QUASIHYPERBOLIC GEOMETRY

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In comparison to Riemann spaces the theory of Finsler spaces developed on a very meager store of concrete examples, a fact which greatly hampered progress. Any new special, non-artificial metric is likely to provide important hints for the general theory. The present paper bears this contention out by studying the *quasihyperbolic plane,* which is the only two-dimensional Finsler space with a transitive group of motions which has not yet been thoroughly investigated ([1]).

We first briefly recall the definition and known properties, then procure a *model* and determine *all* quasihyperbolic metrics. In contrast to Minkowskian geometry the differentiability rather than the strict convexity of the local unit circle proves decisive. The theory of parallels is exactly as in hyperbolic geometry, but the geodesics satisfy Desargues' Theorem only in the hyperbolic case.

The circles are convex with respect to the orbits of the one-parameter subgroups of the group of motions, but they are in general *not* geodesically convex. However, each circle contains a subarc comprising a semicircle which is convex. This implies the existence of *one-sided perpendiculars,* a phenomenon which does not seem to have been encountered previously. The loci equidistant from a geodesic segment therefore also fail to be convex, and still less has a quasihyperbolic plane negative curvature in the sense of the author, see [1, Section 36].

However, under a mild condition on the local unit circle whith is always

([1]) The other surfaces (= two dimensional spaces) with transitive groups of motions are: the elliptic plane, the sphere, and the plane, cylinder, or torus with Minkowskian metrics, see [1, Theorem (52.7)]. The results of [1] will be freely used.

satisfied by the Finsler spaces usually considered, there is a *unique* angular measure, [1, Section 42], which is invariant under motions and yields a completely additive excess, or equivalently, a *Gauss-Bonnet Theorem.* With this angular measure the area of a triangle is proportional to the excess.

The uniqueness of this measure leads us to the important conclusion that a problem which has interested many mathematicians does not have a solution: *there is no universal* (1) *angular measure for two-dimensional Finsler spaces such that the Gauss-Bonnet Theorem always holds.*

1. Definition and model.

A straight two-dimensional G-space Q is *quasihyperbolic* if it possesses all translations along two geodesics G, H, where G is an asymptote to H, but not parallel to H, see [1, Section 51]. It follows (ibidem) that the asymptotes to H are all asymptotes to each other. The asymptotes in this family F and the limitcircles in the family L which have F as central rays will be called *distinguished.* All translations along each element of F or L exist, a translation along one element of L is a translation along all others. These translations form together a simply transitive group Γ of motions of Q.

Let $p(\tau)$ represent an arbitrary element G_0 of F such that $p(\tau)$ lies for $\tau > 0$ in the interior of the limitcircle L_0 in L through $p(0)$. Let $z(\sigma)$ represent L_0 in terms of arc length with $z(0) = p(0)$. An arbitrary point of Q lies on exactly one geodesic in F which intersects L_0 in a point $z(\sigma)$ and on exactly one limitcircle in L which intersects G_0 in a point $p(\tau)$. In terms of these coordinates σ, τ the group Γ takes the form

$$\sigma' = \sigma \delta^{\beta} + \alpha, \qquad \tau' = \tau + \beta, \tag{1}$$

where δ is a fixed number between 0 and 1 and each pair α, β of real numbers determines a motion in Γ.

Conversely, [1, Theorem (52.9)], if a (σ, τ)-plane is metrized as a G-space for which the transformations (1) are motions, then it is a quasihyperbolic plane Q'. Choose a hyperbolic metrization H of the (σ, τ)-plane such that the parameters σ, τ have for H the same meaning as above for Q. As model for H take the upper halfplane of an (x, y)-plane with the metric $ds^2 = (dx^2 + dy^2)y^{-2}$ with the common endpoint of the distinguished family $x = \text{const}$ at infinity. The orbits of the one-parameter subgroup of Γ (simply called orbits) are the

(1) This will be defined precisely.

circles through this point, hence the euclidean straight lines, or rather their intersections with $y > 0$. The group Γ then becomes the group of conformal mappings of the upper halfplane on itself which leave the point at infinity fixed, i.e.

$$x' = \beta x + \delta, \quad y' = \beta y, \quad \beta > 0, \quad -\infty < \delta < \infty. \tag{2}$$

