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FINITELY ADDITIVE MEASURES ON GROUPS AND RINGS

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On topological groups a natural finitely additive measure can be defined via compactifications. It is closely related to Hartman's concept of uniform distribution on non-compact groups (cf. [3]). Applications to several situations are possible. Some results of M. Paštéka and other authors on uniform distribution with respect to translation invariant finitely additive probability measures on Dedekind domains are transferred to more general situations. Furthermore it is shown that the range of a polynomial of degree ≥ 2 on a ring of algebraic integers has measure 0.

1. Introduction.

A sequence $\mathbf{x} = (x_n)_{n\in\mathbf{N}}$ of integers $x_n \in \mathbf{Z}$ is called uniformly distributed if it is uniformly distributed mod m on each of the finite rings $\mathbf{Z}_m = \mathbf{Z}/(m)$, $(m) = m\mathbf{Z}$. To this concept there corresponds a finitely additive measure on a certain subsystem of the power set of $\mathbf{Z}$. It is the completion of the system of finite unions of remainder classes of the form $k+(m)$ with respect to the finitely additive probability measure μ generated by the requirement $\mu(k + (m)) = m^{-1}$. There is a vast literature on generalizations of these concepts to more general classes of rings R (cf. references). Instead of $(m) \subseteq \mathbf{Z}$ one considers ideals $I \trianglelefteq R$ ($I \triangleleft R$ if the inclusion is strict) with finite index $\#R/I$ and assigns to the classes $r + I$ the measure (or norm) $(\#R/I)^{-1}$, cf. [10], [11], [12].

Here we continue such investigations but take the following point of view. To each family I_j, $j \in J$, of ideals with finite index (norm) there corresponds an embedding

$$\iota : R \to \prod_{j \in J} R/I_j, \quad r \mapsto (r + I_j)_{j \in J},$$

of R into a compact topological ring. Let $C = \overline{\iota(R)}$ be the topological closure of $\iota(R)$ which is again a compact topological ring. C has a natural measure theoretic structure given by the Haar measure μ on the compact group $(C, +)$. For the theory of uniform distribution the suitable concept of measurable sets is that of μ-continuity sets M which are defined by the property $\mu(\delta M) = 0$ for their topological boundary δM. The preimages $\iota^{-1}(M)$ of μ-continuity sets M form a set algebra (in general not a σ-algebra) which essentially coincides with the system of measurable sets from the first paragraph if all ideals with finite index occur in the product. This is part of Theorem 4 which also asserts that the measures induced by the two constructions coincide.

Note that the embedding ι is not necessarily injective. As an example take any infinite field F, where the only ideal with finite index is F itself. Hence the embedding is the trivial map onto the one element ring and the concept is trivial.

By Pontrjagin's duality (for the structure theory of locally compact abelian groups cf. [4]) this cannot happen if one considers compactifications of the additive group. In terms of projective properties one may compare compactifications in such a way that there is a maximal compactification, the well known Bohr compactification. Among all compactifications the Bohr compactification gives the maximal system of measurable sets. Theorem 1 shows that compactifications, measures and measurability fit together in a natural way. All these ideas are carried out in section 2.

In section 3 the concept is compared with other approaches. In Theorem 2 we show that in the case that R is the ring of integers each set which is measurable in the sense of compactifications has a density equal to its measure. The converse is not true as Theorem 3 implies. There are sets having a density which are not measurable.

Theorems 4, 5 and 6 are devoted to the equivalence of the approaches via ideal measures and via compactifications for rings with unity. We conclude section 3 with some examples, especially with a detailed investigation of completion of a ring with unity with respect to a natural metric. This is closely connected with questions concerning uniformly distributed sequences in such rings. In the final section 4 the ideal measures of special sets are computed. This includes the result that the range of a polynomial of degree at least 2 on an algebraic ring of integers has measure 0.

Our results are related to invariant means on topological groups. A deeper understanding of the connection of both approaches should be the aim of investigations in the future.

As a general agreement we suppose all topological spaces to satisfy the Hausdorff separation axiom.

2. Compactifications and measures.

2.1. *Several notions and facts on compactifications*

In the following we present a brief outline on compactifications of topological groups. (Generalizations to a larger class of algebraic structures are possible.) For a more detailed description of the constructions see for instance [2], page 71. Note the analogies with arbitrary compactifications of completely regular topological spaces, especially with the Stone-Čech compactification which is the maximal one.

Let G be any topological group. A pair (C, ι) is called a compactification of G if $\iota : G \to C$ is a continuous homomorphism, C is a compact group and $\iota(G)$ is dense in C. (In this context we do not require that ι is a homeomorphic imbedding.) The compactification is called injective if ι is injective. If one likes, one can force injectivity by considering $G/\ker\iota$ instead of G. (C_1, ι_1) is called smaller than (C_2, ι_2) (we write $(C_1, \iota_2) \leq (C_2, \iota_2)$ via φ) if there is a continuous epimorphism $\varphi : C_2 \to C_1$ such that $\varphi\iota_2 = \iota_1$. The relation $\leq$ is reflexive and transitive. Hence there is a natural notion of equivalence of compactifications: (C_1, ι_1) and (C_2, ι_2) are called equivalent if there exist φ and ψ such that $(C_1, \iota_1) \leq (C_2, \iota_2)$ via φ and $(C_2, \iota_2) \leq (C_1, \iota_1)$ via ψ. In this case $\psi = \varphi^{-1}$. (Obviously this could also be expressed in terms of categories.)

By standard cardinality arguments on appropriate systems of filters on G the equivalence classes of compactifications of G can be represented by a set. Hence we are allowed to fix a set $\mathcal{C}$ which contains exactly one representative of each equivalence class of compactifications of G. Any compactification may be identified with its equivalent copy in $\mathcal{C}$.

