongly continuous. Indeed let $x_n \to x$ weakly in X. Then, because X embeds compactly in $L^4(Z, R)$, we have $x_n \to x$ in $L^4(Z, R)$. We will show that $v(x_n) \to v(x)$ in X^*.

Suppose not. Then by passing to a suitable subsequence, if necessary, we may assume that there exists $\varepsilon > 0$ and $\{y_n\}_{n \geq 1} \subset X$ with $\|y_n\| \leq 1$ such that

$$\langle v(x_n) - v(x), y_n \rangle \geq \varepsilon, \quad \text{for all} \quad n \geq 1.$$

We may assume that $y_n \to y$ weakly in X and so $y_n \to y$ in $L^4(Z, R)$. Then we have

$$\langle v(x_n) - v(x), y_n\rangle =$$

$$\int_Z \left[r\sum_{k=1}^{N} \sin x_n(z) D_k x_n(z) - r\sum_{k=1}^{N} \sin x(z) D_k x(z) \right] y_n(z) dz =$$

$$r\sum_{k=1}^{N}\int_Z [\sin x_n(z) - \sin x(z)] D_k x_n(z) y_n(z) dz +$$

$$r\sum_{k=1}^{N}\int_Z \sin x(z) D_k x_n(z)[y_n(z) - y(z)] dz +$$

$$r\sum_{k=1}^{N}\int_Z \sin x(z)[D_k x_n(z) - D_k x(z)] y(z) dz +$$

$$r\sum_{k=1}^{N}\int_Z \sin x(z) D_k x(z)[y(z) - y_n(z)] dz.$$

Recall that $|\sin x_n(z) - \sin x(z)| \le |x_n(z) - x(z)|$. So applying Hölder's inequality with three factors we have

$$\left|\int_Z [\sin x_n(z) - \sin x(z)] D_k x_n(z) y_n(z) dz\right| \le$$

$$\le \|x_n - x\|_{L^4(Z,R)} \|x_n\| \, \|y_n\|_{L^4(Z,R)} \le$$

$$\le M_1 \|x_n - x\|_{L^4(Z,R)} \to 0, \quad \text{as} \quad n \to \infty,$$

$$\left|\int_Z \sin x(z) D_k x_n(z)[y_n(z) - y(z)] dz\right| \le$$

$$\le M_2 \|x_n\| \, \|y_n - y\|_{L^4(Z,R)} \le$$

$$\le M_3 \|y_n - y\|_{L^4(Z,R)} \to 0, \quad \text{as} \quad n \to \infty,$$

$$\left|\int_Z \sin x(z)[D_k x_n(z) - D_k x(z)]y(z)dz\right| \to 0, \quad \text{as} \quad n \to \infty,$$

since $D_k x_n \to D_k x$ weakly in $L^2(Z, R)$, and

$$\left|\int_Z \sin x(z) D_k x(z)[y(z) - y_n(z)]dz\right| \to 0, \quad \text{as} \quad n \to \infty,$$

since $y_n \to y$ weakly in $L^2(Z, R)$.

Thus finally we have

$$\langle v(x_n) - v(x), y_n \rangle \to 0, \quad \text{as} \quad n \to \infty,$$

a contradiction to the choices of y_n's. Hence $v(.)$ is strongly continuous from X into X^* as claimed. Now note that

$$|v(x_n)|^2 = \int_Z \left(r \sum_{k=1}^{N} (\sin x_n(z)) D_k x_n(z) \right)^2 dz,$$

therefore $|v(x_n)| \leq M_4 \|x_n\| \leq M_5$, for all $n \geq 1$. So by passing to a subsequence, if it is necessary, we may assume that $v(x_n) \to w$ weakly in H. But, since $v(x_n) \to v(x)$ in X^* for the original sequence, we conclude that $w = v(x)$ and $v(x_n) \to v(x)$ weakly in H; i.e. $v(.)$ is sequentially continuous from X_w into H_w, as claimed.

