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THE GEOMETRY OF ZEROS OF TRINOMIAL EQUATIONS

HARRIET FELL

Ce travail est consacré à la première discussion *complète* du comportement des racines de l'équation trinomiale

$$\alpha z^{r+s} + (1 - \alpha) z^r - 1 = 0 \ (r, s \in \mathbf{N}) \text{ pour } \alpha \text{ variant } -\infty \text{ à } +\infty.$$

The original intent of this work was to study the transition of the roots of the equation $\alpha z^{r+s} + (1 - \alpha) z^r - 1 = 0$ as α varies between 0 and 1. This equation is a weighted sum of the two binomial equations, $z^{r+s} - 1 = 0$ and $z^r - 1 = 0$ so that when $\alpha = 1$ its roots are the $r + s^{th}$ roots of unity and when $\alpha = 0$ its roots are the r^{th} roots of unity. The question arose as to what happens in between, i.e. $0 < \alpha < 1$. Which of the $r + s^{th}$ roots of unity move over to r^{th} roots and what kind of paths do they follow? What trajectories are taken by those roots that move out to infinity?

While investigating this equation it was noticed that a large class of trinomials could, by means of a simple substitution, be put into his form and that the remaining trinomials could be put into the even simpler form: $z^{r+s} + z^r + k = 0$ where k is a positive real. Therefore it seemed reasonable to broaden the scope of the paper to include the location of the roots of arbitrary trinomials.

In the early 1900'*s*, trinomial equations were discussed in many papers and by a variety of authors. The techniques found and investigated by these authors where for the location of the roots of general polynomials and the trinomial, with its few terms, offered a good example for the application of these methods. Nekrasoff [8] considers all trinomials, complex coefficients included, by putting them into the normalized form: $u^m - p u^n - q$ where $m > n$ and p and q may be complex.

He showed that the roots of this equation have their moduli bounded by $r\,|q|^{1/m}$ and $\rho\,|q|^{1/m}$ where r and ρ satisfy:

$$\rho \geq 1 \text{ and } \rho^{m-n} - \rho^{-n} = (m/n)^{n/m}\left(\frac{m}{m-n}\right)^{(m-n)/m}$$

$$r \leq 1 \text{ and } r^{-n} - r^{m-n} = (m/n)^{n/m}\left(\frac{m}{m-n}\right)^{(m-n)/m}.$$

As for the arguments of the roots, he showed only that, taking p and q real, there is exactly one root of the equation in each of the sectors:

$$(2k+1)\pi/m \leq \theta \leq (2k+3)\pi/m \qquad k = 0, \ldots, m-1,$$

except under certain conditions when the roots belonging to two adjacent sectors actually both lie on the boundary common to these two sectors. There are no sectors excluded nor clues given as to whether a root in a given sector is likely to be of large or small modulus.

By normalizing the constant term and making a substitution of the form $z = kx$, any trinomial can be put in the form $1 + x^r + ax^n = 0$ where a and k may both be complex. Biernaki [1] stated that any equation of this form must always have r of its roots with magnitude less than $(r/(n-r))^{1/r}$. When applied to an equation not already in this special form, this does not give a coefficient free or necessarily very tight bound. For example, for $-\alpha z^n + (\alpha - 1) z^r + 1 = 0$ we must first put

$$z = x \cdot \frac{1}{(\alpha - 1)^{1/r}}$$

to arrive at the fact that

$$0 = \frac{-\alpha}{(\alpha - 1)^{n/r}} x^n + x^r + 1$$

has r roots of magnitude less than $(r/(n-r))^{1/r}$ which in turn implies that our original equation has r roots of magnitude less than $(r/(n-r))^{1/r}(1/(\alpha-1))^{1/r}$ so that as α nears 1, this bound approaches infinity while it is obvious that all the roots of original equation will have magnitude near one. Again there is no information as to which roots are likely to be of small or large magnitude. Neither is this objection met by the class of theorems which state that p roots of a polynomial (of degree $n > p$) have magnitude less than an exhibited bound. Pellet's Theorem (Marden [7] p. 99) which produces an annulus that separates the p roots of smallest magnitude from the $n - p$ roots of largest magnitude is not applicable to all trinomials. For $a z^n + z^r + 1 = 0$ to satisfy its hypotheses for separating out the r roots of smallest magnitude, one must have:

$$(r/n)^{r/(n-r)} - (r/n)^{n/(n-r)} > |a|^{r/(n-r)}.$$

Kempner [5] does give a method for determining sectors in which the roots of a given trinomial must lie. His method works for trinomials with arbitrary complex coefficients and it produces n sectors, smaller than those of Nekrasoff and sharing at most one common boundary, such that each of these sectors must contain one of the n roots of the trinomial. In this paper, we will restrict our attention to trinomials with real coefficients. We will produce, though by different methods, similar bounds to those of Kempner. We will, however, describe these bounds in a more closed form and at the same time discuss the magnitude of the roots in the various sectors.

Preliminaries.

Given an arbitrary trynomial equation, $a z^n + b z^k + c z^m = 0$, we can assume, without losing anything more than a few easily retrievable roots, that $m = 0$ and $a = 1$. We shall therefore take as our general form of a trinomial equation:

$$x^n + b x^k + c = 0$$

where b and c are nonzero reals. We shall put this equation into one of two normalized forms depending on whether or not it has a real root. First, if the equation does have a real root, r_0, then we divide by $-c$ and substitute $x = r_0 z$ to put the equation in the form:

$$\alpha z^n + (1 - \alpha) z^k - 1 = 0.$$

Finally, to simplify the discussion later in this paper, we rewrite this equation as:

$$\alpha z^{r+s} + (1 - \alpha) z^r - 1 = 0. \tag{1}$$

Returning to our general trinomial equation, $x^n + b x^k + c = 0$, we observe that: if c is negative, the equation has a real root; if n is odd, the equation has a real root; if n and k are both even we can reduce the degree of the equation by substituting $y = x^2$. Therefore, we need only worry about the case where c is positive, n is even and k is odd. Assuming that these conditions hold, we make the substitution, $x = r_0 z$ where r_0 is a real satisfying $r_0^{n-k} = b$. It is possible to find such a real, even if b is negative, since by our assumption $n - k$ is odd. We then divide by r_0^n and make the same change of exponents that we did above to obtain our second normalized form of trinomial equation:

$$z^{r+s} + z^r + k = 0 \tag{2}$$

where k is a positive constant since c is positive and $n = r + s$ is even.

In this paper, we will describe the trajectories of the roots of equation (1) as α varies from $-\infty$ to $+\infty$ and of the roots of equation (2) as k varies from 0 to $+\infty$. We will present some purely numerical results which are necessary for the later description; second, we will give a qualitative description of what happens; third, we will present the mathematical justification of our description.

Part 1. LEMMA. *Let $m \geq 2$ be an integer. Define*

$$A_m = \left\{\frac{i}{m} \mid 1 \leq i \leq m\right\}$$

and

$$B_m = \left\{\frac{2i+1}{m} \mid 1 \leq 2i + 1 \leq m\right\}.$$

Then for r and s integers, $r, s \geq 1$, and for j any integer, the closed interval $I = \left[\frac{2j-1}{2(r+s)}, \frac{2j-1}{2(r+s)}\right]$ contains exactly one element from each of the following four sets:

1) $A_r \cup B_{2s}$

2) $A_s \cup B_{2r}$

3) $A_r \cup A_s$

4) $B_{2r} \cup B_{2s}$.

Proof. We shall prove the lemma only for set 1). For 2) the statement then obviously follows by symmetry. The proof for sets 5) and 4) is exactly like that for set 1).

We shall often use in this proof the fact:

(*) *Given any four positive integers x, x', y, y', one has that $(x + x')/(y + y')$ belongs to the open interval bounded by x/y and x'/y' unless $x/y = x'/y' = (x + x')/(y + y')$.*

We first show that our interval contains at most one element of $A_r \cup B_{2s}$. It surely cannot contain two elements of A_r or two or B_{2s} since the length of the interval, I, is $1/(r+s)$ which is smaller than both $1/r$ and $1/s$, the respective distances between consecutive elements of A_r, and B_{2s}. Suppose then that I contains one element of each of these sets, i.e. $k/r \in I$ and $(2i+1)/(2s) \in I$. Since $k/r = 2k/2r$, we have that $(2k+2i+1)/(2(r+s)) \in I$. This implies that $2j-1 \leq 2k+2i+1 \leq 2j+1$. It is impossible for both of these inequalities to be strict so that equality must hold at one end, but this implies, by (*), that $k/r = (2i+1)/(2s)$ and that we still have only one element of $A_r \cup B_{2s}$ in I.

It now remains to show that there really is an element of $A_r \cup B_{2s}$ in I. Suppose, for the sake of contradiction, that this is not the case. Then there would exist integers k and i such that:

$$\frac{k}{r} < \frac{2j-1}{2(r+s)} < \frac{2j+1}{2(r+s)} < \frac{k+1}{r},$$

$$\frac{2i-1}{2s} < \frac{2j-1}{2(r+s)} < \frac{2j+1}{2(r+s)} < \frac{2i+1}{2s}.$$

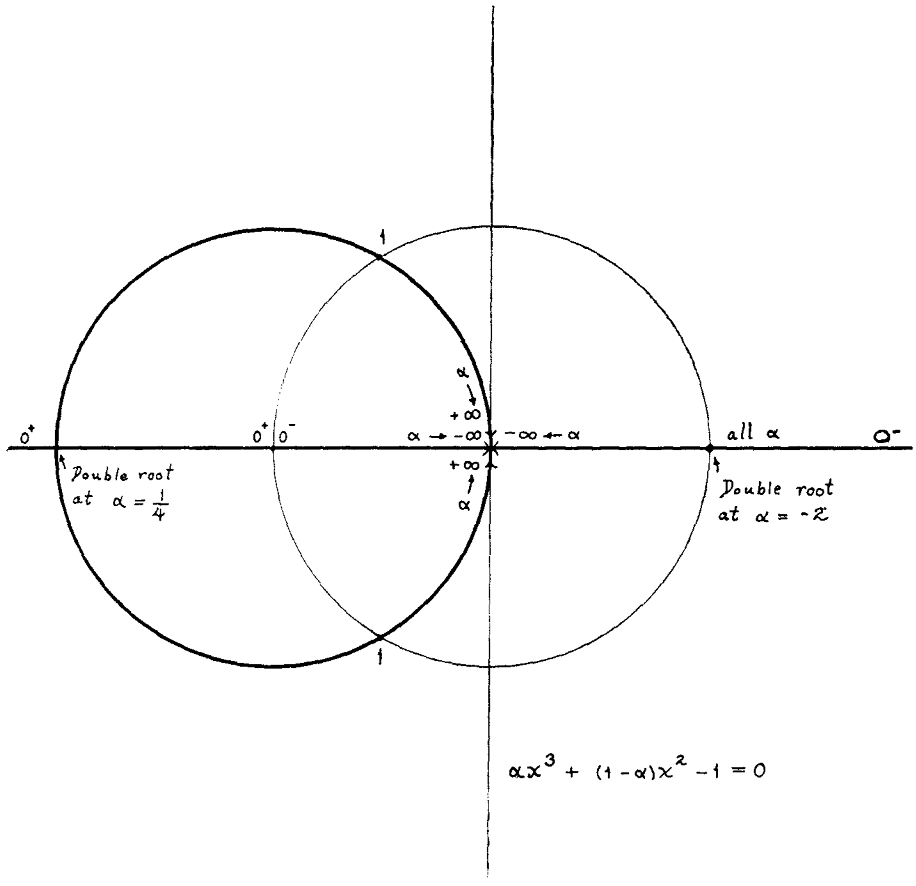
Combining these two inequalities and using (*) yields:

$$\frac{2k+2i-1}{2(r+s)} < \frac{2j-1}{2(r+s)} < \frac{2j+1}{2(r+s)} < \frac{2k+2i+3}{2(r+s)}.$$

The difference between the first and last numerators is 4. All of the numerators are odd and as the inequalities are strict, the first and last must differ by at least 6 which is a contradiction.