The relation $\leq$ is a partial order relation on $\mathcal{C}$. But far more is true: If (C_i, ι_i), $i \in I$, is any family of compactifications we may consider the direct product. Let $P = \prod_{i \in I} C_i$ and define the continuous homomorphism $\iota : G \to P$ by $g \mapsto (\iota_i(g))_{i \in I}$. If C is the closure of $\iota(G)$ in the compact group P then (C, ι) turns out to be the least common upper bound of the (C_i, ι_i). Note that the trivial compactification onto the one element group is the least upper bound of the empty set. Thus $(\mathcal{C}, \leq)$ in fact is a complete lattice. (Since we do not need this fact in full generality we do without the somewhat tedious proof.) In particular there is a maximal compactification, called the Bohr compactification $(bG, b\iota)$. It follows from Pontrjagin's duality that, if G is a discrete abelian or, more generally, a locally compact abelian group, the Bohr compactification is injective and can be obtained by taking the dual $\widehat{\widehat{G}_d}$ of the discretely topologized dual $\widehat{G}_d$ of G.

For us there is a second compactification of special interest. If one considers compactifications (C, ι) of the additive group $(R, +)$ of a ring it might be convenient to extend the ring multiplication from R to C in a continuous way. If this is possible, the continuation of ring multiplication (as addition) is, by density, uniquely determined by ι. We then call (C, ι) a ring compactification. The maximal ring compactification - we denote it by rbR- may happen to be not injective as the extreme example of an infinite field F shows where $rbF = \{0\}$. This follows from the fact that finite fields are the only compact topological rings that are fields, because compact rings with identity have an ideal topology, cf. [17] 32.3 and 32.5. This means that there is a neighbourhood base for $0 \in R$ consisting of clopen (= closed and open) ideals.

2.2. *Compactifications and finitely additive measures.*

On every compact group C there is a unique regular probability measure which is a left and right invariant Borel measure. Let μ_C denote its completion which is called the Haar measure. For our purposes this gives rise to the following notions:

Let G be any topological group and (C, ι) a compactification of G. Then we call a subset $T \subseteq G$ measurable with respect to (C, ι) if it is the preimage under ι of a μ_C-continuity set M, i.e. $T = \iota^{-1}(M)$ with $\mu_C(\bar{M} \cap \overline{(C \backslash M)}) = 0$. Thus a set $M \subseteq C$ is a μ_C-continuity set if and only if its topological boundary δM is a zero set. It is easy to check that the system $\mathcal{S}_C$ of all μ_C-continuity sets in C and the system $\mathcal{S}_{(C,\iota)}$ of their preimages $T \subseteq G$ - which we call the sets measurable with respect to, for short w.r.t., (C, ι) - form a set algebra (in general not a σ-algebra) on C resp. on G. Obviously every $T \in \mathcal{S}_{(C,\iota)}$ satisfies $T = \iota^{-1}(\iota(T))$. Note that $T \in \mathcal{S}_{(C,\iota)}$ implies that $\overline{\iota(T)} \in \mathcal{S}_C$ but that the converse is not true. (Consider for instance the compact group $G = C = \mathbf{R}/\mathbf{Z}$, ι the identity and T a dense set which is not a continuity set.) Considering the open kernel of $\overline{\iota(T)}$ and of its complement which, together, form the complement of its boundary, we observe for $T = \iota^{-1}(\iota(T))$: $T \in \mathcal{S}_{(C,\iota)}$ if and only if there are disjoint open sets O_1, $O_2 \subseteq C$ with $\iota(T) \cap O_2 = \emptyset$, $\iota(G \backslash T) \cap O_1 = \emptyset$ and $\mu_C(O_1) + \mu_C(O_2) = 1$. The definition $\mu_{(C,\iota)}(T) = \mu_C\overline{(l(T))}$ or, equivalently, $\mu_{(C,\iota)}(\iota^{-1}(M)) = \mu_C(M)$, transfers the measure μ_C on $\mathcal{S}_C$ to the system $\mathcal{S}_{(C,\iota)}$, and gives the natural finitely additive measure on G w.r.t. (C, ι), defined for all $T \in \mathcal{S}_{(C,\iota)}$. A similar situation is investigated in the papers of Paštéka.

If the compactification is not injective then the measure $\mu_{(C,\iota)}$ is in general not complete. Nevertheless, since μ_C is complete, the following similar statement holds: If the family $T_i \in \mathcal{S}_{(C,\iota)}$, $i \in I$, satisfies $\inf_{i \in I} \mu_{(C,\iota)}(T_i) = 0$, then every T with $T = \iota^{-1}(\iota(T))$ which is contained in the intersection of the T_i is in $\mathcal{S}_{(C,\iota)}$ and has measure 0. To see this, take closed sets $M_i = \overline{\iota(T_i)} \in \mathcal{S}_C$ satisfying $T_i \subseteq \iota^{-1}(M_i)$ and $\mu_C(M_i) = \mu_{(C,\iota)}(T_i)$. Since C is fixed we may consider the intersection M of the M_i. M is a closed set of measure 0 which contains $\overline{\iota(T)} \supseteq \iota(T)$. Thus $\iota(T) \in \mathcal{S}_C$ since μ_C is complete, implying $T = \iota^{-1}(\iota(M)) \in \mathcal{S}_{(C,\iota)}$ with measure 0.

The maximal compactification (Bohr compactification) $(C, \iota) = (bG, b\iota)$ of G will play a special rôle. Hence we write μ_G for $\mu_{(C,\iota)}$.