The proof of [1, Theorem (52.9)] shows that $y = \text{const}$ are the distinguished limitcircles for both H and Q', and that the family F of Q' need not coincide with lines $x = \text{const}$, but may consist of the curves equidistant in the hyperbolic sense to the lines $x = \text{const}$ at the same distance from, and on the same side of, these curves, i.e. F may be any family of parallel lines other than $y = \text{const}$. It was already observed in [1, l.c.] that it means no restriction on the generality of a quasihyperbolic metric, if we assume that F is the family $x = \text{const}$. In fact, the hyperbolic line element in the form $(E\,dx^2 + 2F\,dx\,dy + G\,dy^2)y^{-2}$, where E, F, G are constant, has for a proper choice of E, F, G a given family of parallel lines (other than $y = \text{const}$) as geodesics.

This will be our *model* M for quasihyperbolic geometry, i.e. the geometry is defined for $y > 0$, the distinguished families F and L consist of the lines $x = \text{const.}$ and $y = \text{const.}$, and the group Γ is given by (2).

We now investigate the shape of the *ordinary* (= non-distinguished) *geodesics* in M. The proof of [1, (52.9)] implies that an ordinary geodesic intersects an orbit at most twice. Since an equidistant curve to a distinguished geodesic is in the quasihyperbolic sense strictly convex and turns its concave side towards the geodesic, [1, (51.1)], the ordinary geodesics in M are strictly convex curves in the euclidean sense which turn their concave sides downwards. If $p(\tau)$ represents an ordinary geodesic, then $p(\tau)$ tends for $\tau \to \infty$ or $\tau \to -\infty$ to a point on the x-axis. These two points will be called the endpoints of $p(\tau)$. A line $x = \text{const.}$ being a geodesic intersects $p(\tau)$ at most once, so that $p(\tau)$ can be written in the form $y = f(x)$. All other ordinary geodesics are obtained from one by the transformations (2), hence have the form

$$y = \beta^{-1} f(\beta x + \delta), \quad \beta > 0. \tag{3}$$

We may therefore assume that $f(x)$ is defined for $-1 \leq x \leq 1$ with $f(-1) = f(1) = 0$, the points $(-1, 0)$ and $(1, 0)$ being the endpoints (of course $f(x) > 0$ for $-1 < x < 1$).

It follows that the geodesics with a common endpoint form a (complete) family of asymptotes, so that the asymptote relation is symmetric and transitive.

The general ordinary geodesic with $(-1, 0)$ as left endpoint is given by $y = \beta^{-1}f(\beta x + \beta - 1)$. For $\beta \to \infty$ this geodesic must tend to $x = -1$. This means that the (one-sided) tangent of $y = f(x)$ at $x = -1$ is perpendicular to the x-axis. The same holds at $x = 1$.

PROPERTY 1. *In the model* M *the ordinary geodesics have the form* $y=\beta^{-1}f(\beta x+\delta)$, $\beta > 0$, $-\infty < \delta < \infty$, *where* $y = f(x)$, $-1 \leq x \leq 1$, *is a strictly concave curve with* $f(-1) = f(1) = 0$, *whose right tangent at* $(-1, 0)$ *and left tangent at* $(1, 0)$ *are perpendicular to the x-axis.*

2. Determination of all quasihyperbolic geometries.

We are now going to show that *every* system of curves satisfying the conditions of Property 1 occurs as system of ordinary geodesics in a quasihyperbolic geometry.

4. If $f(x)$ behaves as in Property 1 then any two points in the upper half plane which do not have the same abscissa lie on exactly one curve $y = \beta^{-1}f(\beta x + \delta)$, $\beta > 0$.

For if (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ and $y_i > 0$ are given, then the curve $y=f(x)$ possesses a family of parallel chords with slope $(y_1 - y_2)(x_1 - x_2)^{-1}$. Among these chords there is exactly one, say from (x_1', y_1') to (x_2', y_2'), such that the trapezoid with vertices $(x_1', 0)$, (x_1', y_1'), (x_2', y_2'), $(x_2', 0)$ is homothetic to the trapezoid with vertices $(x_1, 0)$ (x_1, y_1), (x_2, y_2), $(x_2, 0)$, hence there is exactly one transformation (2) which sends $y = f(x)$ into a curve through (x_1, y_1) and (x_2, y_2).

The hyperbolic distance (induced by the line element $ds^2 = (dx^2 + dy^2)y^{-2}$) of two points p, q will be denoted by $h(p, q)$, the quasihyperbolic distance simply by pq, finally the euclidean distance by $e(p, q) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$, when $p = (x_1, y_1)$, $q = (x_2, y_2)$.

