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# An Upper Bound on the Coupling Constant of Scalar Mesons.

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**Summary.** — Assuming the scattering amplitude of scalar neutral mesons to be crossing symmetric and to have a certain amount of analyticity, then unitarity leads to an upper bound on the coupling constant for three scalar neutral mesons.

There has recently been considerable interest ([1-3]) in the question of whether or not one can deduce absolute upper bounds on the strength of strong interactions as consequences of general principles without using direct experimental information or specific models. GESHKENBEIN and IOFFE ([1]), and MEIMAN ([2]), based their treatments of this problem on the Källén-Lehmann spectral representation for propagators. MARTIN ([3]) derived an absolute upper bound on the pion-pion scattering amplitude from the assumptions of crossing symmetry and analyticity of the scattering amplitude together with unitarity.

This note can be considered as an Appendix to Martin's work ([3]). Following the method developed there, it is shown that there exists an upper bound for the coupling constant associated with the interaction of three scalar neutral particles. Although the basic concept is very similar to that of Martin, certain modifications are required for our derivation which is therefore presented briefly.

We consider elastic scattering of scalar neutral particles of mass $m$. Intro-

([1]) B. V. GESHKENBEIN and B. L. IOFFE: *Sov. Phys. JETP.*, **17**, 820 (1963).
([2]) N. N. MEIMAN: *Sov. Phys. JETP*, **17**, 830 (1963).
([3]) A. MARTIN: *An Absolute Upper Bound on the Pion-Pion Scattering Amplitude* preprint Stanford University (August 1964).

duce the usual Mandelstam variables $s$, $t$ and $u$ with $s+t+u=4m^2$; the scattering amplitude $F(s, t, u)$ can be expanded in partial waves, for example, in the $s$ channel:

$$F(s,t,u)=16\pi\sum_{l,\,\text{even}}(2l+1)f_l(s)P_l\left(1+\frac{t}{2k^2}\right), \tag{1}$$

where $k^2$ is related to $s$ by $s=4(k^2+m^2)$. In the physical region of the $s$ channel $s=x+io$, $x\geqslant 4m^2$, unitarity requires

$$\operatorname{Im} f_l(x+io)\geqslant \varrho(x)\,|f_l(x+io)|^2 \tag{2}$$

with

$$\varrho(x)=\frac{1}{2}\sqrt{\frac{x-4m^2}{x}}\,. \tag{3}$$

Following MARTIN (3), we make the assumptions on the scattering amplitude $F(s, t, u)$:

$A$) complete crossing symmetry, that is invariance of $F(s, t, u)$ under any permutation of the variables $s$, $t$ and $u$;

$B$) analyticity i) with $|t|<4m^2$ fixed, $F(s, t, u)$ is an analytic function in $s$ having poles at $s=m^2$ and $u=m^2$ and cuts from $4m^2\leqslant s<+\infty$ and $4m^2\leqslant u<+\infty$; and it is furthermore polynomially bounded at infinity; ii) for fixed real $s\geqslant 4m^2$ the scattering amplitude $F(s, t, u)$ is analytic in $t$ for $|t|\leqslant 4m^2$ except for poles at $t=m^2$ and $u=m^2$.

Within the assumptions stated above, JIN and MARTIN (4) proved that there are at most two subtractions required in a fixed momentum transfer dispersion relation with $|t|\leqslant 4m^2$. Hence,

$$F(s,t,u)=-g^2\left\{\frac{1}{s-m^2}+\frac{1}{t-m^2}+\frac{1}{u-m^2}\right\}+C(t)+\\ +\frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}x\,\frac{A(x,t)}{x^2}\left\{\frac{s^2}{x-s}+\frac{u^2}{x-u}\right\}. \tag{4}$$

The coupling constant $g$ of three scalar neutral mesons introduced in (4) has the dimension of a mass. It is simply related to the corresponding coupling constant $\lambda$, defined by $\mathscr{L}(x)=\lambda:\varphi^3(x):$ in a Lagrangian field theory

$$\frac{\lambda^2}{4\pi}=\frac{1}{36}\frac{g^2}{4\pi}\,. \tag{5}$$

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(4) Y. S. JIN and A. MARTIN: *Phys. Rev.*, **135**, B 1375 (1964).