2.3. *Compatibility of compactifications and measures.*

As expected, the measure $\mu_{(C,\iota)}(T)$ of a set $T \subseteq G$ does not depend on the compactification (C, ι), in the following sense:

THEOREM 1. *Let* (C_1, ι_1) *and* (C_2, ι_2) *be compactifications of* G. *On the intersection* $\mathcal{S}_{(C_1,\iota_1)} \cap \mathcal{S}_{(C_2,\iota_2)}$ *the measures* $\mu_1 = \mu_{(C_1,\iota_1)}$ *and* $\mu_2 = \mu_{(C_2,\iota_2)}$ *coincide.* $(C_1, \iota_1) \leq (C_2, \iota_2)$ *via* φ *implies* $\mathcal{S}_{(C_1,\iota_1)} \subseteq \mathcal{S}_{(C_2,\iota_2)}$. *In this case,* $M \in \mathcal{S}_{C_1}$ *implies* $\varphi^{-1}(M) \in \mathcal{S}_{C_2}$.

Proof. We proceed in several steps. Under the additional assumption $(C_1, \iota_1) \leq (C_2, \iota_2)$ via φ we prove the following facts.

1. $\overline{\varphi^{-1}(M)} \subseteq \varphi^{-1}(\bar{M})$ for every $M \subseteq C_1$: Continuity of φ.

2. $\delta(\varphi^{-1}(M)) \subseteq \varphi^{-1}(\delta(M))$ for all $M \subseteq C_1$: Using the first fact we observe

$$\delta(\varphi^{-1}(M)) = \overline{\varphi^{-1}(M)} \cap \overline{C_2 \backslash \varphi^{-1}(M)} \subseteq \varphi^{-1}(\bar{M}) \cap \varphi^{-1}\overline{(C_1 \backslash M)} =$$

$$= \varphi^{-1}(\bar{M} \cap \overline{C_1 \backslash M}) = \varphi^{-1}(\delta M).$$

3. $\mu_{C_2}(\varphi^{-1}(M)) = \mu_{C_1}(M)$ for every measurable set $M \subseteq C_1$: The left hand side of the equality defines a translation invariant probability measure on the Borel sets of C_1, hence has to coincide with the unique Haar measure μ_{C_1}.

4. $M \in \mathcal{S}_{C_1}$ implies $\varphi^{-1}(M) \in \mathcal{S}_{C_2}$: By the assumption, $\mu_1(\delta(M)) = 0$, hence by 2. and 3.

$$\mu_{C_2}(\delta(\varphi^{-1}(M)) \leq \mu_{C_2}(\varphi^{-1}(\delta(M))) = \mu_{C_1}(\delta(M)) = 0.$$

5. $\mathcal{S}_{C_1,\iota_1} \subseteq \mathcal{S}_{C_2,\iota_2}$: By the criterion in Section 2.2 $T \in \mathcal{S}_{C_1,\iota}$ implies that there are disjoint open sets $O_1, O_2 \subseteq C_1$ with $\iota_1(T) \cap O_2 = \emptyset = \iota_1(G \backslash T) \cap O_1$ and $\mu_{C_1}(O_1) + \mu_{C_1}(O_2) = 1$. Using 4. we get that $O_i' := \varphi^{-1}(O_i)$, $i = 1, 2$, play the same rôle in C_2. $\ker \iota_2 \subseteq \ker \varphi\iota_2 = \ker \iota_1$ and $\iota_1^{-1}\iota_1(T) = T$ implies $\iota_2^{-1}\iota_2(T) = T$. Thus, again by the same criterion, we conclude $T \in \mathcal{S}_{(C_2,\iota_2)}$.

6. $T \in \mathcal{S}_{(C_1,\iota_1)}$ implies $\mu_1(T) = \mu_2(T)$: By 5., $T \in \mathcal{S}_{(C_2,\iota_2)}$. Hence for both values $i = 1, 2$ we have the relation

$$\mu_{C_i}\overline{(\iota_i(T))} + \mu_{C_i}\overline{(\iota_i(G \backslash T))} = 1.$$

Using 1. we get

$$\overline{\iota_2(T)} \subseteq \overline{\varphi^{-1}\varphi\iota_2(T)} \subseteq \varphi^{-1}\overline{(\varphi\iota_2(T))} = \varphi^{-1}\overline{(\iota_1(T))},$$

hence by 3. $\mu_{C_2}\overline{(\iota_2(T))} \leq \mu_{C_1}\overline{(\iota_1(T))}$. The same holds for $G\backslash T$ instead of T which, together with the above relations, is possible only if $\mu_1(T) = \mu_2(T)$.

We have proved everything for the case $(C_1, \iota_1) \leq (C_2, \iota_2)$. The general case follows since two compactifications have a common upper bound, for instance $(bG, b\iota_G)$ with measure μ_G:

$$\mu_1(T) = \mu_G(T) = \mu_2(T) \qquad \blacksquare$$

Theorem 1 has the following consequence for further investigations. If a set $T \subseteq G$ is measurable w.r.t. any compactification (C, ι) then it is measurable w.r.t. all bigger compactifications. The value $\mu_{(C,\iota)}(M)$ does not depend on the compactification as long as T is measurable w.r.t. it. Thus the Bohr compactification and the corresponding finitely additive probability measure $\mu_G = \mu_{(bG,b\iota)}$ tells us everything about measures of sets $T \subseteq G$. Let us call μ_G the *Hartman measure* on G, cf. [3], and the corresponding measurable sets $T \subseteq G$ *group measurable* or *Hartman measurable.* If we replace the Bohr compactification by the ring Bohr compactification we call the measurable sets *ring measurable.*

We are interested in measurability properties of subsets $T \subseteq G$. The concepts are nontrivial since it is possible to construct sets $T \subseteq G$ which are not Hartman measurable, which follows from Theorem 3. Furthermore there are Hartman measurable sets that are not ring measurable. Consider for instance a ring which is an infinite discrete field. In this case Bohr and ring Bohr compactification are non-equivalent in an extreme way.