The lines $G_\alpha : x \cos\alpha = y \sin\alpha$, $\alpha \neq 0$, are the equidistant curves to $x = 0$ both in the hyperbolic and the quasihyperbolic sense; they go into themselves under the motions $x' = \beta x$, $y' = \beta y$. Therefore the hyperbolic and the quasihyperbolic arclengths along G_α differ only by a factor $c(\alpha)$, which is the same for any line parallel to G_α because $x' = x + \delta$, $y' = y$ is a motion. A slightly different argument or continuity proves the same for the lines $y = \text{const}$.

The ratio of length to chord along G_α tends to 1 almost everywhere, see [1, (5.14)]; using the motions $x' = \beta x$, $y' = \beta y$ shows that it equals 1 everywhere and that

$$\lim pq . h^{-1}(p, q) = c(\alpha) > 0,$$

whenever $p \neq q$ lie on G_α and approach the same point $r = (x_r, y_r)$. Since

$$\lim h(p, q) e^{-1}(p, q) = y_r^{-1}$$

it follows that

$$\lim pq . e^{-1}(p, q) = c(\alpha) y_r^{-1}.$$

If we define

$$F(u, v) = (u^2 + v^2)^{1/2} . c(\arctan u v^{-1}), \qquad F(0, 0) = 0,$$

then $F(ku, kv) = |k| F(u, v)$ for all real k and $F(u, v) > 0$ for $(u, v) \neq (0, 0)$. The triangle inequality for quasihyperbolic geometry yields in a standard way that $F(u, v)$ is a convex function of u and v. The line element of quasihyperbolic geometry has therefore the form

(5) $$ds = F(dx, dy) y^{-1}.$$

We have to investigate whether $F(u, v)$ must have any additional properties.

The curves G_α have a family of quasihyperbolic geodesics as orthogonal trajectories in the sense that a point on a geodesic H in this family has its intersection with G_α as foot on G_α, [1, (51.1)]. The other elements of the family are obtained from H by the motions $x' = \beta x$, $y' = \beta y$. Consequently G_α intersects H transversally, or a line through $(0, 0)$ parallel to a supporting line of H at $G_\alpha \cap H$ must intersect the curve

$$K \text{ with equation } F(x, y) = 1$$

at a point where K possesses a supporting line parallel to G_α. Using the results of [2] we obtain the following very surprising fact:

PROPERTY 2. *The line element ot the model* M *has the form* $ds = F(dx, dy) y^{-1}$.

The geodesics are the (intersections with $y > 0$ *of the) solutions of the isoperimetric problem for the Minkowski metric* $F(x_1 - x_2, y_1 - y_2)$ *in the* (x, y)*-plane whose centers lie on the x-axis.*

If perpendicularity (or transversality) in this Minkowski metric is symmetric, then the geodesics are the curves $F(x - \delta, y) = k$, $y > 0$.

Since, see [2], K is obtained from H by a polar reciprocity and a revolution through $\pi/2$, the strict convexity of H implies the differentiability of K, hence of $F(u, v)$ for $(u, v) \neq (0, 0)$. The fact that K has $(0, 0)$ as center implies that

(6) *The orthogonal trajectories of the lines* $x \cos \alpha = y \sin \alpha$ *are the curves* $y = \beta^{-1} f(x\beta)$.

Since the one-sided tangents of H at its endpoints are perpendicular to the x-axis, a whole solution of the isoperimetric problem possesses at its intersections with the x-axis a tangent. This means for K that its tangents at its intersections k_1, k_2 with the y-axis are perpendicular to the y-axis and touch K only at k_1 or k_2. Also, since H determines the shape of K:

PROPERTY 3. *If one ordinary geodesic of a quasihyperbolic metric is known, then the metric is determined up to a factor (when range and* Γ *are given).*

It will now be shown that we have obtained all characteristic properties of $F(u, v)$. Since the metric (5) is obviously invariant under the transformations (2), this assertion is (because of [1, Theorem (52.9)]) trivial as soon as $F(u, v)$ has the properties used in the calculus of variations to show that the extremals have the properties of the geodesics of a G-space. Therefore we will merely indicate very briefly how this result can be established without the assumptions of the calculus of variations.