The lemma tells us, for example, that if C is the set of roots of $x^{r+s}-1$, A the set of roots of x^r-1 and B the set of roots of x^s+1, then for each element, c of C, there is exactly one element, b, of $A \cup B$ such that $\arg(c) - \arg(b)$ is less than or equal to $2\pi/2(r+s)$. This b is that member of $A \cup B$ which is nearest to c. From the proof of the lemma we can also see that two elements c_1 and c_2 of C can have the same b as their respective nearest neighbor in $A \cup B$ if and only if b lies in $A \cap B$ and if it is equidistant from c_1 and c_2.

Part 2. We will consider, first, the equation (1) $\alpha z^{r+s} + (1-\alpha) z^r - 1 = 0$. Before we indulge in a verbal description of the trajectories of the roots of this equation, we present some sketches of what takes place for some special cases of r and s (see the figures that follow).



1
α
+∞
0^+
0^+ 0^-
α → −∞
−∞ ← α
all α
0^-
+∞
α
Double root at $\alpha = \frac{1}{4}$
Double root at $\alpha = -2$
1
$\alpha x^3 + (1-\alpha)x^2 - 1 = 0$

Fig. 1

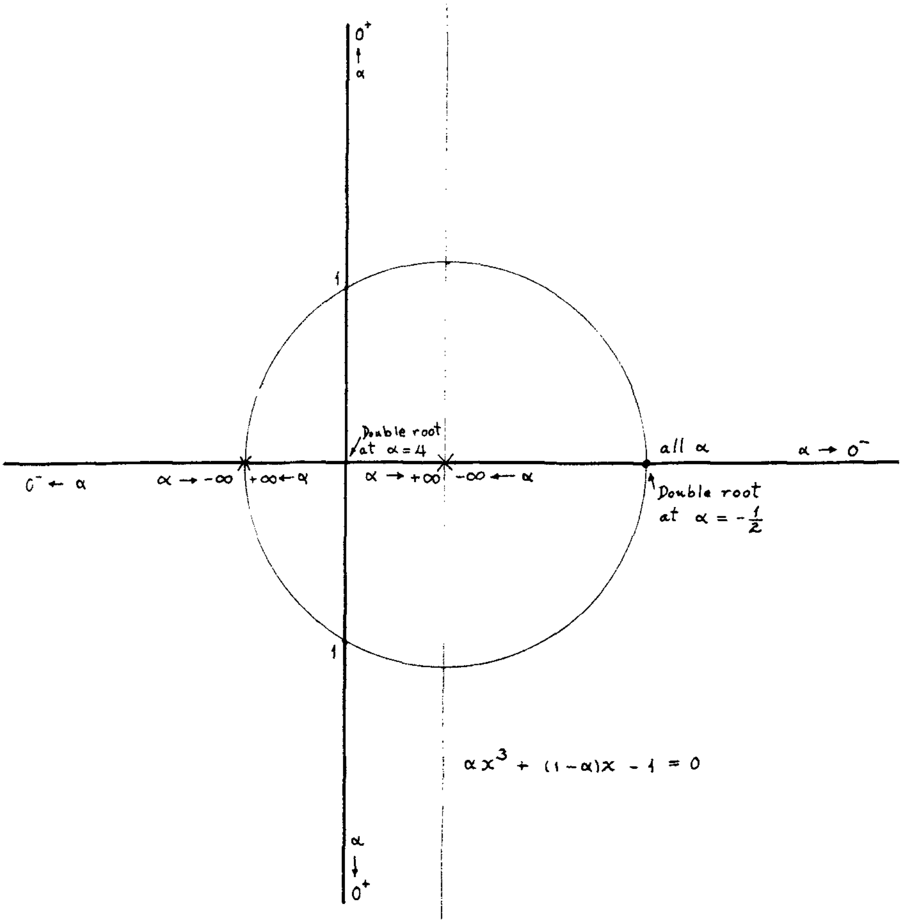

0+
α
1
Double root
at α = 4
all α
α → 0−
0− ← α
α → −∞
+∞ ← α
α → +∞
−∞ ← α
Double root
at α = −1/2
1
αx³ + (1 − α)x − 1 = 0
α
0+

Fig. 2

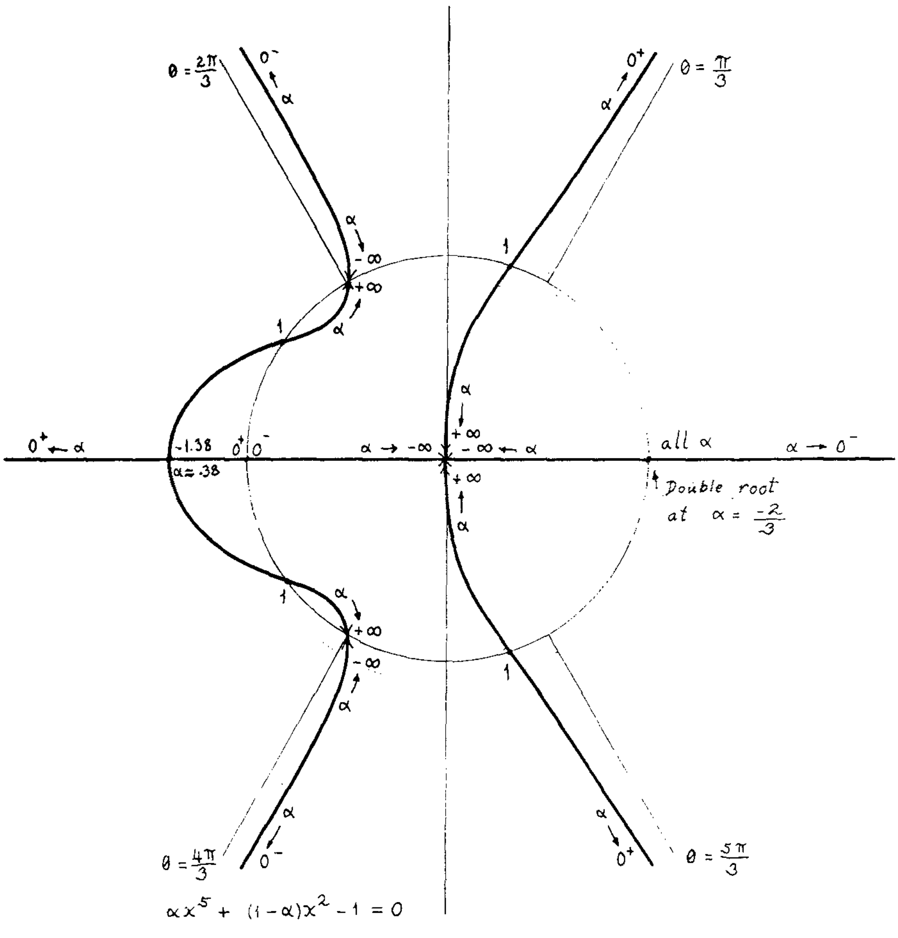

θ = 2π/3
θ = π/3
θ = 4π/3
θ = 5π/3
0⁺ ← α
-1.38
α ≈ .38
α → -∞
-∞ ← α
all α
α → 0⁻
Double root
at α = -2/3
αx⁵ + (1 − α)x² − 1 = 0

Fig. 3

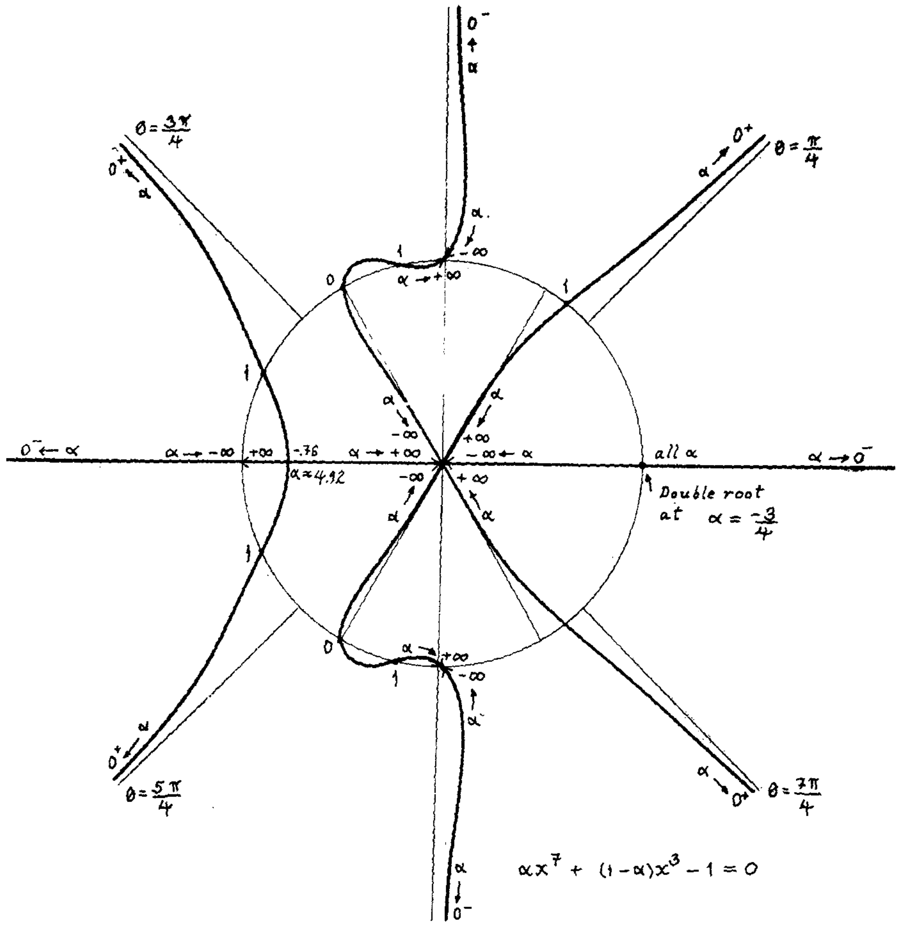

$\theta = \frac{3\pi}{4}$
$\theta = \frac{\pi}{4}$
$\theta = \frac{5\pi}{4}$
$\theta = \frac{7\pi}{4}$
all α
Double root at α = -3/4
α ≈ 4.92
-.76
$\alpha x^7 + (1-\alpha)x^3 - 1 = 0$