3. Comparing several concepts.

3.1. *Sets of integers: Hartman measurability and density.*

In the special case $R = \mathbf{Z}$ one has a further nontrivial and natural finitely additive measure, the density. For a set $T \subseteq \mathbf{Z}$ of integers consider, for any finite set $S \subseteq \mathbf{Z}$, the number $A(T, S) = \dfrac{\#T \cap S}{\#S}$. If, for the sets $S_N = \{n \in \mathbf{N} | n \leq N\}$ and $-S_N = \{-n | n \in S_N\}$, both sequences $A(T, S_N)$ and $A(T, -S_N)$ tend, for $N \to \infty$, to the same limit, we denote this common limit by $\mathrm{dens}(T)$ and call it the density of T.

It turns out that every Hartman measurable set has a density coinciding with its Hartman measure. The converse is not true, since the density of Hartman measurable sets is even uniform in the following sense:

Let $T \subseteq \mathbf{Z}$ be a set of integers. We say that T has the (unique) uniform density $\mathrm{dens}(T)$ (also called Banach density) if the following holds: For every $\varepsilon > 0$ there is a positive integer N_ε such that every set $I_{k_1,k_2} = \{n \in \mathbf{Z} | k_1 < n \leq k_2\}$ with $k_2 - k_1 \geq N_\varepsilon$ fulfills

$$\mathrm{dens}(T) - \varepsilon \leq A(T, I_{k_1,k_2}) \leq \mathrm{dens}(T) + \varepsilon.$$

Of course if the uniform density exists then also the density of T exists and both values are equal.

THEOREM 2. *Every Hartman measurable set $T \subseteq \mathbf{Z}$ of integers has a uniform density $dens(T)$ with $dens(T) = \mu_G(T)$.*

Proof. Consider the Bohr compactification $(C, \iota) = (b\mathbf{Z}, b\iota)$ which can be realized by

$$\iota(k) = (k\alpha)_{\alpha \in \mathbf{R}/\mathbf{Z}} \in C \subseteq \prod_{\alpha \in \mathbf{R}/\mathbf{Z}} (\overline{\chi_\alpha(\mathbf{Z})}, \chi_\alpha).$$

Here every character $\chi_\alpha : G \to \mathbf{R}/\mathbf{Z}$, $k \mapsto k\alpha$, of the integers represents a compactification. A topological base $\mathcal{B}$ is given by all sets $B \subseteq C$ of the type

$$B = B(\alpha_1, \ldots, \alpha_k, I_1, \ldots, I_k) = \{(x_\alpha)_{\alpha \in \mathbf{R}/\mathbf{Z}} \in C | x_{\alpha_j} \in I_j,\ j = 1, \ldots, k\},$$

where $k \in \mathbf{N}$, $\alpha_j \in \mathbf{R}/\mathbf{Z}$, and the I_j are connected open subsets (intervals) in $\mathbf{R}/\mathbf{Z}$. Note that all these base sets are μ-continuity sets where μ is the Haar measure on C. The same is true if we consider the base sets of a smaller compactification (instead of (C, ι)) where not all $\alpha \in \mathbf{R}/\mathbf{Z}$ occur.

Suppose $T \in \mathcal{S} = \mathcal{S}_{(C,\iota)}$, i.e. $T = \iota^{-1}(M)$ for some $M \subseteq C$ with $\mu(\delta M) = 0$. This means $\mu(M^0) = \mu(M) = \mu(\bar{M})$ for the open kernel M^0 and the closure $\bar{M}$.

First we show, for given $\varepsilon > 0$, that there are μ-continuity sets M_i, $i = 0, 1$, being finite unions of base sets such that $M_0 \subseteq M \subseteq M_1$ and $\mu(M_1 \setminus M_0) < \varepsilon/2$. To see this use regularity of μ to get a compact set K and open sets O and V with

$$K \subseteq O \subseteq \bar{O} \subseteq M^0 \subseteq \bar{M} \subseteq V$$

and $\mu(V\backslash K) < \varepsilon/2$. Consider a finite covering of $\bar{O}$ by open base sets $B_i \subseteq M^0$, $i = 1, \ldots, n_1$, and put $M_0 = \bigcup_{i=1}^{n_1} B_i$. Add a finite covering of $\bar{M}\backslash O$ by sets $B_i \subseteq V\backslash K$, $i = n_1+1, \ldots, n_1+n_2$, and put $M_1 = M_0 \cup U$ with $U = \bigcup_{i=n_1+1}^{n_1+n_2} B_i$. Since all base sets are continuity sets M_0 and M_1 have the required properties. Note furthermore that $\mu(U) < \varepsilon/2$.

The definition of the involved sets uses only finitely many B_i, each of them involving only finitely many α_j, $j = 1, \ldots, s$. Thus we may consider the projection $\pi : C \to C_\varepsilon = \overline{\pi(C)}$, $\iota_\varepsilon = \pi\iota$, onto the occuring components corresponding to α_j, $j = 1, \ldots, s$. Hence $(C_\varepsilon, \iota_\varepsilon)$ is a finite dimensional compactification of $\mathbf{Z}$, generated by the characters χ_{α_j}, $j = 1, \ldots, s$. This means that $C_\varepsilon \subseteq (\mathbf{R}/\mathbf{Z})^s$ is (in the topological sense) generated by the element $\alpha = (\alpha_1, \ldots, \alpha_s) \in C_\varepsilon$. Write $A' = \pi(A)$ for arbitrary $A \subseteq C$. Since all B_i depend only on the components corresponding to the α_j, $j = 1, \ldots, s$, we have $\pi^{-1}(B_i') = B_i$ for $i = 1, \ldots, n_1 + n_2$. This implies $\pi^{-1}(U') = U$ and $\pi^{-1}(M_i') = M_i$ for $i = 0, 1$. With $T_i = \iota^{-1}(M_i)$ we conclude

$$\mu_G(T_0) = \mu_{C_\varepsilon}(M_0') \leq \mu_G(T) \leq \mu_G(T_1) = \mu_{C_\varepsilon}(M_1').$$