Let $F(u, v)$ have all the mentioned properties, and represent a solution of the isoperimetric problem (in $y > 0$) with center (0, 0) belonging to $F(x_1 - x_2, y_1 - y_2)$ in terms of polarcoordinates ρ, α by $\rho = g(\alpha)$, $0 < \alpha < \pi$. This curve is cut transversally by all lines G_α. If $\rho = h(\alpha)$, $0 < \alpha < \pi$, represents a second curve cut transversally by the G_α, then $g'(\alpha)$ and $h'(\alpha)$ exist except at the most denumerable many values α which correspond to segments lying on K and $g'(\alpha)/g(\alpha) = h'/(\alpha)/h(\alpha)$. Since $\rho = g(\alpha)$ is a convex curve we have enough information regarding the behaviour of $g'(\alpha)/g(\alpha)$ (and hence of $h'(\alpha)/h(\alpha)$) in the neighborhood of the exceptional values to conclude that $h(\alpha)$ and $g(\alpha)$ differ only by a factor. Thus the curves $\rho = kg(\alpha)$ are the only curves cut transversally by the G_α.

Consider now a curve $\rho = q(\alpha)$, $\alpha_1 \leqslant \alpha \leqslant \alpha_2$ connecting two points (ρ_1, α_1), (ρ_2, α_2), $\rho_i = g(\alpha_i)$. Choose values $w_0 = \alpha_1 < w_1 < \cdots < w_n = \alpha_2$ such that the direction perpendicular to each G_{w_i} in the sense of the Minkowskian geometry is unique for $i \neq 0, n$. If (x_i, y_i) is the point $(q(w_i), w_i)$ and the partition (w_i) is sufficiently fine then $\Sigma F(x_i - x_{i+1}, y_i - y_{i+1})y_i^{-1}$ is arbitrarily close to the quasihyperbolic length of $\rho = q(\alpha)$. If (x_i', y_i') is the foot of (x_i, y_i) on $G_{w_{i+1}}$ then, unless $\rho = q(\alpha)$ is cut transversally by all G_α (i.e $q(\alpha)$ coincides by the preceding argument with $g(\alpha)$)

$$\sum F(x_i - x_i', \ y_i - y_i')y_i^{-1} < \sum F(x_i - x_{i+1}, \ y_i - y_{i+1})/y_i - \varepsilon,$$

where $\varepsilon > 0$ is the same for all sufficiently fine partitions (w_i). On the other hand, if u_i, v_i are the cartesian coordinates of the point $(g(w_i), w_i)$ and (u_i', v_i') its foot on $G_{w_{i+1}}$, then

$$\sum F(u_i - u_i', \ v_i - v_i') v_i^{-1} = \sum F(x_i - x_i', \ y_i - y_i') y_i^{-1}.$$

Because $\rho = g(\alpha)$ is cut transversally by the G_α the sum on the left differs for fine partitions (w_i) arbitrarily little from $\Sigma F(u_i - u_{i+1}, \ v_i - v_{i+1}) v_i^{-1}$, which in turn is arbitrarily close to the length of $\rho = g(\alpha)$, $\alpha_1 \leqslant \alpha \leqslant \alpha_2$. This length is therefore smaller than that of $\rho = q(\alpha)$. The estimate clearly only improves for curves connecting (ρ_1, α_1) and (ρ_2, α_2) and not representable in the form $\rho = q(\alpha)$.

Thus, if two points can be connected by an arc of a solution of the isoperimetric problem, then this arc is the unique shortest connection of the two points. But under our assumption on K the hypotheses of (4) are satisfied, which proves that the space is a G-space. Thus:

PROPERTY 4. *Every quasihyperbolic geometry can be obtained from a line element* $ds = F(dx, dy) y^{-1}$ *in* $y > 0$, *where* $|k|^{-1} F(kx, ky) = F(x, y)$ *is convex, differentiable and positive for* $(x, y) \neq (0, 0)$, *and the tangents of* $F(x, y) = 1$ *at its intersections with the y-axis are perpendicular to the y-axis and touch* $F(x, y) = 1$ *at only one point.*

Conversely, every line element of this form yields a quasihyperbolic geometry.