Fig. 4

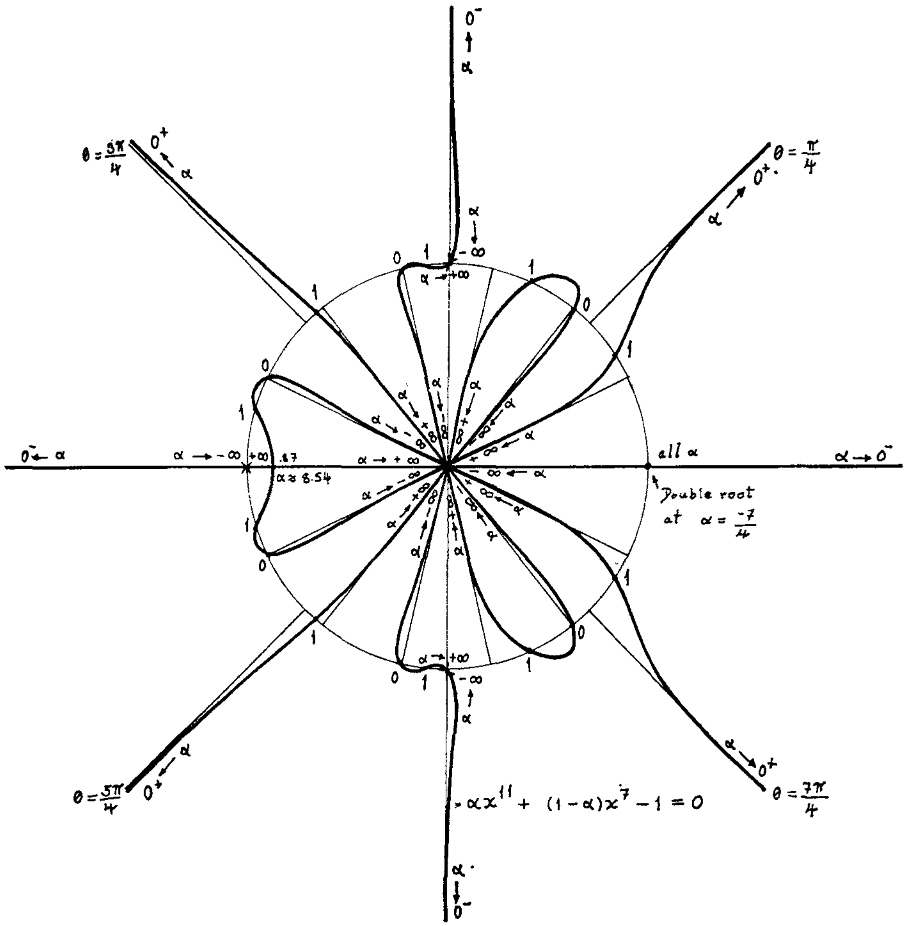

θ = 3π/4
θ = π/4
θ = 5π/4
θ = 7π/4
all α
Double root
at α = -7/4
α ≈ 8.54
.87
αx^11 + (1 − α)x^7 − 1 = 0

Fig. 5

Now, to describe the locus of roots of equation (1), we first notice that for $\alpha = 1$, the equation has $r + s$ roots, the $r + s$ roots of unity. We can then describe that part of our locus with α restricted to $[0, 1]$ as trajectories of particles starting at these $r + s$ roots. As α charges from 1 to zero, they move continuously until at $\alpha = 0$, r of them have moved into r^{th} roots of unity and s of them have disappeared. We must first be able to tell whether or not a given $(r + s)^{th}$ root of unity lies on a trajectory that passes through an r^{th} root of unity. We sum this up in the following:

Descriptive Claim I: As in the remark after the numerical lemma, let

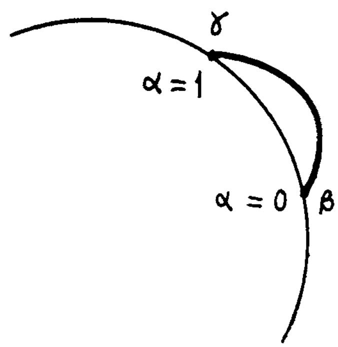
$A = \{r^{th} \text{ roots of unity}\}$

$B = \{s^{th} \text{ roots of } -1\}$

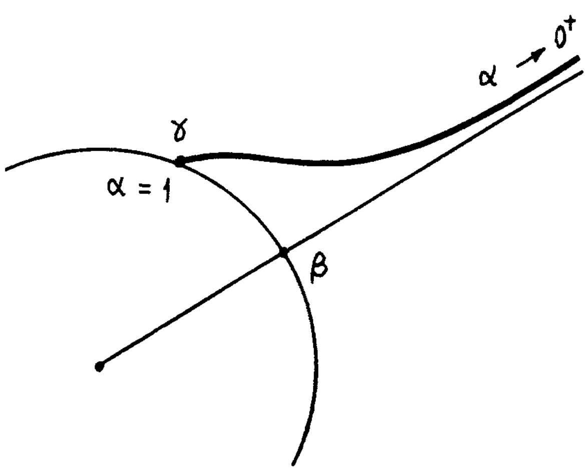
$C = \{(r + s)^{th} \text{ roots of unity}\}$ $(r > 1, s > 1)$.

Let $\gamma \in C$ and let β be the unique nearest neighbor of γ in $A \cup B$. Then:

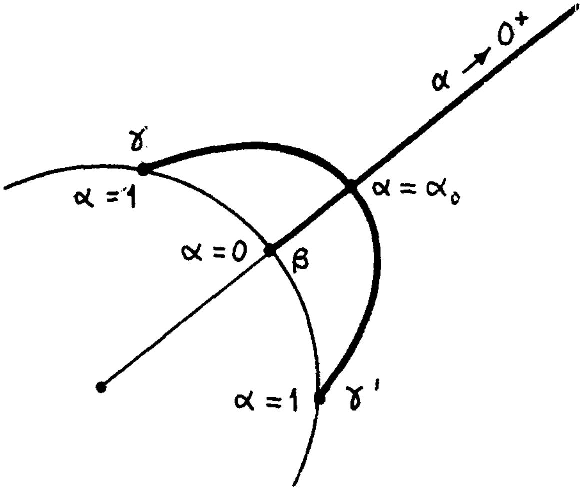
1. If $\beta \in A$, $\beta \notin B$, then the trajectory of a particle starting at γ when $\alpha = 1$, passes through β when $\alpha = 0$. The argument θ of the particles position changes monotonically with α, while the modulus never gets very far from 1.



2. If $\beta \in B$, $\beta \notin A$, a particle starting at γ when $\alpha = 1$ moves out to infinity as $\alpha \to 0$ along a curve asymptotic to the ray $\theta = \text{argument}(\beta)$.



3. If $\beta \in A \cap B$, then there is a $\gamma' \in C$ such that β is equidistant from γ and γ'. In this case, there is an α_0, $0 < \alpha_0 < 1$, such that two particles starting at γ and γ' when $\alpha = 1$ move so that both the arguments and moduli of their positions change monotonically with α until $\alpha = \alpha_0$ when they meet on the ray $\theta = \arg(\beta)$. As α changes from α_0 to 0, both roots stay on this ray, one heading out to infinity, the other heading in to β.



Note that as long as $0 \leq \alpha \leq 1$, the moduli of the roots are greater than or equal to 1 and that in those sectors of type « 1 » the moduli do not get very big. Later, we shall produce bounds to make this statement more explicit.

We can now see what happens when $1 < \alpha < \infty$, by means of a transformation on our equation $\alpha z^{r+s} + (1 - \alpha) z^r - 1 = 0$. We put $a = 1/\alpha$, $y = 1/z$ to get:

$$\frac{1}{a} y^{-(r+s)} + \left(1 - \frac{1}{a}\right) y^{-r} - 1 = 0,$$

or

$$a y^{r+s} + (1 - a) y^s + 1 = 0.$$

Therefore, if we believe or verify Descriptive Claim 1, we must automatically believe or accept as proved:

Descriptive Claim II: Let

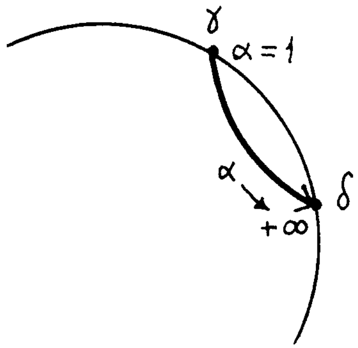
$$C = \{(r + s)^{th} \text{ roots of unity}\}$$

$$D = \{s^{th} \text{ roots of unity}\}$$

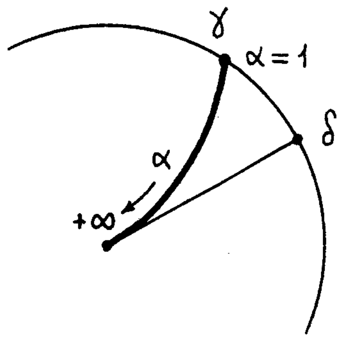
$$E = \{r^{th} \text{ roots of } -1\}$$

Let $\gamma \in C$ and δ be the unique nearest neighbor of γ in $D \cup E$, then we have the following pictures:

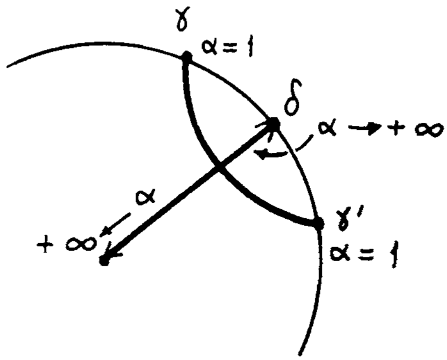
1. $\delta \in D, \delta \notin E$



2. $\delta \in E, \delta \notin D$



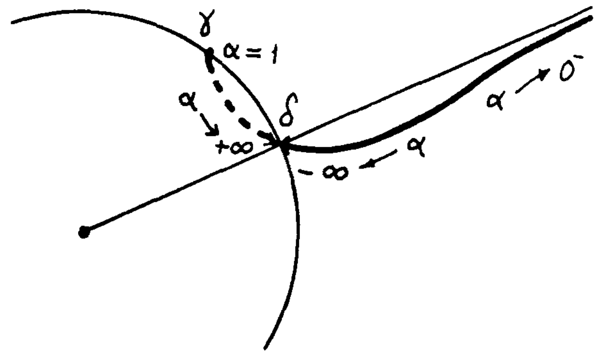
3. $\delta \in E \cap D$

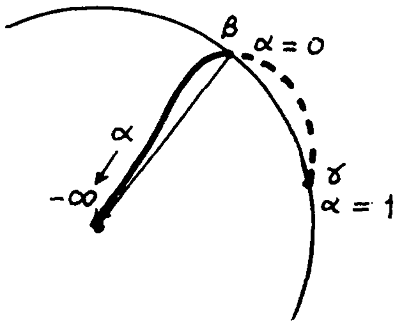


Lastly, we must see what happens when $-\infty < \alpha < 0$. When $\alpha = 0$, the equation (1) has only r roots but as α becomes negative, the s roots that « disappeared » as approached 0 from the positive side (i.e. moved into the point at infinity) reappear. These s trajectories must approach the s roots of unity as α approaches $-\infty$. Similarly the r roots that collapsed into zero as α approached $+\infty$ reappear when α is near $-\infty$ and these r trajectories must approach the r^{th} roots of unity as α approaches 0.