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Using unitarity (2), an upper bound on the scattering amplitude in the physical region can be deduced ([3]) in terms of the absorptive part with $t$ fixed in the interval $0 < t < 4m^2$,

$$(6)\qquad \begin{cases} 0 < \hat{t} < 4m^2\,, \\ t \leqslant 0\,, \quad s = x + io\,, \quad x \geqslant 4m^2 - t\,, \\ A(s, \hat{t}) \geqslant |F(s, t, u)|^2 K(s, \hat{t})\,, \end{cases}$$

with

$$K(s, \hat{t}) = \varrho(x)\left[16\pi \sum_{l,\,\text{even}} \frac{2l+1}{P_l(1 + \hat{t}/2k^2)}\right]^{-1}.$$

We first require a slight extension ([3]) of a theorem due to SZEGÖ ([5]), later rederived by MEIMAN ([2]) and DRELL, FINN and HEARN ([6]). We will present it in a version suitable for our problem. Defining the function

$$(7)\qquad \tilde{F}(s, t, u) = (s - m^2)(t - m^2)(u - m^2)\,F(s, t, u)\,,$$

we conclude from the properties of $F(s, t, u)$ that for fixed $t \leqslant 0$ it is analytic in $s$ with cuts on the real axis in the range $-\infty < s \leqslant -t$ and $4m^2 \leqslant s < +\infty$. Because of crossing symmetry and unitarity we furthermore observe that the discontinuity of $\tilde{F}$ in the unphysical part $4m^2 \leqslant s < 4m^2 - t$ of the right-hand cut is positive. $[\operatorname{Im} \tilde{F}(s, t, 4m^2 - s - t) = \operatorname{Im} \tilde{F}(s, 4m^2 - t - s, t) > 0.]$ Therefore, we can majorize the unknown discontinuity in the derivation of ref. ([6]) and perform a suitable conformal mapping, again using crossing symmetry, to obtain the theorem in the following form:

$$t \leqslant 0, \quad s_1 \text{ real} < 4m^2, \quad f(s) \text{ real} > 0\,,$$

$$(8)\qquad \frac{1}{\pi}\int_{4m^2-t}^{\infty} \mathrm{d}s\, f(s)\,|\tilde{F}(s, t, u)|^2 > \left(\frac{\sqrt{s_1(4m^2 - t - s_1)} - \sqrt{-4m^2 t}}{\sqrt{s_1(4m^2 - t - s_1)} + \sqrt{-4m^2 t}}\right)^2 (\tilde{F}(s_1, t, u))^2 \cdot$$
$$\cdot \exp\left[\frac{1}{\pi}\int_{4m^2-t}^{\infty} \mathrm{d}s\, \varphi(s) \ln\left(\frac{f(s)}{\varphi(s)}\right)\right]$$

with

$$\varphi(s) = \frac{(2s - 4m^2 + t)\sqrt{s_1(4m^2 - t - s_1)}}{\sqrt{s(s - 4m^2 + t)}\,[s(s - 4m^2 + t) + s_1(4m^2 - t - s_1)]}\,.$$

([5]) G. SZEGÖ: *Orthogonal Polynomials* (New York, 1959), p. 299.
([6]) S. D. DRELL, A. C. FINN and A. C. HEARN: *Phys. Rev.*, **136**, B 1439 (1964).