Now note that

$$\begin{aligned} |\langle v(x), x \rangle| &\leq |r| \sum_{k=1}^{N} \left| \int_Z (\sin x(z))(D_k x(z)) x(z) dz \right| \leq \\ &\leq |r| \sum_{k=1}^{N} \left(\int_Z |D_k x(z)|^2 dz \right)^{1/2} |x| \\ &\leq |r| \|x\|^2 \end{aligned}$$

$\Rightarrow \langle v(x), x \rangle \geq -|r| \, \|x\|^2$.

Also let $F_1 : T \times X \to P_{fc}(H)$ be defined by

$$F_1(t, x) = \{u \in H : f_1(t, z, x(z)) \leq u(z) \leq f_2(t, z, x(z)) \text{ a.e. on } Z\}.$$

Now we want to prove that $t \mapsto F_1(t, x)$ is measurable from T into H. To this end we fix $y \in H$, and we observe that (cf. [10], Theorem 2.2)

$$[d(y, F_1(t, x))]^2 = \inf\left\{\int_Z |y(z) - h(z)|^2 dz : h \in F_1(t, x)\right\} =$$

$$\int_Z \inf\{|y(z) - v|^2 : v \in F_0(t, z, x(z))\} dz = \int_Z d(y(z), F_0(t, z, x(z)))^2 dz,$$

where $F_0 : T \times Z \times R \to P_{fc}(R)$ is the multifunction defined by

$$F_0(t, z, x) = [f_1(t, z, x), f_2(t, z, x)].$$

Since from $H(f)$-ii) we have that $(t, z) \mapsto F_0(t, z, x(z))$ is measurable, by using Fubini's theorem we obtain that $t \mapsto d(y, F_1(t, x))$ is measurable and so it is measurable the multifunction $t \mapsto F_1(t, x)$.

Then define $F(t, x) = v(x) + F_1(t, x)$. Evidently $F(t, x)$ satisfies hypothesis $H(F)$.

Rewrite the problem (3) in the following equivalent evolution inclusion form:

$$\text{(5.1)} \qquad \begin{cases} \dot{x}(t) + A(t, x(t)) + F(t, x(t)) \ni \hat{h}(t), & \text{a.e. on } T \\ x(0) = x(b). \end{cases}$$

Here $\hat{h}(t) = h(t, .) \in H$. Applying theorem 3.1 on (5.1) we get a solution $x \in L^2(T, H_0^1(Z, R)) \cap C(T, L^2(Z, R))$, with $\dfrac{\partial x}{\partial t} \in L^2(T, H^{-1}(Z, R))$, of (3).

Example 2. In this example we consider an optimal control problem for a system driven by a nonlinear parabolic equation. So let $T = [0, b]$ and $Z \subset R^N$ a bounded domain with a C^1-boundary Γ. Fixed $p \geq 2$, let $q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. As before $D_k = \dfrac{\partial}{\partial z_k}$, $k = 1, ..., N$, α will denote a multi-index (by which we mean an N-uple of nonnegative integers), $D^\alpha = D_1^{\alpha_1} ... D_k^{\alpha_N}$, $|\alpha|$ is the length of the multi-index, i.e. $|\alpha| = \sum_{k=1}^{N} \alpha_k$, for $x \in W^{m,p}(Z, R)$, $\eta(x) = (D^\alpha x : |\alpha| \leq m)$ and $\xi(x) = (D^\beta x : |\beta| \leq m - 1)$.