Descriptive Claim III: Let A, B, C, D, and E be as before. Let $\gamma \in C$, β, δ as above.

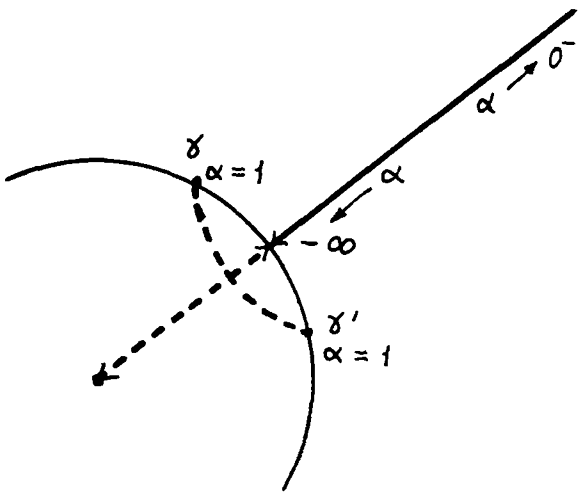
1. $\delta \in D$, $\delta \notin E$. As α moves from $-\infty$ to 0, there is a trajectory starting at δ such that ρ changes from 1 to $+\infty$, θ as a function of α has only one turning point on this part of the trajectory. As ρ goes to $+\infty$, the curve is asymptotic to the ray $\theta = \arg(\delta)$. As in the picture this part of the trajectory lies entirely on that side of the ray opposite from γ.



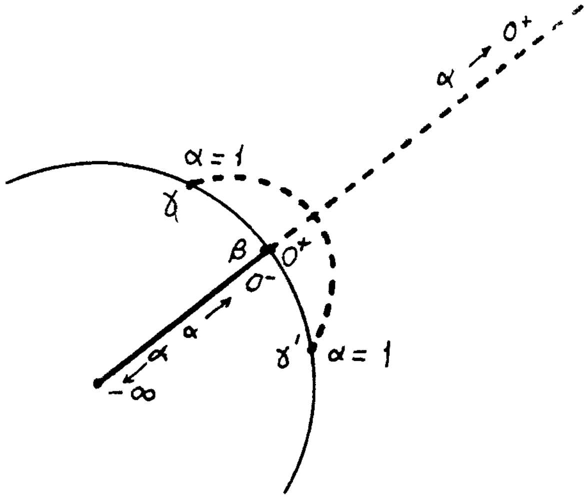


2. $\beta \in A$, $\beta \notin B$. As α moves from 0 to $-\infty$, there is a trajectory starting at β such that ρ moves from 1 to 0. θ as a function of α has only one turning point in this range of θ. The trajectory is tangent to the ray $\theta = \arg(\beta)$ and it lies on that side of this ray opposite from γ.

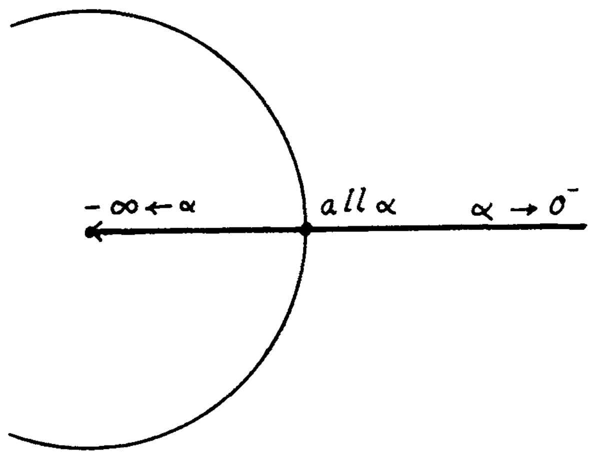
3. $\delta \in D \cap E$. As α moves from 0 to $-\infty$, ρ decreases from $+\infty$ to 1. $\theta = \arg(\delta)$.



4. $\beta \in A \cap B$. As α moves from 0 to $-\infty$, ρ changes from 1 to 0. $\theta = \arg(\beta)$.



5. $\theta = 0$. As α moves from 0 to $-\infty$, ρ decreases monotonically from $+\infty$ to 0. There is always a root at $z = 1$.



We will now consider equation (2), $z^{r+s} + z^r + k = 0$. Again, we first present sketches of some special cases. We will always take r and s to be odd and relatively prime. After the sketches, we will describe the trajectories of the roots of this equation for positive k in Descriptive Claim IV.

