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FRACTAL DIMENSIONS OF SOME SETS IN [0,1)*

In Memory of Professor M. T. Cheng

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Abstract

A kind of fractal sets, non-regular sets with 0-Lebesgue measure, is introduced by considering binary expansion of real numbers in [0,1). *Their fractal dimensions are also studied in this paper.*

1 Introduction

At first, we introduce some notation in this paper.

$\dim_H E$: Hausdorff dimension of a set E,

$\dim_B E$: the box dimension of a set E,

$\dim_P E$: the packing dimension of a set E,

$\# E$: the cardinal number of a set E,

$[x]$: the integer part of x.

Let $x \in [0,1)$, consider the binary expansion of x (we agree that the rational numbers have finite expansions), $x = 0.x_1x_2\cdots$, where $x_i \in \{0,1\}$, $i = 1,2,\cdots$.

For $n \in \mathcal{N}$, we define

$$f_n(x) = \frac{\#\{i; x_i = 1, i = 1,2,\cdots,2n\}}{2n},$$

$$E_n = \{x \in [0,1); f_n(x) = \frac{1}{2}\},$$

$$E = \{x \in [0,1); \lim_{n\to\infty} f_n(x) = \frac{1}{2}\},$$

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$$\bar{E} = \overline{\lim_{n\to\infty}} E_n = \{x \in [0,1): f_n(x) = \frac{1}{2} \text{ for infinitely many } n's\},$$

$$\underline{E} = \underline{\lim_{n\to\infty}} E_n = \{x \in [0,1): \exists\ N \geqslant 1, \text{ s.t. } f_n(x) = \frac{1}{2} \text{ for } n \geqslant N\},$$

$$F = \{x \in [0,1): f_n(x) = \frac{1}{2} \text{ for finitely many } n's\},$$

$$G = \{x \in [0,1): f_n(x) \neq \frac{1}{2} \text{ for any } n\}.$$

From ref. [1], we have $\dim_H E = 1$, by the above definitions, it is obvious to see $\bar{E}$, $\underline{E}$, F and G are fractal sets which are different from E, thus the other kind of fractal sets is constructed, and the fractal dimensions of these sets are studied in this paper.

For the definitions and properties of the Hausdorff dimension, box dimension and packing dimension, see ref. [2].

2 Main results and proofs

Firstly, we list some results (ref. [3]) about the limit set of infinite iterated function systems consisting of similarity maps.

Let D be a closed subset of R^n. A mapping $S: D \to D$ is called a contracting similarity map of R^n if there is a number c with $0<c<1$ such that $|S(x) - S(y)| = c|x-y|$ for all x, y in D. We say that $\{S_i\}_{i=1}^{\infty}$ satisfies the open set condition if there exists a non-empty bounded open set V such that $\bigcup_{i=1}^{\infty} S_i(V) \subset V$ with $S_i(V) \cap S_j(V) = \varnothing$ for $i \neq j$.

Theorem A. [3] *Suppose that the open set condition holds for similarities $\{S_i\}_{i=1}^{\infty}$ on R^n with ratios $\{c_i\}_{i=1}^{\infty}$. If A is the invariant set satisfying $A = \bigcup_{i=1}^{\infty} S_i(A)$, and $\dim_H A = s$, then $s = \inf_t \{t: \sum_{i=1}^{\infty} c_i^t \leqslant 1\}$. Moreover, if A_n denotes the invariant set of $\{S_i\}_{i=1}^{n}$, $A_n = \bigcup_{i=1}^{n} S_i(A_n)$, and $\dim_H A_n = s_n$, then $s = \lim_{n\to\infty} s_n$.*

For $n \in \mathscr{N}$, define

$$L_n = \{x \in E_n: x = 0.x_1x_2\cdots x_{2n}\},$$

$$L_n^* = \{x \in L_n: x \notin E_k, k < n\},$$

$$N_k(L_n) = \#\{x \in L_n: x \in E_k \text{ and } x \notin E_m, m < k\},$$

$$N_k^* = N_k(L_k).$$

Theorem 2.1. *Suppose that* $\dim_H \bar{E} = s$, *then* $s = \inf_t \{t: \sum_{k=1}^{\infty} N_k^* (2^{-2k})^t \leqslant 1\}$.