3. Parallels. Desarguesian spaces.

There are two positive constants d_1, d_2 such that

$$d_1 (x^2 + y^2)^{1/2} \leqslant F(x, y) \leqslant d_2 (x^2 + y^2)^{1/2}. \tag{7}$$

Let $p(\tau) = (x(\tau), y(\tau))$, $\tau \geqslant 0$ represent any half-geodesic with $p(\tau) \to (0, 0)$ for $\tau \to \infty$. If $q(\tau) = (x'(\tau), y'(\tau))$, $\tau \geqslant 0$ represents any other half-geodesic R with the same endpoint (0, 0), (i.e., is a co-ray to $p(\tau)$, see [1, Section 22]) then the fact that the tangent of these geodesics at their endpoint (0, 0) is the y-axis implies

$$x(\tau)/y(\tau) \to 0 \quad \text{and} \quad x'(\tau)/y'(\tau) \to 0 \quad \text{for} \quad \tau \to \infty.$$

If $\tau'(\tau)$ is chosen such that $y'(\tau') = y(\tau)$ then $\tau' \to \infty$ for $\tau \to \infty$ and by (7)

$$p(\tau) q(\tau) < d_2 |x(\tau) - x'(\tau')| y^{-1}(\tau) \to 0,$$

hence $p(\tau)R \to 0$ for $\tau \to \infty$. If the endpoint of R is different from (0, 0) then the hyperbolic distance $h(x(\tau), R)$ tends because of (7) with the order of τ

to ∞. Applying (7) again we find $x(\tau)R \geqslant d_1 h(x(\tau), R)$. We have found, for the terminology see [1, Section 23]:

PROPERTY 5. *A quasihyperbolic geometry satisfies the hyperbolic parallel axiom. If* $p(\tau)$, $\tau \geqslant 0$ *represents any geodesic ray and* R *is a co-ray to* $p(\tau)$ *then* $p(\tau)R \rightarrow 0$ *for* $\tau \rightarrow \infty$. *If* R *is not a co-ray to* $p(\tau)$ *then* $p(\tau)R$ *tends with the order of* τ *to* ∞.

We next investigate in which quasihyperbolic geometries the geodesics satisfy Desargues' Theorem; since the intersections necessary for the usual Theorem of Desargues need not exist, we mean more precisely the Desargues Property as formulated in [1, Section 13]. If the Desargues Property holds, then the quasihyperbolic plane can be mapped on an open convex subset of the affine plane such that the geodesics go into the intersections of the affine straight lines with this convex set, [1, (13.1)]. Because of Property 5 the set can neither be the whole or a half plane nor a strip bounded by two parallel lines, hence will after a suitable choice of the line at infinity be a set bounded by a closed convex curve C.

Let $a \in C$ be the common endpoint of the distinguished family F of geodesics. The motions (2) will now be induced by projectivities which map C and its interior on themselves. Since a suitable translation along a distinguished limit-circle carries a given point $c \neq a$ on C into another given point $c' \neq a$ on C, the curve C has, except possibly at a, a continuous non-vanishing curvature. Let T be one of the one-sided tangents of C at a and T' the tangent of C at $b \neq a$. The translations along the geodesic G with endpoints a, b leave a, b fixed and take C into itself. They also leave $d = T \cap T'$ (possibly at infinity) fixed. If a, b, d are taken as points (1, 0, 0), (0, 0, 1), (0, 1, 0) of a projective coordinate system x_1, x_2, x_3, then a projectivity which leaves a, b, d fixed has the form $\rho x_1' = \lambda x_1$, $\rho x_2' = \mu x_2$, $\rho x_3' = \nu x_3$, so that C must have the form $x_1^{\alpha} x_3^{\beta} = k x_2^{\gamma}$, where α, β, γ are positive, $\alpha + \beta = \gamma$, $k \neq 0$, see [3, § 41]. If we return to affine coordinates by putting $x_3 = 1$ the equation of C takes the form $x_1 = k x_2^{\delta}$, $\delta > 1$. Among all these curves only the parabola $x_1 = c x_2^2$ has non-vanishing curvature at (0, 0). Hence C is a conic. It follows now readily from Property 3 that the metric is hyperbolic.

PROPERTY 6. *The hyperbolic geometry is the only Desarguesian quasihyperbolic geometry.*

If the line element (5) satisfies $F(x, y) \equiv F(-x, y)$ then the space possesses reflections in the distinguished geodesics. But the space cannot admit any other additional motions without becoming hyperbolic. For any such motion carries some distinguished geodesic into an ordinary geodesic, so that all translations along this geodesic also exist. It follows from [1, (51.5)] that then the metric is hyperbolic.