We consider the following nonlinear optimal control problem:

$$(4)\quad \begin{cases} \int_0^b \int_Z L(t,z,x(t,z),u(t,z))dzdt \to \inf = m \\ \text{such that} \\ \frac{\partial x}{\partial t} - \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,z,\eta(x(t,z)))+ \\ \qquad f(t,z,x,\xi(x(t,z)))u(t,z) = h(t,z) \\ \text{a.e on } \ T\times Z \\ |u(t,z)| \le M, \quad \text{a.e. on } \ T\times Z,\ u(.,.) \text{ measurable.} \end{cases}$$

We will need the following hypotheses on the data of (4),

$H(A)_1$: $A_\alpha : T\times Z\times R^{N_m} \to R$ $\left(N_m = \frac{(N+M)!}{N!m!}\right)$ are function such that

i) $\forall \eta \in R^{N_m}$, $(t,z) \mapsto A_\alpha(t,z,\eta)$ is measurable;

ii) $\forall (t,z) \in T\times Z$, $\eta \to A_\alpha(t,z,\eta)$ is continuous;

iii) $\sum_{|\alpha|\le m} (A_\alpha(t,z,\eta) - A_\alpha(t,z,\eta'))(\eta_\alpha - \eta'_\alpha) \ge 0$, $\forall (t,z) \in T\times Z$, $\forall \eta, \eta' \in R^{N_m}$;

iv) $\exists a_1 \in L^q(T\times Z, R_0^+))$ and $c_1 \in L^\infty(T, R_0^+)$:

$$|A_\alpha(t,z,\eta)| \le a_1(t,z) + c_1(t)\|\eta\|^{p-1},$$

a.e. on $T\times Z, \forall \eta \in R^{N_m}$, $\forall \alpha : |\alpha| \le m$,

v) $\exists c > 0 : \sum_{|\alpha|\le m} A_\alpha(t,z,\eta) \ge c\|\eta\|^p$, a.e.on $T\times Z$, $\forall \eta \in R^{N_m}$.

$H(f)_1$: $f : T\times Z\times R\times R^{N'_m} \to R$, where $N'_m = N_{m-1} - 1 = \frac{(N+m-1)!}{N!(m-1)!} - 1$, is a function such that

j) $\forall (x,\xi) \in R\times R^{N'_m}$, $(t,z) \mapsto f(t,z,x,\xi)$ is measurable;

jj) $\forall (t,z) \in T\times Z$, $(x,\xi) \mapsto f(t,z,x,\xi)$ is continuous;

jjj) $\exists a_2 \in L^q(T, L^2(Z, R_0^+))$ and $c_2 \in L^\infty(T, R_0^+)$:

$$|f(t,z,x,\xi)| \le a_2(t,z) + c_2(t)(|x|^{(p-1)/2} + \|\xi\|^{(p-1)/2},$$

a.e. on $T \times Z$.

jv) $\exists \hat{c} > 0$: $f(t,z,x,\xi)x \ge -\hat{c}$, a.e. on $T \times Z$, $\forall x \in R$, $\forall \xi \in R^{N'_m}$.

$H(L)$: $L : T \times Z \times R \times R \to R \cup \{+\infty\}$ is a function such that

1) $(t,z,x,u) \mapsto L(t,z,x,u)$ is measurable;
2) $\forall (t,z) \in T \times Z$, $(x,u) \mapsto L(t,z,x,u)$ is l.s.c.;
3) $\forall (t,z,x) \in T \times Z \times R$, $u \mapsto L(t,z,x,u)$ is convex;
4) $\exists \nu \in L^1(T \times Z, R)$ and $\beta > 0$:

$$L(t,z,x,u) \ge \nu(t,z) - \beta(|x| + |u|), \quad \text{a.e. on } T \times Z.$$

THEOREM 5.2. *If hypotheses $H(A)_1$, $H(f)_1$, $H(L)$ hold, $x_0 \in L^2(Z, R)$ and $h \in L^2(T \times Z, R)$, then problem (4) has an optimal solution*

$$[x,u] \in (L^p(T, W_0^{m,p}(Z,R)) \cap C(T, L^2(Z,R))) \times L^\infty(T \times Z, R).$$

Proof. In this case the evolution triple consists of $X = W_0^{m,p}(Z,R)$, $H = L^2(Z,R)$ and $X^* = W^{-m,q}(Z,R)$. Again the Sobolev embedding theorem tells us that X embeds compactly into H.