It is known from the theory of uniform distribution on monothetic groups (cf. [5], p. 269, Corollary 4.1) that the sequence $(k\alpha)$ is well distributed in C_ε for every generating element α. In our case this means that there is a positive integer N_ε such that for all $k \in \mathbf{Z}$ and all $N \geq N_\varepsilon$ we have

$$\mu_{C_\varepsilon}(M_i') - \varepsilon/2 < \frac{1}{N}\#\{n \in \mathbf{Z} | k < n \leq k + N, n\alpha \in M_i'\} < \mu_{C_\varepsilon}(M_i') + \varepsilon/2$$

for $i = 0, 1$. Furthermore we have

$$\mu_{C_\varepsilon}(M_1') \leq \mu_{C_\varepsilon}(M_0') + \mu_{C_\varepsilon}(U') < \mu_{C_\varepsilon}(M_0') + \frac{\varepsilon}{2}.$$

Thus we get for every $I = I_{k_1,k_2}$ with $k_2 - k_1 \geq N_\varepsilon$

$$\frac{\#I \cap T}{\#I} \leq \frac{\#I \cap T_1}{\#I} < \mu_{C_\varepsilon}(T_1) + \frac{\varepsilon}{2} < \mu_{C_\varepsilon}(M_0') + \varepsilon < \mu_G(T) + \varepsilon,$$

and similarly $\frac{\#I \cap T}{\#I} \geq \mu_G(T) - \varepsilon$, proving the theorem. ■

As a corollary we get that sets with a density are not necessarily measurable.

THEOREM 3. *There are sets of integers which have a density but are not Hartman measurable.*

Proof. Since the density of a Hartman measurable set is uniform by Theorem 2, it suffices to construct a set T which has a density but not a uniform density. Take $T_k = \{k^2, k^2+1, ..., k^2+k\}$, $T^+ = \bigcup_{k \in \mathbb{N}} T_k$, $T^- = \{-n | n \in T^+\}$ and $T = T^+ \cup T^-$ then it is easy to check that T has the density $\operatorname{dens}(T) = 1/2$ which is not uniform. ■

3.2. *Compactifications and families of ideals of finite index.*

In this subsection we show that the approach via ring compactifications is equivalent to that via ideals of finite index.

For an arbitrary topological ring R with identity let $\mathcal{J} = \{I_j | j \in J\}$ be a family of clopen ideals $I_j \trianglelefteq R$ of finite index $\#R/I_j$ and suppose that $\mathcal{J}$ is closed under intersections. It is known (cf. for instance [10], [11]) that $\mathcal{J}$, if it is closed under finite intersections, defines a finitely additive measure $\mu_{\mathcal{J}}$ on R in the following way:

For every subset $T \subseteq R$ which is a finite union $T = \bigcup_{i=1}^{n} r_i + I_{j_i}$ with pairwise disjoint $r_i + I_{j_i}$ the number $\mu_{\mathcal{J}}(T) = \sum_{i=1}^{n} \#R/I_{j_i}$ is independent of the representation of T. Let us call such sets $\mathcal{J}$-definable. The set function $\mu_{\mathcal{J}}$ is a finitely additive measure $\mu_{\mathcal{J}}$ on the set algebra of $\mathcal{J}$-definable sets. $\mu_{\mathcal{J}}$ can be uniquely extended to the so-called ideal measure (induced by $\mathcal{J}$) on the set algebra of all subsets $T \subseteq R$ which can be approximated in the following sense:

T is called measurable w.r.t. $\mathcal{J}$ if there is a (unique) number $\mu_{\mathcal{J}}(T)$ such that for each $\varepsilon > 0$ there are $\mathcal{J}$-definable sets A_ε, $B_\varepsilon \subseteq R$ with $A_\varepsilon \subseteq T \subseteq B_\varepsilon$ and

$$\mu_{\mathcal{J}}(T) - \varepsilon < \mu_{\mathcal{J}}(A_\varepsilon) \leq \mu_{\mathcal{J}}(B_\varepsilon) < \mu_{\mathcal{J}}(T) + \varepsilon.$$

We show that this approach essentially leads to the same concepts as the compactification $(C_{\mathcal{J}}, \iota_{\mathcal{J}})$ defined by

$$\iota_{\mathcal{J}} : R \to \prod_{j \in J} R/I_j, \qquad r \mapsto (r + I_j)_{j \in J}.$$

For notational convenience call $\mathcal{J}$ point separating if $\bigcap_{j \in J} I_j = \{0\}$. Note that this is the case if and only if $\iota_{\mathcal{J}}$ is injective if and only if $T = \iota_{\mathcal{J}}^{-1} \iota_{\mathcal{J}}(T)$ for all $T \subseteq R$.

THEOREM 4. *Suppose* $T \subseteq R$. *If* $T \in \mathcal{S}_{(C_{\mathcal{J}}, \iota_{\mathcal{J}})}$ *then it is measurable w.r.t.* $\mathcal{J}$. *The converse holds if* $T = \iota_{\mathcal{J}}^{-1} \iota_{\mathcal{J}}(T)$. *If both values* $\mu_{\mathcal{J}}(T)$ *and* $\mu_{(C_{\mathcal{J}}, \iota_{\mathcal{J}})}\mathcal{J}(T)$ *are defined they coincide. Hence* $\mu_{\mathcal{J}} = \mu_{(C_{\mathcal{J}}, \iota_{\mathcal{J}})}$ *if* $\mathcal{J}$ *is point separating.*

Proof. First note that the $\mathcal{J}$-definable sets $T \subseteq R$ can be represented as preimages $T = \iota^{-1}(M_T)$ of clopen sets M_T forming a topological base of $C_{\mathcal{J}}$. A set is clopen if and only if it has empty topological border. Hence all these sets are measurable w.r.t. $\mathcal{J}$ as well as w.r.t. the compactification $(C_{\mathcal{J}}, \iota_{\mathcal{J}})$. By translation invariance of the Haar measure it is obvious that $\mu_{\mathcal{J}}(T) = \mu_{(C_{\mathcal{J}}, \iota_{\mathcal{J}})}(T)$ for such sets.