PROPERTY 7. *The only motions other than the translations along the distinguished geodesics or limit circles which a quasihyperbolic geometry can possess without being hyperbolic are the reflections in the distinguished geodesics.*

4. Convexity properties.

The quasihyperbolic circles $K(p, \rho)$ (i.e. the loci $px = \rho$) *are strictly convex with respect to the orbits* (hence strictly convex in the euclidean sense in M). For if an orbit contained either a proper subarc of $K(p, \rho)$ or intersected $K(p, \rho)$ in three points, then a sufficiently small translation along this orbit would produce a circle $K(p', \rho)$, $p' \neq p$, which has three or more common points with $K(p, \rho)$, which contradicts [1, (10, 11)]. Therefore the distinguished limit circle L through p intersects $K(p, \rho)$ in exactly two points and divides $K(p, \rho)$ into two arcs $K_1(p, \rho)$ and $K_2(p, \rho)$ one of which, say $K_1(p, \rho)$, lies except for its endpoints outside (or in the model M below) L, the other inside (or above) L. The arc $K_1(p, \rho)$ properly contains a semicircle, because the supporting geodesic to L at p lies outside L. The arc $K_1(p, \rho)$ is geodesically convex. For at a given point $q \in K_1(p, \rho)$ there exists a supporting orbit D, and there is (unless $D \in F$) a supporting geodesic G of D at q which lies on the other side of $K(p, \rho)$, so that $K_1(p, \rho)$ is convex (see [1, (25.2)]).

The other arc $K_2(p, \rho)$ need not be convex. For any ordinary geodesic G decomposes the quasihyperbolic plane into two domains of which one is convex with respect to the orbits and the other not. We call these domains the orbit convex and orbit concave sides of G. Consider supporting orbits of G at a point $q \in G$. If D is not a limit circle, then it intersects the x-axis in a point $(x_0, 0)$ and the lines through $(x_0, 0)$ form a family of curves equidistant to $x = x_0$, with a family of geodesics as orthogonal trajectories. Exactly one of these, H_q, passes through q. Every point u of H_q has q as foot on D. If u lies on the orbit concave side of G, then q is also the only foot of u on G.

PROPERTY 8. *Every point on the orbit concave side of an ordinary geodesic G has exactly one foot on G. The points on this side with the same foot q on G form a half geodesic H'_q and $H'_q \cap H'_r = 0$ for $q \neq r$.*

If the circles $K(p, \rho)$ are convex, then perpendiculars (see [1, (20.10)]) to a given geodesic at a given point exist and perpendiculars to the same geodesic at different points do not intersect. These perpendiculars are the lines containing the above constructed half geodesics.

However, the geodesics containing H'_q and H'_r of Property 8 will in general intersect on the orbit convex side of G. This can readily be verified for a geodesic G which approximates the polygon with vertices $(-1, 0)$, $(-1, 10^{-1})$, $(0, 1)$, $(1, 10^{-1})$, $(1, 0)$, such that its curvature is large in the neigborhood of $(-1, 10^{-1})$, $(0, 1)$, $(1, 10^{-1})$ and small elsewhere; the points q, r are to be taken close to $(1,0)$.

PROPERTY 9. *A circle $K(p, \rho)$ is divided by the distinguished limit circle L through p into two arcs, $K_1(p, \rho)$ outside L and $K_2(p, \rho)$ inside L. The arc $K_1(p, \rho)$ properly contains a semicircle and is convex, $K_2(p, \rho)$ need not be convex.*

According to the general theory (see [1, (20.5)]) there is a number $\sigma \geqslant 0$ (the values 0 and ∞ are admitted) such that $K(p, \rho)$ is convex for $\rho \leqslant \sigma$ and not convex for $\rho > \sigma$. It is not hard to see that $\sigma = 0$ when the geodesics in M are not differentiable (or $F(x, y) = 1$ is not strictly convex). If $F(x, y)$ satisfies the usual assumptions of the calculus of variations then $\sigma > 0$, see Whitehead [4].

We mention without proof the following implications: An ordinary geodesic G is for any $\delta > 0$ equidistant from the locus of points on orbit concave side of G which have distance δ from G, but G is equidistant from the analagous locus on the orbit convex side only for $\delta \leqslant \sigma$.

When the circles are not all convex, then the loci equidistant from segments or straight lines are not all convex and *still less can the space have negative curvature*, see [1, Section 36].