For the actual derivation, we consider the scattering amplitude at two points with the same momentum transfer:

$$\left.\begin{aligned} F(1) &= F(4m^2-\hat{t},\, \hat{t},\, 0) \\ F(2) &= F(4m^2-\hat{t}+\Lambda,\, \hat{t},\, -\Lambda) \end{aligned}\right\} \text{ with } \left\{\begin{aligned} &2m^2<\hat{t}<3m^2, \\ &0<\Lambda<m^2. \end{aligned}\right.$$

Then, using the dispersion relation (4), we form the difference

$$\Delta F = F(2)-F(1) = g^2\sigma(\hat{t},\Lambda) + \frac{1}{\pi}\int_{4m^2}^{\infty} \mathrm{d}x\,\alpha(x,\hat{t},\Lambda)\,A(x,\hat{t}) \tag{9}$$

eliminating thus the unknown subtraction constant $C(\hat{t})$. We introduced in the above

$$\sigma(\hat{t},\Lambda) = g^2\left\{-\left[\frac{1}{m^2}-\frac{1}{m^2+\Lambda}\right]+\left[\frac{1}{3m^2-\hat{t}}-\frac{1}{3m^2-\hat{t}+\Lambda}\right]\right\},$$

$$\alpha(x,\hat{t},\Lambda) = \frac{1}{x^2}\left\{\frac{(4m^2-\hat{t}+\Lambda)^2}{x-4m^2+\hat{t}-\Lambda}-\frac{(4m^2-\hat{t})^2}{x-4m^2+\hat{t}}+\frac{\Lambda^2}{x+\Lambda}\right\}.$$

For the variables in the intervals stated above, $\sigma(\hat{t},\Lambda)>0$ and $\alpha(x,\hat{t},\Lambda)>0$. Using inequality (6) and the function $\tilde{F}$ defined in (7) we can write (9) in the form

$$\Delta F > g^2\sigma + \frac{1}{\pi}\int_{4m^2-t}^{\infty}\mathrm{d}x\,\frac{K(x,\hat{t})\alpha(x,\hat{t},\Lambda)\,|\tilde{F}(x,t,4m^2-t-x)|^2}{(x-m^2)^2(t-m^2)^2(3m^2-t-x)^2} \tag{10}$$

with $t\leqslant 0$. Because of crossing symmetry

$$F(1) = F(\hat{t},\,0,\,4m^2-\hat{t}),$$

$$F(2) = F(\hat{t},\,-\Lambda,\,4m^2-\hat{t}+\Lambda).$$

Therefore we can apply theorem (8) to the right-hand side of (12), putting $s_1=\hat{t}$ and $t=0$ and $t=-\Lambda$, resepctively, to obtain the inequalities

$$\left\{\begin{aligned} F(2)-F(1) &> g^2\sigma + c_1(F(1))^2, \\ F(2)-F(1) &> g^2\sigma + c_2(F(2))^2. \end{aligned}\right. \tag{11}$$

Using $|F(1)|+|F(2)|\geqslant F(2)-F(1)$, we deduce from (11), since $\sigma>0$ and

$c_i > 0$, the upper bound on the coupling constant:

$$g^2 < \max\left\{\frac{1}{\sigma c_1}\,;\quad \frac{1}{\sigma c_2}\right\}.$$

The best number obtained in the region considered is $g^2/4\pi < 1.5\cdot 10^6\ \text{m}^2$ which yields $\lambda^2/4\pi < 4.2\cdot 10^4\ \text{m}^2$. These huge numbers indicate that considerable improvement could be achieved applying a refined technique developed by MARTIN (7) and ŁUKASZUK (8). But as the scattering of scalar neutral particles is anyway an academic problem, only the existence of an upper bound may be of some interest.

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(7) A. MARTIN: Proceedings of the 1965 *Trieste International Seminar on Elementary Particles and High-Energy Physics*, and to be published.
(8) L. ŁUKASZUK: *Nuovo Cimento*, **41** A, 67 (1966).

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## RIASSUNTO (*)

Nell'ipotesi che l'ampiezza di scattering dei mesoni scalari neutri sia a simmetria incrociata e che abbia una certa quantità di analiticità, l'unitarietà porta ad un limite superiore della costante di accoppiamento per tre mesoni scalari neutri.

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(*) *Traduzione a cura della Redazione.*