Assume now that $T = \iota^{-1}(M_T)$ is measurable w.r.t. the compactification $(C_{\mathcal{J}}, \iota_{\mathcal{J}})$ with $\mu_{C_{\mathcal{J}}}(\delta M_T) = 0$. In the same way as in the proof of Theorem 2 one finds, for $\varepsilon > 0$, finite unions M_0 and M_1 of clopen base sets such that $M_0 \subseteq M_T \subseteq M_1$ and $\mu_{C_{\mathcal{J}}}(M_1 \backslash M_0) < \varepsilon$. The preimages $T_i = \iota^{-1}(M_i)$, $i = 0, 1$, are $\mathcal{J}$-definable and play the rôle of A_ε and B_ε in the definition of measurability w.r.t. $\mathcal{J}$, showing that T has this property. Furthermore this argument shows that in this case the measures $\mu_{\mathcal{J}}(T) = \mu_{(C_{\mathcal{J}}, \iota_{\mathcal{J}})}(T)$ coincide.

It remains to prove that $T \in \mathcal{S}_{(C_{\mathcal{J}}, \iota_{\mathcal{J}})}$, provided that T is measurable w.r.t. $\mathcal{J}$ and $T = \iota_{\mathcal{J}}^{-1}(M)$ with $M = \iota_{\mathcal{J}}(T)$. For each $\varepsilon > 0$, there are $\mathcal{J}$-definable sets A_ε and B_ε with $A_\varepsilon \subseteq T \subseteq B_\varepsilon$ and $\mu_{\mathcal{J}}(B_\varepsilon \backslash A_\varepsilon) < \varepsilon$. They can be written as the preimages $A_\varepsilon = \iota^{-1}(M_{A_\varepsilon})$ resp. $B_\varepsilon = \iota^{-1}(M_{B_\varepsilon})$ of clopen sets satisfying $M_{A_\varepsilon} \subseteq M \subseteq M_{B_\varepsilon}$. Note that $\delta M \subseteq M_{B_\varepsilon} \backslash M_{A_\varepsilon}$. Furthermore their Haar measure is equal to the ideal measure. It follows immediately that $\mu_{C_1}(\delta M) = 0$. Thus $T = \iota_{\mathcal{J}}^{-1}(M)$ is measurable with respect to $(C_{\mathcal{J}}, \iota_{\mathcal{J}})$. ■

Remark. For each $I \trianglelefteq R$ $d_I(r_1, r_2) = 0$ if $r_1 - r_2 \in I$ and $d_I(r_1, r_2) = 1$ otherwise defines a pseudometric d_I on R. To a given $\mathcal{J}$ there corresponds the system of all d_I with $I \in \mathcal{J}$. This system is point separating if and only if $\mathcal{J}$ is point separating. The completion with respect to the uniformity of this system of pseudometrics turns out to be equivalent to $(C_{\mathcal{J}}, \iota_{\mathcal{J}})$. The construction and the proof are standard. If $\mathcal{J}$ is countable the system of pseudometrics can be replaced by a metric. This special case is discussed in detail in section 3.3.

It is clear that the class of $\mathcal{J}$-measurable sets increases with $\mathcal{J}$. This is an immediate consequence of the following theorem together with Theorem 1.

THEOREM 5. *$\mathcal{J}_1 \subseteq \mathcal{J}_2$ implies* $(C_{\mathcal{J}_1, \iota_{\mathcal{J}_1}}) \leq (C_{\mathcal{J}_2, \iota_{\mathcal{J}_2}})$.

Proof. As one checks easily, the mapping

$$\varphi : \iota_{\mathcal{J}_2}(R) \to \iota_{\mathcal{J}_1}(R), \ (r + I)_{I \in \mathcal{J}_2} \mapsto (r + I)_{i \in \mathcal{J}_1}$$

is well defined and can be uniquely extended to a continuous epimorphism $C_{\mathcal{J}_2} \to C_{\mathcal{J}_1}$. ■

Hence it remains to investigate the largest possible choice for $\mathcal{J}$. Let $\mathcal{F}$ be the system of all ideals $I \trianglelefteq R$ with finite index. Note that $\mathcal{F}$ is closed under finite intersections (cf. [10], [11]). The situation is explained by the following theorem.

THEOREM 6. *Let R be a ring with identity. Then the ring Bohr compactification $(C, \iota) = (rbR, rb\iota)$ and the compactification $(C_{\mathcal{F}}, \iota_{\mathcal{F}})$ are equivalent.*

Proof. Since $(C_{\mathcal{F}}, \iota_{\mathcal{F}})$ is a ring compactification there is a continuous epimorphism $\varphi : C \to C_{\mathcal{F}}$ from the maximal ring compactification C onto $C_{\mathcal{F}}$ with $\varphi\iota = \iota_{\mathcal{F}}$. Since C is compact, every continuous injection into a Hausdorff space is a homeomorphism. Thus it suffices to prove $\ker\varphi = \{0\}$.