Let $A : T \times X \to X^*$ be defined by

$$\langle A(t,x), y\rangle = \sum_{|\alpha| \le m} A_\alpha(t,z,\eta(x(z)))D^\alpha y(z)dz.$$

Using hypothesis $H(A)_1$ it is routine that $A(t,x)$ satisfies hypothesis $H(A)$.

Then we define $F : T \times X \to P_{fc}(H)$ by

$$F(t,x) = \{v \in H : v(z) = f(t,z,x(z),\xi(x(z)))u(z)$$
$$\text{a.e. on } Z,\ u \text{ meas.}, |u(z)| \le M, \text{ a.e. on } Z\}$$

then we can easily see that $F(t,x)$ satisfies hypothesis $H(F)$.

Finally let $\hat{L} : T \times H \times H \to R \cup \{+\infty\}$ be defined by

$$\hat{L}(t,x,u) = \int_Z L(t,z,x(z),u(z))dz.$$

From our conditions it is possible to find a sequence of Caratheodory integrands $L_k : T \times Z \times R \times R \to R$ such that $\hat{\nu}(t,z) - \beta(|x|+|u|) \leq L_k(t,z,x,u) \leq k$ (where $\hat{\nu}(t,z) = \min\{\nu(t,z), 1\}$) and $L_k \uparrow L$. Set $\hat{L}_k(t,x,u) = \int_Z L_k(t,z,x(z),u(z))dz$. Evidently by Fubini's theorem and by the dominated convergence theorem, $(t,x,u) \mapsto \hat{L}_k(t,x,u)$, $k \geq 1$, is a Caratheodory function, hence jointly measurable. Since by the monotone convergence theorem we have that $\hat{L}_k \uparrow L$ as $k \to \infty$, we deduce that $(t,x,u) \mapsto \hat{L}(t,x,u)$ is measurable. Also by Fatou's lemma $(x,u) \to \hat{L}(t,x,u)$ is lower semicontinuous and $u \mapsto \hat{L}(t,x,u)$ is convex (cf. hypothesis $H(L)$-3). Moreover

$\hat{L}(t,x,u) \geq \tilde{\nu}(t) - \tilde{\beta}(|x|+|u|)$, a.e. on T with $\tilde{\nu} \in L^1(T,R)$, $\tilde{\beta} > 0$. Then the cost functional of (4) is equivalent to

$$J(x,u) = \int_0^b \hat{L}(t,x(t),u(t))dt$$

which, by theorem 2.1 of Balder (cf. [2]), is sequentially lower semicontinuous on $L^1(T,H) \times L^1(T,H)_w$.

Now let $[x_n, u_n]$, $n \geq 1$, be a minimizing sequence of admissible state-control pairs of (4). But $x_n \in L^p(T, W_0^{m,p}(Z,R)) \cap C(T, L^2(Z,R))$, $n \geq 1$, are solutions of the equivalent evolution inclusion

$$(5.2) \qquad \begin{cases} \dot{x}(t) + A(t,x(t)) + F(t,x(t)) \ni \hat{h}(t), & \text{a.e. on } T \\ x(0) = x_0, \end{cases}$$

where $\hat{h}(t)(.) = h(t,.)$.

Since the solution set of (5.2) is compact in $L^p(T,H)$ we may assume that $x_n \to x$ in $L^p(T,H)$. Also we can say that $u_n \to u$ weakly in $L^1(T,H)$.

Thus $J(x,u) \leq \liminf_{n\to\infty} J(x,u_n) = m \Rightarrow J(x,u) = m$, i.e. $[x,u]$ is optimal.

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Nikolaos S. Papageorgiou
National Technical University
Department of Mathematics
Zografou Campus
Athens 15773, Greece

Francesca Papalini
Department of Mathematics
of Perugia University
Via Vanvitelli, 1 - Perugia, 06123 Italy

Francesca Renzacci
Department of Mathematics
of Perugia University
Via Vanvitelli, 1 - Perugia, 06123 Italy