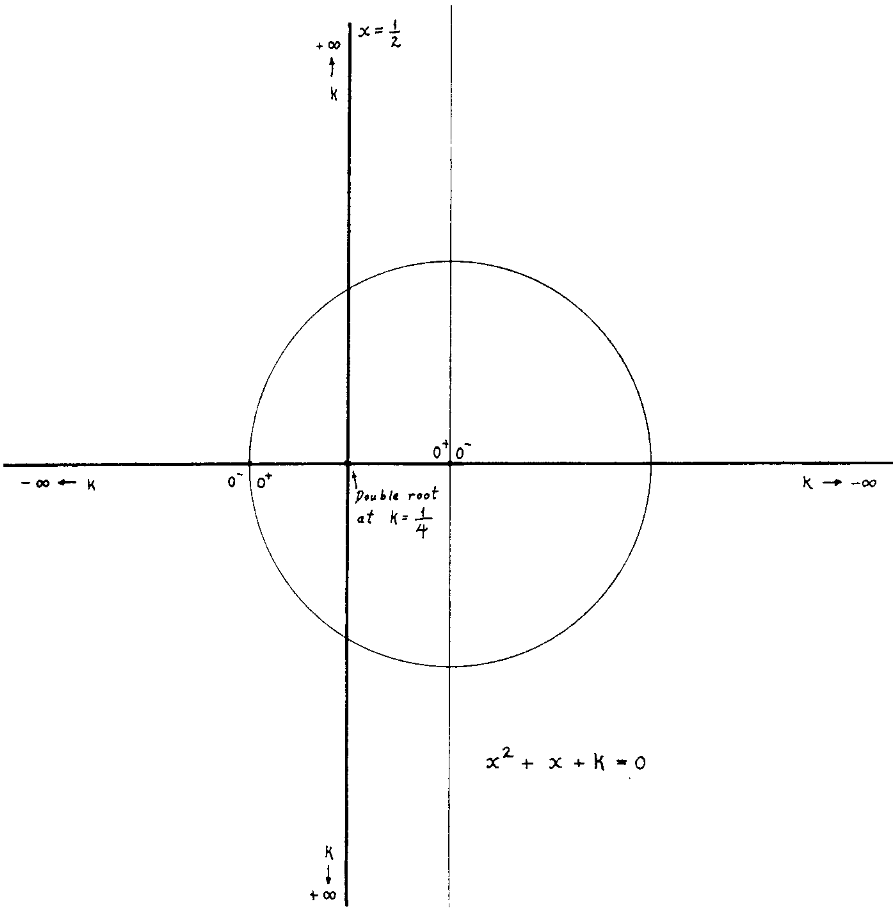


Fig. 6

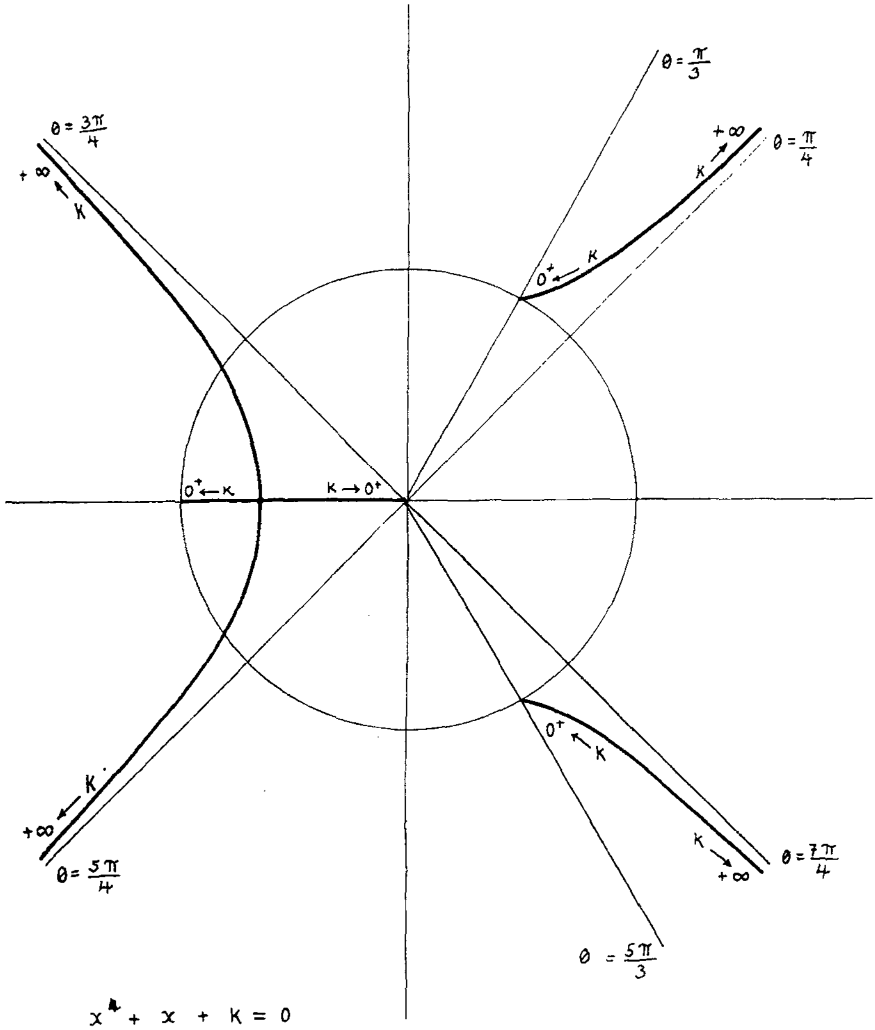


Fig. 7

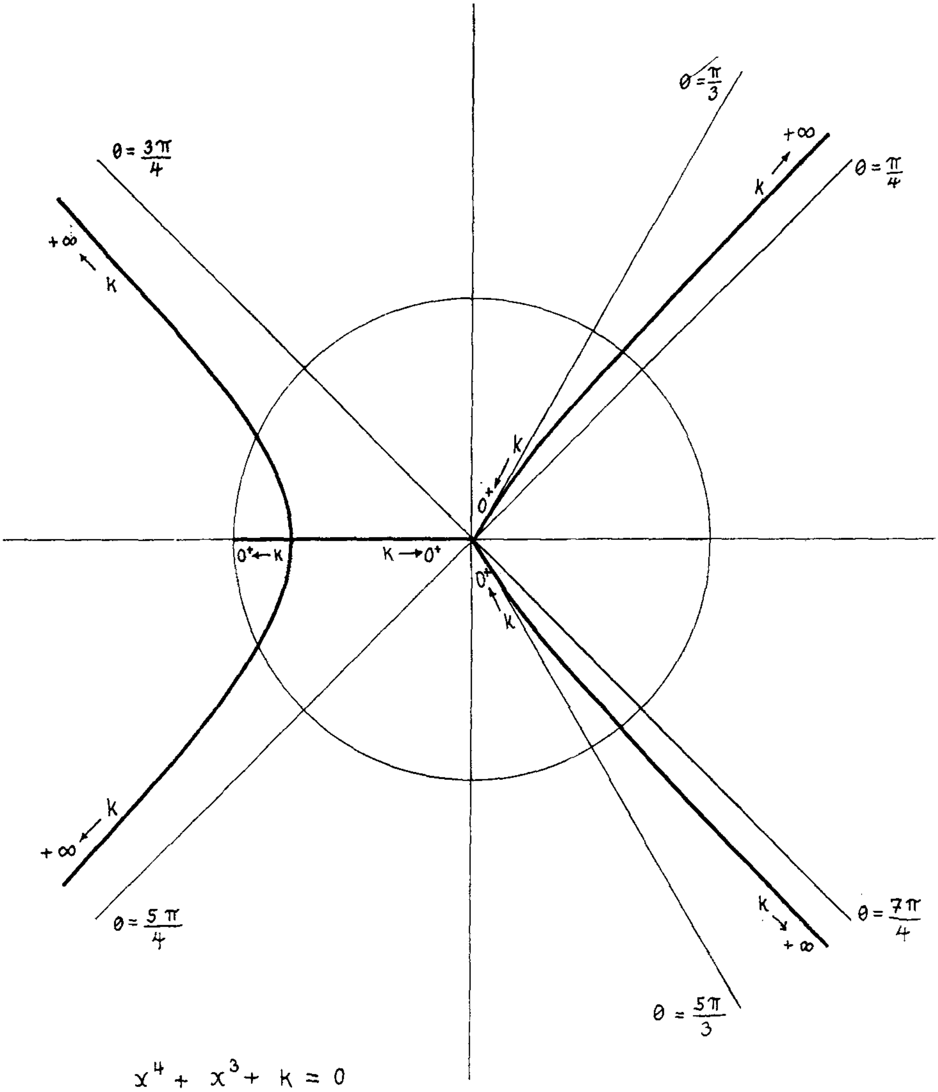


Fig. 8

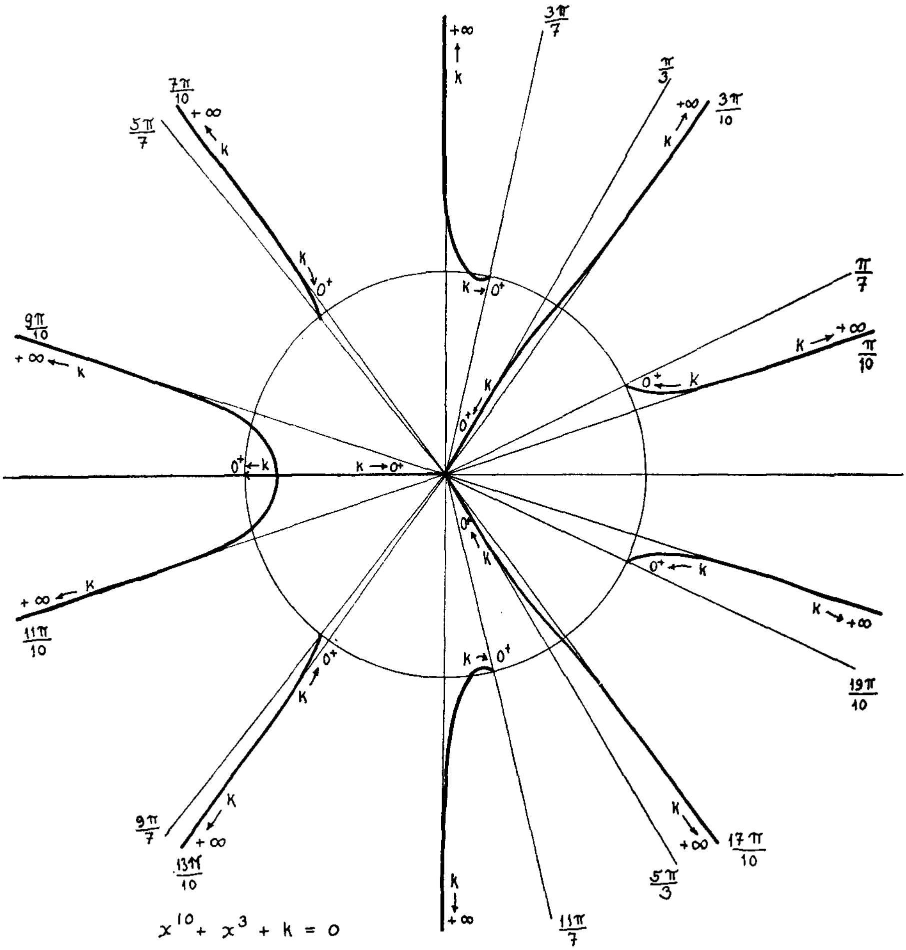


Fig. 9

Descriptive Claim IV: In keeping with the notation that we have already used, we let

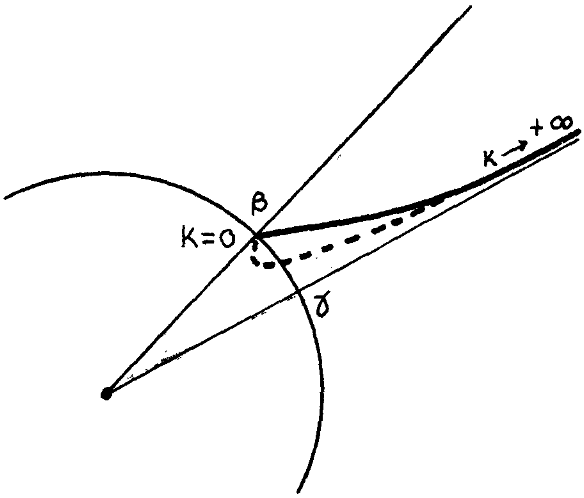
$$B = \{s^{th} \text{ roots of } -1)\}$$

$$E = \{r^{th} \text{ root of } -1\}$$

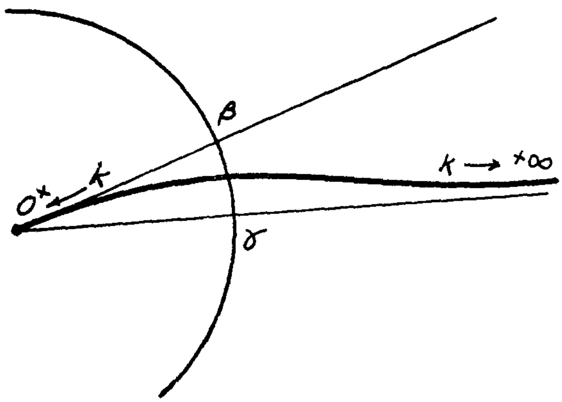
$$G = \{(r+s)^{th} \text{ roots of } -1\}.$$

Let $\gamma \in G$ and let β the unique nearest neighbor of γ in $E \cup B$. Then:

1. If $\beta \in B$, $\beta \notin E$, a particle starting at β when $k = 0$ moves with the argument of its position changing monotonically with k. The modulus might fall below one before the particle reaches ∞. If $(r+s) > 2$, the particle approaches the ray $\theta = \arg(\gamma)$ as $k \to +\infty$.

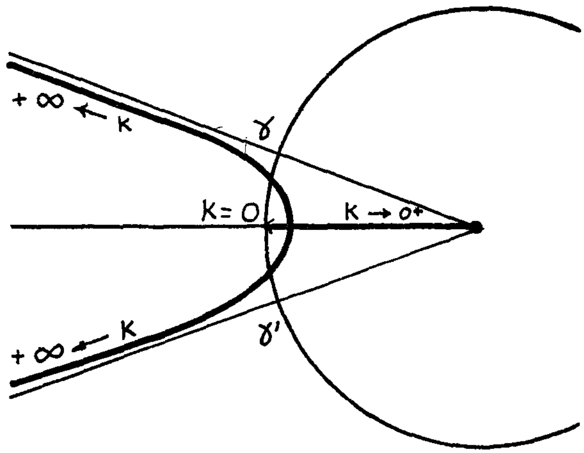


2. If $\beta \in E$, $\beta \notin B$, there is a root of equation (2) for each θ in the sector bounded by arg (β) and arg (γ). θ changes monotonically with k. As $k \to 0$, $\rho \to 0$ and the curve is tangent to the ray $\theta = \arg(\beta)$. As $k \to +\infty$, $\rho \to +\infty$ and the curve is asymptotic to the ray $\theta = \arg(\gamma)$.



3. If $\beta = -1$ there is a γ' in G such that β is equidistant from γ and γ'. The trajectories of roots in the sector bounded by $\arg(\gamma)$ and $\arg(\gamma')$ is as shown in the diagram with the motion being monotonic in k either side of the double point.

These are all the roots of equation (2) with k positive.



Part 3. In this section, we will verify the descriptive claims that we made in the previous section. As we will handle equations (1) and (2) separately, we will subdivide this section into Part 3 *A* and Part 3 *B*.

Part 3 *A*: Here we shall study only equation (1), $\alpha z^{r+s} + (1-\alpha) z^r - 1 = 0$.

a) Isolation of the sectors:

We start by taking the imaginary and real parts of equation (1).