Let U be any open neighbourhood of $0 \in C$. It follows from the structure theory of compact rings with unity (cf. [17], Theorem 32.3 and 32.5) that C has a topological base of clopen ideals which, since C is compact, must have finite index. Let $I \trianglelefteq C$ be such a clopen ideal with

$I \subseteq U$. It follows that $I_0 = \iota^{-1}(I) \trianglelefteq R$ with $|R/I_0| = |C/I|$. Hence $I_0 \in \mathcal{F}$ and $\ker \iota_{\mathcal{F}} \subseteq I_0$. We conclude

$$\iota^{-1}(\ker \varphi) = \ker(\iota\varphi) = \ker \iota_{\mathcal{F}} \subseteq I_0$$

and thus

$$\ker \varphi = \iota\iota^{-1}(\ker \varphi) \subseteq \iota(I_0) = I \subseteq U.$$

Since this holds for an arbitrary neighbourhood U of $0 \in C$ and since we have adopted the Hausdorff separation axiom we have $\ker \varphi = \{0\}$, proving the theorem. ■

Theorem 6 implies that $\mu_{(C_{\mathcal{F}},\iota_{\mathcal{F}})}$ and $\mu_{(rbR,rb\iota)}$ are defined on the same set algebra and therefore, by Theorem 1, coincide. We will call this finitely additive measure the ideal measure on the ring.

3.3. *Examples.*

Let us investigate subsets $T \subseteq \mathbf{Z}$ of the integers and their measurability w.r.t. several compactifications. Take any $\alpha \in \mathbf{R}$ and consider the compactification (C, χ_α) where $C \leq \mathbf{R}/\mathbf{Z}$ is the torus group and χ_α is the unique character with $\chi_\alpha(1) = \alpha + \mathbf{Z}$. If $\alpha = \frac{p}{q}$ is rational with integers $q > 0$ and p relatively prime, then the measurable sets are exactly the unions of residue classes w.r.t. the cyclic factor group $\mathbf{Z}/(q)$ modulo q. One gets the concept of uniform distribution modulo q.

If one looks at the supremum of all (C_α, ι_α) where α runs through all $\frac{1}{p^n}$ where p is a prime number one gets the injective p-adic completion of the integers. All singletons are measurable zero sets. But also all powers of p form a measurable zero-set w.r.t. this compactification.

If $n = \frac{1}{\alpha}$ runs through all positive integers one gets the classical concept of uniform distribution in $\mathbf{Z}$, cf. [8]. This compactification coincides with the ring Bohr compactification and is closely related to the Banach-Buck measure on the positive integers (cf. [1], [5] p. 313-315, [6], [9], [10] and [11]).

In the following we discuss in more detail the situation where R is a ring with 1 and $\mathcal{J} : I_1 \supseteq I_2 \supseteq I_3 \ldots \supseteq I_n \supseteq \ldots$ is a sequence of ideals satisfying

$$\bigcap_{n=1}^{\infty} I_n = \{0\}.$$

A metric $d(x, y) = \|x - y\|$ can be introduced by the usual norm

$$\|x\| = \sum_{n=1}^{\infty} \frac{1 - \chi_{I_n}(x)}{2^n},$$

where χ_E denotes the characteristic function of a set E. If $\mathcal{J}$ is uncountable one has a system of pseudometrics inducing the corresponding uniform structure, cf. section 3.2.

Obviously $d(x_n, y_n) \to 0$ if and only if for every $N \in \mathbf{N}$ there exists an $n_0 = n_0(N)$ such that for all $n \geq n_0$ the relation $x_n \equiv y_n \bmod I_N$ holds. This yields that the ring operations are continuous. Denote by (Ω, d) the completion of the metric space (R, d). (A standard construction as it is carried out in the last section, paragraph 170, in the second volume of van der Waerden's book [16] shows that ring operations can be uniquely extended to Ω continuously.) Denote by $\bar{S}$ the closure of a set S in Ω. Then the following elementary properties can be established.

(i) For any ideal I in R the closure $\bar{I}$ is an ideal in Ω.

(ii) $x + \overline{I_n} = \overline{x + I_n}$ for $x \in R$, $n = 1, 2, \ldots$

(iii) For every $\alpha \in \Omega$ there exists an $x \in R$ such that $\alpha + \overline{I_n} = x + \overline{I_n}$.

(iv) For all $\alpha \in \Omega$, $n = 1, 2, \ldots$ the set $\alpha + \overline{I_n}$ is open and closed.

(v) $(x + \overline{I_n}) \cap R = x + I_n$ for all $x \in R$, $n = 1, 2, \ldots$

(vi) $\bar{S} = \bigcap_{n=1}^{\infty} (S + \overline{I_n})$ for every $S \subseteq \Omega$.

(vii) The system $\{x + \overline{I_n} | x \in \Omega, n = 1, 2, \ldots\}$ is a closed open base in Ω.

(viii) Ω is compact if and only if each factor ring R/I_n is finite.

In the following we suppose that each factor ring is finite, i.e. Ω is compact. Following the general approach of section 2 we consider the group $(\Omega, +)$ with Haar measure μ_Ω. Furthermore we define for arbitrary $S \subseteq R$ a set function $\bar{\mu}_{\mathcal{J}}$ by $\bar{\mu}_{\mathcal{J}}(S) = \mu_\Omega(\bar{S})$. $\bar{\mu}_{\mathcal{J}}$ is a finitely additive outer measure and is called the *covering density* induced by $\mathcal{J}$. It can easily be seen that for $A, B \subseteq R$

$$\bar{\mu}_{\mathcal{J}}(A \cup B) + \bar{\mu}_{\mathcal{J}}(A \cap B) \leq \bar{\mu}_{\mathcal{J}}(A) + \bar{\mu}_{\mathcal{J}}(B).$$