5. **Existence of individual Bonnet measures.**

We now investigate *angular measures*. The exact definition is found in [1, Section 42]; the essential points are additivity for angles with the same vertex and that measure π characterizes straight angles.

A natural angular measure must be *invariant under motions*. But this does not characterize the measure: because the mappings (2) are conformal, the ordinary euclidean angle (at least when the geodesics in M are differentiable) has this property and suitable functions of the euclidean angle will also be invariant and satysfy the axioms for angular measure.

A very important property of the angle in the Riemannian case is the

Theorem of Gauss-Bonnet or, equivalently, that the excess (see [1, Section 42]) which is defined on polygonal regions can be extended to a completely additive set function defined on all Borel sets. If angular measure is invariant under motions, then this set function will be too and must from the theory of Haar measure be proportional to the hyperbolic (or quasihyperbolic) area. A continuous (see [1, Section 42]) angular measure for which the excess can be extended to a completely additive set function will be called a *Bonnet measure.*

Consider a distinguished geodesic G, say $x = 0$ in the model M, and the geodesic H_y with a fixed end point $(a, 0)$, $a > 0$, intersecting G at $(0, y)$. Denote by α_y the measure of the convex angle A_y formed by the two rays of G and H_y beginning at $(0, y)$, and ending at $(0, 0)$, $(a, 0)$. Then $\alpha_y < \alpha_{y'}$ for $y > y'$. For the translation T along G which takes $(0, y)$ into $(0, y')$ will take $(a, 0)$ into $(b, 0)$ with $0 < b < a$, and H_y into the geodesic through $(y', 0)$ and $(b, 0)$. Therefore $A_y T$ is properly contained in $A_{y'}$, so that by invariance under motion the measure of $A_y T$ equals α_y and is less than $\alpha_{y'}$. Hence $\lim_{y\to\infty} \alpha_y = \alpha$ exists. For an arbitrary $b > 0$ there is a translation along G that takes $(a, 0)$ into $(b, 0)$, so that α is independent of a.

For a fixed $y > 0$ the measure of the convex angle between the rays from $(0, y)$ to $(b, 0)$ and to $(0, 0)$ tends to 0 for $b \to 0$, if the angular measure is continuous. Hence $\alpha = 0$. Similarly $\lim_{y\to\infty} \alpha_y$ exists and equals π. Using the additivity of angular measure we conclude: if $p(\tau)$ represents a geodesic, then for any point q the measure of the angle $p(0)p(\tau)q$ tends to 0 when $\tau \to \infty$. This is an important property, not shared by arbitrary angular measures, and is called non-extendability, [1, Section 42].

PROPERTY 10. *A continuous angular measure which is invariant under motions is non-extendable.*

Consider the asymptote triangle formed by the two lines $x = -1$ $x = 1$ and the ordinary geodesic $y = f(x)$ with end points $(-1, 0)$ and $(1, 0)$. If $\varepsilon > 0$ is sufficiently small, it follows from the continuity of angular measure and the preceding arguments, that the sum of the measures of the angles in the geodesic triangle with vertices $(-1 + \varepsilon, f(-1 + \varepsilon))$, $(-1 + \varepsilon, \varepsilon^{-1})$, $(1 - \varepsilon, f(1 - \varepsilon))$ is arbitraliry small, hence the excess of the triangle (i.e. the angle sum minus π) is negative. Consequently, for a Bonnet measure the excess of a geodesic triangle is negative and the total excess of the above asymptote triangle equals π. Under the assumption that we have a Bonnet measure denote by β_{x_0}, $-1 \leqslant x_0 \leqslant 1$, the measure of the convex angle between the rays

$y \geqslant f(x_0)$ of $x = x_0$ and $x \geqslant x_0$ of $y = f(x)$. The hyperbolic area, with respect to the metric $(dx^2 + dx^2)y^{-2}$, of the domain $y \geqslant f(t)$, $-1 \leqslant t \leqslant x$ is

$$\int_{-1}^{x} \int_{f(t)}^{\infty} \frac{dt\,dy}{y^2} = \int_{-1}^{x} \frac{dt}{f(t)}.$$

The total excess of this domain is $\pi - \beta_x - \pi = -\beta_x$. Since $\beta_1 = \pi$ we find

$$\beta_x = \pi \int_{-1}^{x} \frac{dt}{f(t)} \Bigg/ \int_{-1}^{1} \frac{dt}{f(t)}, \tag{8}$$

provided these integrals exist. This measure is clearly invariant under (2). The angle between any two geodesics is now determined by additivity. Standard arguments of hyperbolic geometry show that with this angular measure the hyperbolic area of any geodesic triangle is proportional to the excess. The same holds for quasihyperbolic area which differs from the hyperbolic area only by the constant factor $\pi / \int_0^{\pi} F^{-2}(\cos\varphi, \sin\varphi)\, d\varphi$.