(3) $$\alpha \rho^{r+s} \sin(r+s)\theta + (1-\alpha)\rho^r \sin r\theta = 0$$

and

(4) $$\alpha \rho^{r+s} \sin(r+s)\theta + (1-\alpha)\rho^r \cos r\theta - 1 = 0.$$

From equation (3), we get:

(5) $$\rho^s = \frac{\alpha - 1}{\alpha} \frac{\sin r\theta}{\sin(r+s)\theta}.$$

By first dividing equation (1) by z^{r+s} and then taking the imaginary part, we obtain in a similar manner:

$$\rho^r = \frac{\sin(r+s)\theta}{(1-\alpha)\sin s\theta} \quad (6)$$

Since ρ is always positive, we can use these last two equations to deduce the follow relations between the signs (abbreviated sgn) of our sin functions over various ranges of α. From equation (5), we see that if $\alpha < 0$ then $(\alpha - 1)/\alpha$ is positive so $\operatorname{sgn}(\sin r\theta) = \operatorname{sgn}(\sin(r+s)\theta)$; if $0 < \alpha < 1$ then $(\alpha - 1)/\alpha$ is negative so $\operatorname{sgn}(\sin r\theta) = -\operatorname{sgn}(\operatorname{sir}(s+s))$; if $1 < \alpha$ then $(\alpha - 1)/\alpha$ is positive so $\operatorname{sgn}(\sin r\theta) = \operatorname{sgn}(\sin(r+s)\theta)$. From equation (6), we have that if $\alpha < 1$ then $1 - \alpha$ is positive so $\operatorname{sgn}(\sin s\theta) = \operatorname{sgn}(\sin(r+s)\theta)$; if $1 < \alpha$ then $1 - \alpha$ negative so $\operatorname{sgn}(\sin s\theta) = -\operatorname{sgn}(\sin(r+s)\theta)$. Putting this information together gives:

$$\begin{array}{ll} \alpha < 0 & \operatorname{sgn}(\sin r\theta) = \operatorname{sgn}(\sin(r+s)\theta) = \operatorname{sgn}(\sin s\theta) \\ 0 < \alpha < 1 & -\operatorname{sgn}(\sin r\theta) = \operatorname{sgn}(\sin(r+s)\theta) = \operatorname{sgn}(\sin s\theta) \\ 1 < \alpha & \operatorname{sgn}(\sin r\theta) = \operatorname{sgn}(\sin(r+s)\theta) = -\operatorname{sgn}(\sin s\theta). \end{array}$$

Solving for α in equations (3) and (4) and equating the two expressions yields the following α-free equation for the trajectories of the roots of (1):

$$\rho^{r+s}\sin s\theta - \rho^s \sin(r+s)\theta + \sin r\theta = 0. \quad (7)$$

If θ is such that it satisfies either the second or third equation on signs above, in particular, if $\sin r\theta$ and $\sin s\theta$ have opposite signs, then equation (7) will have leading coefficient positive and value at zero negative or vice-versa. In either case, it will have a positive root ρ.

Now, let us look at an interval of length less than or equal to $\frac{2\pi}{2(r+s)}$, bounded on one side by $\arg(\beta)$ and on the other by $\arg(\gamma)$, where γ is an $(r+s)^{th}$ root of unity and β is an r^{th} root of unity or an s^{th} root of -1 but not both. In such an interval, the second set of equalities in satisfied. We shall see in the next section of this part that θ is monotonic in α in these intervals with α ranging from 0 at $\arg(\beta)$ to 1 at $\arg(\gamma)$. Hence, for each α between 0 and 1 there will be a root of (1) in each of these sectors. If β is both an r^{th} root of unity and an s^{th} root of -1, then we will again have monotonicity between α and θ on the interval but α will only take on the values less than 1 and such that

$$\frac{|(1-\alpha)|^{r+s}}{\alpha^s} < \frac{(r+s)^{r+s}}{r^r s^s}.$$

For smaller positive $\alpha's$ there will be first a double root on the ray $\theta = \arg(\beta)$ and then the two roots on this ray. For any α between zero and one, we have now accounted for $r + s$ roots of equation (1). Since there are no other roots to worry about, we have shown that for this range of α, the roots of (1) lie in the sectors that we specified in the previous part. The transformation $\alpha \to 1/a$, $x \to 1/y$, takes care of our assertions for α bigger than one.

For α negative, we look at intervals of θ of length $\leq \frac{2\pi}{2(r+s)}$ bounded on one side by $\arg(\beta)$, where β is an r^{th} or s^{th} root of unity and on the other side by $\arg(\gamma)$ where γ is an $(r+s)^{th}$ root of -1. In such an interval, the first set of equalities hold. This does not imediately imply that for each negative α there is a root of (1) with argument in this interval. Though this is true, the demonstration will take a bit more machinery so that we leave it for the next section of this part.

b) Investigation of the change in θ with respect to α:

As long as we keep in mind that our search for roots must be restricted to intervals where one of our three sets of equalities hold, we can replace equation (5) by

$$\rho = \left|\frac{\alpha - 1}{\alpha}\right|^{1/s} \frac{|\sin r\theta|^{1/s}}{|\sin(r+s)\theta|^{1/s}}. \tag{8}$$

Substituting into equation (4) the expression given for ρ by equation (8) yields:

$$\alpha \left|\frac{\alpha - 1}{\alpha}\right|^{(r+s)/s} \frac{|\sin r\theta|^{(r+s)/s}}{|\sin(r+s)\theta|^{(r+s)/s}} \cos(r+s)\theta +$$

$$+ (1-\alpha)\left|\frac{\alpha - 1}{\alpha}\right|^{r/s} \frac{|\sin r\theta|^{r/s}}{|\sin(r+s)\theta|^{r/s}} \cos r\theta = 1.$$

With patience, we proceed to simplify this expression:

$$\frac{\sin r\theta|^{r/s}}{|\sin(r+s)\theta|^{r/s}} \left|\frac{\alpha - 1}{\alpha}\right|^{r/s} \times$$

$$\times \left(\frac{\alpha\,|\alpha - 1|}{|\alpha|} \frac{|\sin r\theta|}{\sin(r+s)\theta} \cos(r+s)\theta + (1-\alpha)\cos r\theta\right) = 1.$$

We note that the three sets of equalities tell us that if:

$$\frac{|\sin r\theta|}{|\sin (r+s)\theta|} = \frac{\sin r\theta}{\sin (r+s)\theta} \text{ then } \underset{\alpha<0}{\overset{\alpha>1}{\text{or}}} \text{ but then } \frac{|\alpha-1|}{\alpha} = \frac{\alpha-1}{\alpha}$$

$$\frac{|\sin r\theta|}{|\sin (r+s)\theta|} = \frac{-\sin r\theta}{\sin (r+s)\theta} \text{ then } 0<\alpha<1 \text{ and } \left|\frac{\alpha-1}{\alpha}\right| = \frac{1-\alpha}{\alpha}.$$

In either case, the equation becomes:

$$\frac{|\sin r\theta|^{r/s}}{|\sin (r+s)\theta|^{r/s}} \left|\frac{\alpha-1}{\alpha}\right|^{r/s} (1-\alpha)(\sin (r+s)\theta \cos r\theta - \sin r\theta \cos (r+s)\theta) =$$

$$= \sin (r+s)\theta.$$

This in turn reduces to:

$$\frac{|\sin r\theta|^{r/s}}{|\sin (r+s)\theta|^{r/s}} \left|\frac{\alpha-1}{\alpha}\right|^{r/s} (1-\alpha) \sin s\theta = \sin (r+s)\theta.$$

As long as one of the three sets of equalities is satisfied, we have,

$$\left|\frac{(1-\alpha)\sin s\theta}{\sin (r+s)\theta}\right| = \frac{(1-\alpha)\sin s\theta}{\sin (r+s)\theta}.$$

And we can change our equation to:

$$\frac{|1-\alpha|^{(r+s)/s}}{|\alpha|^{r/s}} = \frac{|\sin (r+s)\theta|^{(r+s)/s}}{|\sin r\theta|^{r/s}\,|\sin s\theta|}.$$

Taking the s^{th} powers of both sides, we have more simply:

$$\text{(9)} \qquad \frac{|1-\alpha|^{r/s}}{|\alpha|^{r}} = \frac{|\sin (r+s)\theta|^{r+s}}{|\sin r\theta|^{r}\,|\sin s\theta|^{s}}.$$

Now, differentiating both sides with respect to θ and leaving out the algebraic manipulations, we get:

$$\text{(10)} \qquad \frac{d\alpha}{d\theta} = \frac{(1-\alpha)\,\alpha\,|\alpha|^{r}\,|\sin (r+s)\theta|^{r+s} A}{(s\alpha+r)\,|1-\alpha|^{r+s}\,|\sin r\theta|^{r}\,|\sin s\theta|^{s} \sin (r+s)\theta\, \sin r\theta \sin s\theta}.$$

where $A = r^2 \sin^2 s\theta + s^2 \sin^2 r\theta - 2rs(\cos(r+s)\theta \sin r\theta \sin s\theta)$. So that A lies between the two squares $(r \sin s\theta - s \sin r\theta)^2$ and $(r \sin s\theta + s \sin r\theta)^2$. So A could be zero only if both $\cos(r+s)\theta$ was $+1$ and the appropriate square was zero. This can happen only if $r = s$, a case which doesn't concern us at the moment since we are insisting that $\gcd(r, 2s) = 1$. The derivative, $d\alpha/d\theta$ will be zero or go to infinity only at points where one of the following occurs. $\sin s\theta = = 0$, $\sin r\theta = \theta$, $\sin (r+s)\theta = 0$, $\alpha = 0$, $\alpha = 1$, or $\alpha = -r/s$. For positive α then, $d\alpha/d\theta$ does not have any singularities in the interior of these intervals we described in the last section. Therefore, as we claimed, θ is a monotonic function of α in those intervals.

We must now see what happens when α is negative. $\alpha = -r/s$ is surely of particular interest. First notice that it is the α between $-\infty$ and 0 at which the left hand side of equation (9) takes on its minimum value, $\dfrac{(r+s)^{r+s}}{r^s s^r}$.

We must see what happens to the right side of this equation in an interval of length less than $2\pi/2(r+s)$ which is bounded on one side by $\arg(\beta)$ and one other by $\arg(\gamma)$, where γ is an $(r+s)^{th}$ root of -1, β is an r^{th} root of unity or an s^{th} root of unity. At $\arg(\beta)$ the right hand side of (9) is at $+\infty$ and it decreases monotonically to 0 as θ moves from $\arg(\beta)$ to $\arg(\gamma)$. If θ_0 is the unique point in this interval such that

$$|\sin (r+s)\theta_0|^{r+s}/|\sin r\theta_0|^r \, |\sin s\theta_0|^s = (r+s)^{r+s}/r^s s^r,$$

then for each negative α, there will be a θ in the interval bounded by $\arg(\gamma)$ and θ_0 such that α, θ satisfy equation (9). Each θ in the inferior of the interval will be good for two $\rho' s$.

c) Investigation of the change in ρ with respect to θ:

We would like to show that ρ changes monotonically with respect to θ and hence with respect to α in those sectors where θ lies between $\arg(\beta)$ and $\arg(\gamma)$ where γ is an $(r+s)^{th}$ root of unity and β is an s^{th} root of -1. In fact, we have never demonstrated that this is the case but we present the following work which leads to an equation in θ alone that is satisfied by those points on the trajectories where $d\rho/d\theta = 0$.