It follows from measure theoretic and topological standard arguments that the system

$$\mathcal{D}_{\mathcal{J}} = \{S \subseteq R | \bar{\mu}_{\mathcal{J}}(S) + \bar{\mu}_{\mathcal{J}}(R \backslash S) = 1\}$$

is a set algebra and the restriction $\bar{\mu}_{\mathcal{J}}$ to $\mathcal{D}_{\mathcal{J}}$ is a finitely additive measure on $\mathcal{D}_{\mathcal{J}}$. (Note that in the finitely additive case the arguments are in fact simpler than in Carathéodory's theory of σ-additive measures. We may omit details.) Denoting by $[S : I_n]$ the number of different residue classes $s + I_n$ with $s \in S$ and putting $N(I_n) = \#R/I_n$ we clearly have

$$\mu_\Omega(\alpha + \overline{I_n}) = \frac{1}{N(I_n)}$$

for $\alpha \in \Omega$, $n = 1, 2, \ldots$. Hence for $S \subseteq R$ the covering density of S can be computed by the limit formula

$$\bar{\mu}_{\mathcal{J}}(S) = \lim_{n \to \infty} \frac{[S : I_n]}{N(I_n)}.$$

Remark. Since every class $\alpha + \overline{I_n}$ is a μ_Ω-continuity set the basic notions of the abstract theory of uniform distribution of sequences can be applied to our situation, and general results in the flavour of [5], chapters 3,4,5 can be shown. For more recent results concerning distribution problems in rings and submeasures we refer to [12]. For instance, a sequence (x_n) in R is called $\mathcal{J}$-well distributed if and only if for each ideal $I_n \in \mathcal{J}$ and $x \in R$ the relation

$$\lim_{m \to \infty} \frac{1}{m} \#\{k | h + 1 \leq k \leq h + m, x_k \equiv x \bmod I_n\} = \frac{1}{N(I_n)}$$

holds uniformly in $h = 1, 2, \ldots$ (cf. also subsection 3.1). Following the ideas of [12] we establish

THEOREM 7. *Let $\mathcal{J}$ be an ideal system as above and $S \subseteq R$ with $\bar{\mu}_{\mathcal{J}}(S) = 1$. Then a $\mathcal{J}$-well distributed sequence can be selected from S.*

Remark. Specific distribution results on linear recurring sequences in Dedekind domains are shown in [13] and [14].

4. Special sets.

In the last section we restrict our investigations to commutative rings with identity. Assume $J_1, J_2, \dots$ to be a sequence of coprime ideals and put $I_n = J_1 \cap \dots \cap J_n$. A set $S \subseteq R$ is called multiplicative if $[S : I \cap J] = [S : I] \cdot [S : J]$ holds for arbitrary coprime ideals I, J. From the Chinese Remainder Theorem we have $N(I_n) = N(J_1) \cdot \ldots \cdot N(J_n)$, hence the limit formula in section 3.3 for the computation of the covering density yields

$$\mu_{\mathcal{J}}(S) = \prod_{n=1}^{\infty} \frac{[S : J_n]}{N(J_n)}$$

for any multiplicative set in R.

As an example of a multiplicative set let us consider a mapping $f : R \rightarrow R$ such that for each ideal I we have $f(x) \equiv f(y) \bmod I$ provided that $x \equiv y \bmod I$. Due to the Chinese Remainder Theorem the mapping $f(x) + I \cap J \mapsto (f(x) + I,\ f(x) + J)$ is a bijection between $R/(I \cap J)$ and $R/I \oplus R/J$, and so the image set $f(R)$ is multiplicative. Therefore the image set of each polynomial in $R[x]$ is multiplicative. Let R^k denote the set of k-th powers of elements of R. Then R^k is multiplicative, too.

Let J be a maximal ideal with finite norm $N(J)$. Then the multiplicative group of the field R/J is cyclic; let $g + J$ be a generator. The elements $x^k + J$, $x \notin J$ form a cyclic subgroup generated by $g^k + J$. The order of this element is $\dfrac{N(J) - 1}{(k, N(J) - 1)}$ and we have $[R^k : J] = \dfrac{N(J) - 1}{(k, N(J) - 1)} + 1$. Thus we have shown

THEOREM 8. *Let J_n, $n = 1, 2, \dots$, be maximal ideals in R (commutative ring with identity). Then*

$$\mu_{\mathcal{J}}(R^k) = \prod_{n=1}^{\infty} \left(\frac{N(J_n) - 1}{(k, N(J_n) - 1)} + 1 \right) \cdot \frac{1}{N(J_n)}.$$

COROLLARY 1. *If $(k, N(J_n) - 1) = 1$, $n = 1, 2, \dots$, then $\mu_{\mathcal{J}}(R^k) = 1$.*

COROLLARY 2. *If* $(k, N(J_n) - 1) > 1$ *for infinitely many* n*, then* $\mu_{\mathcal{J}}(R^k) = 0$.

Our final result is devoted to the ideal measure of the image set $f(R)$ for nonlinear polynomials. Note that the ideal measure is defined via all ideals of finite index as in section 3.2.

THEOREM 9. *Let* R *be the ring of algebraic integers in a number field and let* $f \in R[x]$ *be a non-linear polynomial. Then* $f(R)$ *is of ideal measure 0.*

Proof. Let $n = \deg f$. By a theorem of Niederreiter and Lo [7], there are infinitely many maximal ideals P in R such that f is not a permutation polynomial mod P i.e., the function induced by f on the (finite) residue field R/P is not bijective. For each such P of index $[R : P] = q$, the value set $f(R)$ is contained in the union of at most $q - \frac{q-1}{n}$ residue classes of P, by a theorem of Wan [18].

For different maximal ideals $P_1, \ldots, P_k$ of index $[R : P_i] = q_i$, the ideal measure of the set of elements of R that are for each i in one of N_i given residue classes mod P_i is $\prod_{i=1}^{k} \frac{N_i}{q_i}$, by the Chinese Remainder Theorem. Thus, if f is not a permutation polynomial mod P_i for $i = 1, \ldots, k$ then the image of f is contained in a set of ideal measure at most

$$\prod_{i=1}^{k}\left(1 - \frac{q_i - 1}{nq_i}\right) \le \left(1 - \frac{1}{2n}\right)^k.$$

This value can be made arbitrarily small by considering an infinite sequence of different maximal ideals mod which f is not a permutation polynomial. ∎

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