The integral $\int_{-1}^{0} f^{-1}(x)\,dx$ exists if $f(x)$ possesses at -1 a finite upper curvature (but the condition is not necessary). Similarly for $x = 1$. This means for $F(x, y)$ that $F(x, y) = 1$ has at its intersection with y-axis non-vanishing right and left lower curvatures. If we write $y = f(x)$ in the form $\rho = g(\alpha)$ then

$$\int_{-1}^{x} \frac{dx}{f(t)} = \int_{\pi}^{\alpha} \frac{[g'(\varphi)\cos\varphi - g(\varphi)\sin\varphi]\,d\varphi}{g(\varphi)\sin\varphi} = \int_{\alpha}^{\pi} \left(1 - \frac{g'(\varphi)}{g(\varphi)}\cot\varphi\right) d\varphi.$$

If $F^2(x, y) = x^2 + y^2$ then $g(\varphi)$ is constant and we find the hyperbolic value $\beta_x = \pi - \alpha$. (It is easy but uninteresting to express the last integral in terms of $F(x, y)$). We summarize:

PROPERTY 11. *A quasihyperbolic geometry for which the local unit circle has at the points, where it is tangent to distinguished limit circles non-vanishing right and left lower curvatures, possesses one and only one Bonnet measure which is invariant under motions. In terms of this measure the excess of a geodesic triangle is negative and proportional to the area of the triangle.*

Comparing these results with those of the preceding section, we see that quasihyperbolic geometry furnishes an excellent example to show that *different* *aspects of curvature in Riemann spaces may become entirely dissociated in Finsler spaces.*

6. The non-existence of universal Bonnet measures.

The Bonnet measure which we found for quasihyperbolic geometry is an individual measure adapted to the specific metric, it is not universal.

To make this precise we consider two-dimensional Finsler spaces with the usual properties, which implies in particular that the condition on the local unit circle occurring in Property 11 is automatically satisfied. *A universal angular measure* is characterized by the following property (beyond the conditions for a continuous angular measure):

It depends only on the local Minkowskian geometry, i.e., if the local Minkowskian geometries M_i belonging to the points $p_i \in S_i$ of the two Finsler surfaces S_1, S_2 are isometric, then the angles on S_i at p_i which correspond under an isometry of M_1 on M_2 have equal measures. In particular:

If for two Finsler surfaces S_1, S_2 with integrands $F_1(x, y, dx, dy)$ and $F_2(x, y, dx, dy)$ and two points $(x_i, y_i) \in S_i$ the relation

$$F_1(x_1, y_1, dx, dy) = kF_2(x_2, y_2, dx, dy) \tag{9}$$

holds for all dx, dy, where k does not depend on dx, dy, then the two (convex) angles with vertices (x_1, y_1) in S_1 and (x_2, y_2) in S_2 determined by two arbitrary directions dx, dy and δx, δy have the same measure.

For instance, making the angular measure proportional to the area of the sector of the local unit circle leads to a universal angular measure, and there are many others. A universal measure is always invariant under motion.

A universal Bonnet measure would be a universal measure for which the Gauss-Bonnet Theorem holds (or the set function induced by the excess is completely additive). The discussion of Bonnet measure in quasihyperbolic geometry shows the non-existence of a universal Bonnet measure.

The condition that the tangent of $F(x, y) = 1$ at its intersections with the y-axis is perpendicular to the x-axis is immaterial, because it merely expresses the normalization that the lines $x = \text{const.}$ be the distinguished geodesics rather than $y = m(x + \delta)$. It is easily verified that in general $F(dx, dy)x^{-1}$ in $x > 0$, or more generally $F(dx, dy)(ax + by)^{-1}$ in $ax + by > 0$ leads to a different angular measure if defined as in (8). On the other hand (9) obviously holds for any two points of $F(dx, dy)y^{-1}$ and $F(dx, dy)(ax + by)^{-1}$. Since the