We make heavy use of the α-free equation of the trajectories of equation (1). Recall:

$$\rho^{r+s} \sin s\theta - \rho^s \sin (r+s)\theta + \sin r\theta = 0. \tag{7}$$

From this, we can evaluate $d\rho/d\theta$:

$$d\rho/d\theta = -\frac{(s\rho^{r+s} \cos s\theta - (r+s)\rho^s \cos (r+s)\theta + \cos r\theta}{(r+s)\rho^{r+s-1} \sin s\theta - s\rho^{s-1} \sin (r+s)\theta}. \tag{11}$$

First notice that on an interval of the type that we are considering, $d\rho/d\theta$ as given by equation (11) is well-defined, i.e. the denominator is never 0. If it were zero, we would have

$$\rho^r = \frac{s \sin (r+s)\theta}{(r+s) \sin s\theta},$$

but then substituting this in equation (7) yields:

$$\rho^s \left(\frac{s}{r+s} \sin (r+s)\theta - \sin (r+s)\theta\right) + \sin r\theta = 0,$$

or

$$\rho^s = \frac{(r+s) \sin r\theta}{r \sin (r+s)\theta},$$

but this is impossible as $\operatorname{sgn}(\sin r\theta) = -\operatorname{sgn}(\sin (r+s)\theta)$ when $0 < \alpha < 1$. Notice that this part of the argument applies whenever α is in this interval and that the restriction on θ was not important. To assure monotonicity of ρ in θ, we have only to show that $d\rho/d\theta$ is never zero in the interval. That is, $s\rho^{r+s} \cos s\theta - (r + s)\rho^s \cos (r+s)\theta + r \cos r\theta \neq 0$.

We will now set out to find an equation in θ that will isolate those values of of θ or which $d\rho/d\theta$ is zero. For this, we will first treat this last equation and equation (7) as a pair of simultaneous equations in ρ^s and ρ^{r+s}, to get:

$$\rho^{r+s} = \frac{r \sin s\theta - s \sin r\theta \cos (r+s)\theta}{r \sin s\theta \cos (r+s)\theta - s \sin r\theta}$$

and

$$\rho^s = -\frac{s \sin r\theta \cos s\theta + r \sin s\theta \cos r}{r \sin s\theta \cos (r+s)\theta - s \sin r\theta}.$$

Equating $(\rho^s)^{r+s}$ and $(\rho^{r+s})^s$ gives the ρ-free equation:

$$(-s \sin r\theta \cos s\theta + r \sin s\theta \cos r\theta)^{r+s} =$$

$$= (r \sin s\theta - s \sin r\theta \cos (r+s)\theta)^s (r \sin s\theta \cos (r+s)\theta - s \sin r\theta)^r,$$

which is satisfied by those θ's for which $d\rho/d\theta = 0$.

d) Verification of the asymptotes:

We can see from figure 2, ($r = 1, s = 2$) that the trajectory of the roots of equation (1) is not necessarily asymptotic to the rays $\theta = \arg(\beta)$, where β is an s^{th} root of -1. In fact, as we shall see, this example is really a very special case. If $r > 1$, these rays will in fact be asymptotes. Consider the following case where $\theta_0 = \arg(\beta)$ and p_1 is a point on the trajectory (7), $\rho^{r+s} \sin s\theta - \rho^s \sin(r + s)\theta + \sin r\theta = 0$ We know from the past section that as $\theta \to \theta_0$, $\rho \to \infty$ and we must show that this implies that the distance between p_1 and the ray goes to zero. In other word, we must show that $\rho \sin(\theta_0 - \theta) \to 0$. Dividing equation (1) by ρ^{s+1}, yelds $\rho^{r-1} \sin s\theta - \frac{1}{\theta} \sin(r + s)\theta + \frac{1}{\rho^{s+1}} \sin r\theta + 0$.

Since $\sin(r + s)\theta$ and $\sin r\theta$ are non-zero near θ_0, we have that $\rho \to \infty$ implies that $\rho^{r-1} \sin s\theta \to 0$ so that if $r > 1$ we have $\rho \sin s\theta \to 0$. Since $\cos s\theta_0 = \pm 1$, $\rho \sin(s\theta^0 - s\theta) = \rho(\sin s\theta_0 \cos s\theta = \sin s\theta \cos s\theta_0) \to 0$.

But as $\sin(-\theta_0 - \theta) > \sin(\theta_0 - \theta)$ for θ near θ_0, we finally arrive at $\rho \sin(\theta_0 - \theta) \to 0$.

e) Discussion of the bounds on ρ:

We would like to place a bound on ρ especially in those sectors bounded by $\arg(\beta)$ and $\arg(\gamma)$ where γ is an $(r + s)^{th}$ root of unity and β is an r^{th} root of unity for we claim that for these values of θ, ρ never gets very big.

We start as usual with equation (1), but this time we divide by ρ^{r+s} before taking the imaginary part, so that we have:

$$(1 - \alpha)\rho^s \sin s\theta = \rho^{-(r+s)} \sin(r + s)\theta,$$

or

$$\rho^r = \frac{+1\, |\sin(r + s)\theta|}{|1 - \alpha|\, |\sin s\theta|}.$$

From this, we easily get the first of the inequalities that we will use:

$$\text{(I)} \qquad \rho^r < \frac{1}{|\sin s\theta|\, |1 - \alpha|}.$$

Equation (9) gives us:

$$\frac{|\sin r\theta|^r}{|\sin s\theta|^r\, |\alpha|^r} = \left(\frac{|\sin(r + s)\theta|}{|\sin s\theta|\, |1 - \alpha|}\right)^{r+s}.$$

From this we form the second inequality that we will use:

$$\text{(II)} \qquad \rho^r = \left(\frac{|\sin r\theta|^r}{|\sin s\theta|^r\,|\alpha|^r}\right)^{1/(r+s)} \leq \left(\frac{1}{|\sin s\theta|^r\,|\alpha|^r}\right)^{1/(r+s)}.$$

Now, if $0 < \alpha < 1/2$, then $(1-\alpha) > 1/2$ and from (I) we have $\rho^r < 2/|\sin s\theta|$. On the other hand, for $\alpha > 1/2$, we have from (II) that:

$$\rho^r < \left(\frac{2^r}{|\sin s\theta|^r}\right)^{1/r+s} = \frac{2^{r/r+s}}{|\sin s\theta|^{r/r+s}} < \frac{2}{\sin s\theta}.$$

And so we have that for $0 < \alpha < +\infty$, $\rho^r < 2/|\sin s\theta|$.

If we now restrict our attention to an interval of length less than $2\pi/(r+s)$ which is bounded on one side by $(k/r)2\pi$ and on the other by $(j/(r+s))2\pi$ we notice that on such an interval,

$$|\sin s\theta| \geq \min\{|\sin((r+s)(k/r)2\pi)|, |\sin(r(j/(r+s))2\pi)|\}.$$

We know that this last statement is true since $\sin s\theta$ does not change sign on the interval in question and is concave toward the x-axis. We must now see how small $\sin(r+s)(k/r)2\pi = \sin sk2\pi/r$ and $\sin(rj2\pi/(r+s))$ can be. We assume, as before that $\gcd(r, 2s) = 1$ so that neither of these sin's can be zero. Sin $(sk2\pi/r)$ is smallest when $sk2\pi/r$ is nearest to $n\pi$ for some integer n. That is, when $2sk/r$ is as close to an integer as it can be or when n and k are such that $2sk = rn$ is as small in magnitude as possible. As the smallest that this can be is 1, we have that:

$$\left|\sin(r+s)\frac{k}{r}2\pi\right| \geq \sin\frac{\pi}{r} > \sin\frac{\pi}{r+s}.$$

Similarly, $\sin(rj2\pi/(r+s))$ is smallest when $(rj2\pi/(r+s))$ is as near as possible to $n\pi$ for some integer n. As with the above, we arrive at:

$$\left|\sin\frac{rj2\pi}{(r+s)}\right| \geq \sin\frac{\pi}{r+s} > \frac{\pi}{r+s} - \frac{\pi^3}{6(r+s)^3}.$$

We then have ρ bounded as follows:

$$\rho^r < \frac{2}{|\sin s\theta|} < \frac{2}{\dfrac{\pi}{r+s} - \dfrac{\pi^3}{6(r+s)^3}}.$$

For large $r+s$, this tells us that ρ^r is of the order of $(2(r+s)/\pi)$ but even for small $r+s$, as long as both r and s are greater than or equal to one, it easily gives us that:

$$\rho^r < \frac{4(r+s)}{\pi} \quad \text{or} \quad \rho < \left(\frac{4(r+s)}{\pi}\right)^{1/r}.$$

We now know that the moduli of root of (1) do not get very big in these intervals at least when $r \geq s$.

We can partially take care of the case $s \geq r$ by restricting our attention to α bounded away from 0, for example, $\alpha \geq 1/2$.

From the immaginary part of equation (I), we have:

$$\rho^s = \left| \frac{(1-\alpha)\sin r\theta}{\alpha \sin (r+s)\theta} \right|.$$

By substituting from equation (I) as we have just done, this becomes:

$$\rho = \left| \frac{\sin r\theta}{\alpha \sin s\theta} \right|^{1/(r+s)}.$$

But then, if $\alpha > 1/2$, we arrive at the inequality:

$$\rho < \left| \frac{2}{\sin s\theta} \right|^{1/(r+s)}.$$

Using the same bound on $\sin s\theta$ that we just found, we can now say:

$$\rho < \left| \frac{2}{\dfrac{\pi}{r+s} - \dfrac{\pi^3}{6(r+s)^3}} \right|^{1/(r+s)}.$$

This time we have a bound that in fact goes to one as $r+s$ gets big, independently of any relation between r and s.

Part 3 *B*: We now proceed to verify our claims for the second equation:

$$z^{r+s} + z^r + k = 0, \tag{2}$$

where k is *a* positive real. Following the format of Part 3 *A*, we start with:

a) Isolation of the sectors:

We take the real and imaginary parts of equation (2) to yield:

$$\rho^{r+s} \sin (r+s)\theta + \rho^{r} \sin r\theta = 0 \tag{12}$$

and

$$\rho^{r+s} \cos (r+s)\theta + \rho^{r} \cos r\theta + k = 0. \tag{13}$$

From equation (12), we have

$$\rho^{s} = -\frac{\sin r\theta}{\sin (r+s)\theta}. \tag{14}$$

If we had divided equation (12) by z^{r+s} before taking the imaginary part, we would have obtained in a similar fashion:

$$\rho^{r} = -\frac{k \sin (r+s)\theta}{\sin s\theta}. \tag{15}$$

Since ρ is always positive and we are only worried about positive k, equations (14) and (15) tell us that equation (2) can have solutions only in those sectors where:

$$\operatorname{sgn}(\sin r\theta) = -\operatorname{sgn}(\sin (r+s)\theta) = \operatorname{sgn}(\sin s\theta).$$

These equalities are surely satisfied in any interval of length less than or equal to $(2\pi/2(r+s))$ and with endpoints $\arg(\beta)$ and $\arg(\gamma)$ where γ is an $(r+s)^{th}$ root of -1 and β is an r^{th} or s^{th} root of -1. There are $r+s$ such disjoint sectors if we include the trivial sector $\theta = 0$. We now show that any θ satisfying the equalities on the sgn's of the sin functions is in fact the argument of a solution of (2) and that in each of these $r+s$ sectors, there is a solution for each k.

Suppose that θ does satisfy $\operatorname{sgn}(\sin r\theta) = -\operatorname{sgn}(\sin (r+s)\theta) = \operatorname{sgn}(\sin s\theta)$, then equation (14) gives us a positive ρ such that $z = \rho e^{i\theta}$ is a solution of equation (2) for some k. In addition, we can see from equation (15) that the k in question will be positive. Let us restrict our attention to an interval bounded by $\arg(\beta)$ and $\arg(\gamma)$, as described above. Then from (14),

$$\rho^{s} = -\sin s\theta/\sin (r+s)\theta$$

we see that ρ goes to $+\infty$ as $\theta \to \arg(\gamma)$. ρ changes continuously on the interval, going down to 0 if $\sin r(\arg(\beta)) = 0$ or just down to -1 if $\operatorname{sins} s(\arg(\beta)) = 0$.

Then, from equation (15),

$$k = -\frac{\rho^r \sin s\theta}{\sin(r+s)\theta}$$

so that $k \to \infty$ as $\theta \to \arg\gamma$ (we have $\rho \to +\infty$ from eq. 14), and k changes continuously on the interval, approaching 0 as $\theta \to \arg(\beta)$ either because $\sin s(\arg(\beta)) = 0$ and $\sin((r+s)\arg(\beta)) \neq 0$, $\rho \to 1$; or because $\rho \to 0$, i.e. $\sin(r \arg(\beta)) = 0$ and $\sin(s \arg(\beta)) = -\sin((r+s)\arg(\beta)) \neq 0$. In either case, k does take on all values between 0 and $+\infty$.

All of this presumed, of course that $\sin((r+s)\arg(\beta)) \neq 0$, or that β was not both an r^{th} and an s^{th} root of -1. Now suppose that β is both an r^{th} and s^{th} root of -1 so that it is equidistant from two $(r+s)^{th}$ roots of -1 γ and γ'. We still have that $\rho \to +\infty$ as θ approaches $\arg\gamma$ or $\arg\gamma'$, but now, as θ approaches $\arg(\beta)$, $(-\sin r\theta/\sin(r+s)\theta) \to r/(r+s)$ so that $\rho \to (r/(r+s))^{1/s}$. At this point, $k = (r/(r+s)^{r/s}(s/(r+s))$ and equation (2) has a double root. (2) restricted to the ray $\theta = \arg(\beta)$ is just $\rho^{r+s} - \rho^r + k = 0$ and as long as $k < (r/(r+s))^{r/s}(s/(r+s))$, this equation has two roots between 0 and 1, one either side of $(r/(r+s))^{1/s}$, as $k \to 0$, the equation becomes close to $\rho^{r+s} - \rho^r = 0$ so we can see that one root actually approaches 0 and the other 1.

b) Monotonicity of θ in k:

From equations (14) and (15) we get a ρ-free equation relating k and θ,

$$\left(\frac{-\sin r\theta}{\sin(r+s)\theta}\right)^r = \left(\frac{-k\sin(r+s)\theta}{\sin s\theta}\right)^s.$$

We mentioned in the introduction that we need only worrey about $r+s$ even, and we will assume that now to get rid of the minus signs, so that the equation becomes:

$$k^s = \frac{(\sin r\theta)^r(\sin s\theta)^s}{(\sin(r+s)\theta)^{r+s}}. \tag{16}$$

Differentiating both sides with respect to θ, we find:

$$\frac{dk}{d\theta} = -\frac{(\sin r\theta)^{r-1}(\sin s\theta)^{s-1}}{sk(\sin(r+s)\theta)^{r+s+1}}(r^3\sin^2 s\theta + s^2\sin r\theta + \\ -2rs\sin r\theta\sin s\theta\cos(r+s)\theta). \tag{17}$$

The quantity in the parentheses on the right appeared in $d\alpha/d\theta$ in Part 3 A and here too, it is never zero. Of course $\sin s\theta$, $\sin r\theta$ and $\sin(r+s)\theta$ are never zero in the interval, hence $dk/d\theta$ is non-zero and θ is monotonic in k and vice-versa.

c) Change in ρ with respect to θ:

ρ is given very simply in terms of θ by equation (14): $\rho = -\sin r\theta/\sin(r+s)$. Though ρ appears to be monotonic in θ when β is an r^{th} root of -1 (1), we can easily see from (14) that ρ is not monotonic in θ when β is an s^{th} root of -1, or at least it is not always monotonic. In figure 10 are the three possible forms that the sin functions can take in the interval if β is an s^{th} root of -1 and to the right of each of these is a sketch of the corresponding trajectory.

d) Verification of the asymptotes:

We again use equation (14), $\rho^s = -\sin r\theta/\sin(r+s)\theta$. Then, as long as $s > 1$, $\rho\sin(r+s)\theta = -\sin r\theta/\rho^{s-1}$, and it follows that as $\rho\to\infty$, $\rho\sin(r+s)\theta\to 0$. As in Part 3 A, this implies that the rays $\theta = \arg(\gamma)$ are really asymtotes when $s > 1$.

e) Bounds on ρ:

In the light of section *c)*, we should be interested in finding a lower bound on ρ in those sectors where β is an s^{th} root of unity. This time, we are able to get a result when $s \geq r$. From (14), we get the inequality:

$$\rho^s > \frac{\sin r\theta}{1} > \sin(\pi/(r+s)).$$

Using the same estimate on $\sin(\pi/(r+s))$ that we did in Part 3 A, we have,

$$\rho > \left(\frac{\pi}{r+s} - \frac{\pi^3}{6(r+s)^3}\right)^{1/s}.$$

This bound approaches 1 as s approaches infinity. We can arrive at a bound in the case $r > s$ if we are willing to restrict our interest to k greater than some

(1) I have not succeeded in demonstrating this or the monotonicity of ρ in the α case.

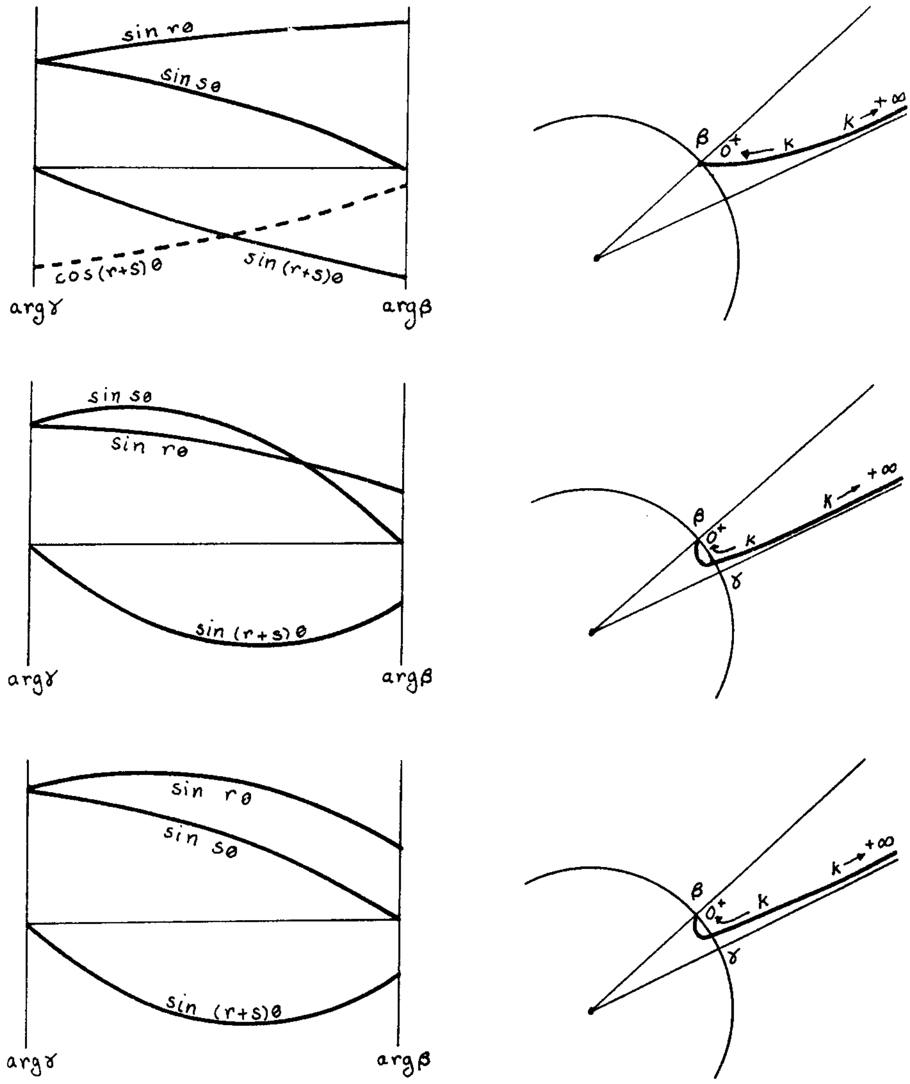

sin rθ
sin sθ
cos (r+s) θ
sin (r+s)θ
arg γ
arg β
β
0+
k
k → +∞
sin sθ
sin rθ
sin (r+s) θ
arg γ
arg β
β
0+
k
k → +∞
γ
sin rθ
sin sθ
sin (r+s)θ
arg γ
arg β
β
0+
k
k → +∞
γ

Fig. 10

constant, k_0. Then, taking the product of equations (14) and (15) we have:

$$\rho^{r+s} = \frac{k \sin r\theta}{\sin s\theta} > k \sin r\theta > k_0 \left(\frac{\pi}{r+s} - \frac{\pi^3}{6(r+s)^3}\right)$$

or,

$$\rho > \left(\frac{\pi}{r+s} - \frac{\pi^3}{6(r+s)^3}\right)^{1/(r+s)} (k_0)^{1/(r+s)}$$

and this bound goes to 1 as $r+s$ approaches infinity.

BIBLIOGRAPHY

[1] Biernaky M., *Sur un nouveau théorème d'Algèbre,* C.R. Académie des Sciences, **177** (1923), 1193-1194.

[2] Dieudonné J., *Sur une équation quadrinome,* Mathematica Cluj, **10** (1934), 37-45.

[3] Dieudonné J., *Sur quelques points de la théorie des polynômes,* Bull. Sci. Math. (2) **58** (1934), 273-296.

[4] Herglotz G., *Ueber die Wurzelanzahl algebraischer Gleichungen innerhalb und auf dem Einheitskreis,* Math. Zeith. **19** (1924), 26-34.

[5] Kempner A.J., *Ueber die Separation komplexer Wurzeln algebraischer Gleichungen,* Math. Ann., **85** (1922), 49-59.

[6] Kempner A.J., *On the complex roots of algebraic equations,* Bull. Amer. Math. Soc., **41** (1935), 809-843.

[7] Marden M., *The Geometry of the Zeros of a Polynomial in a Complex Variable,* Math. Surveys, Amer. Math. Soc., **3** (1949).

[8] Nekrassoff P., *Ueber trinomische Gleichungen,* Math. Ann., **29** (1887), 413-430.

[9] Toda K., *On the location of the roots of linear combinations of some polynomials,* J. Sci. Hirosima Univ., **7** (1937), 141-150.

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Northeastern University
Department of Mathematics
360 Huntington Avenue
Boston, Massachusetts 02115 (USA)