Proof. From the definitions of $N_k(L_n)$ and N_k^*, we have $N_k^* C_{2n-2k}^{n-k} = N_k(L_n)$, also

$$\sum_{k=1}^{n} N_k(L_n) = C_{2n}^{n} = \sum_{k=1}^{n} N_k^* C_{2n-2k}^{n-k}.$$

For each $x \in L_k^*$, we have the correspondence as follows,

$$x \in L_k^* \leftrightarrow [x, x+2^{-2k}) \leftrightarrow \text{ map } S_x : [0,1) \to [x, x+2^{-2k}),$$

for $x, y \in L_k^*, x \neq y$, we get $|x-y| \geqslant 2^{-2k}$, so the correspondence above is one to one, and $S_x(0,1) \cap S_y(0,1) = \varnothing$. Thus we have

$$x \in \bigcup_{k=1}^{\infty} L_k^* \leftrightarrow \{[x, x+2^{-2k})\} \leftrightarrow \{S_x\},$$

i. e., for each k, there exist N_k^* intervals whose length is 2^{-2k}, corresponded to N_k^* similarities $\{S_{ki}\}_{i=1}^{N_k^*}$.

Let $V = (0,1)$, it is easy to check that

$$S_{ki}(V) \cap S_{lj}(V) = \varnothing, \qquad k \neq l \text{ or } i \neq j,$$

$$\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{N_k^*} S_{ki}(V) \subset V,$$

which means that $\{S_{ki}\}$ satisfies the open set condition.

Let A be the invariant set of $\{S_{ki}\}(k=1,2,\cdots, i=1,2,\cdots,N_k^*)$, it is obvious that $A = \bar{E}$, i. e. $\dim_H A = s$, by Theorem A, we know that

$$s = \inf_t \{t : \sum_{k=1}^{\infty} N_k^* \cdot (2^{-2k})^t \leqslant 1\}.$$

The proof is complete.

For $n \in \mathscr{N}$, define

$$G_n = \{x \in [0,1) : x = 0.x_1x_2\cdots x_{2n}, f_k(x) \neq \frac{1}{2}, k \leqslant n\},$$

$$g(n) = \# G_n.$$

Lemma 2.2. $g(n) = C_{2n}^{n}$.

Proof. It is obvious that

$$g(1) = 2^2 - C_2^1,$$

$$g(2) = 2^4 - C_4^2 - C_2^1 \cdot g(1),$$

$$\cdots$$

$$g(n) = 2^{2n} - C_{2n}^{n} - C_{2n-2}^{n-1} \cdot g(1) - C_{2n-4}^{n-2} \cdot g(2) - \cdots - C_2^1 \cdot g(n-1).$$

From (3.90) of ref. [4], we have for $n \in \mathscr{N}$,

$$2^{2n} = \sum_{i=0}^{n} C_{2n-2i}^{n-i} C_{2i}^{i},$$

thus

$$g(1) = C_2^1,$$

$$g(2)=C_4^2,\quad \cdots,$$

$$g(n)=2^{2n}-C_{2n}^n-C_{2n-2}^{n-1}\cdot C_2^1-\cdots-C_2^1\cdot C_{2n-2}^{n-1}=C_{2n}^n.$$

Theorem 2.3. $\dim_B G=1$.

Proof. For $n\in\mathcal{N}$, define

$$K_n=\{[x,x+2^{-2n}),\ x\in G_n\},$$

then $K_1\supset K_2\supset\cdots$, and $\bigcap_{n=1}^{\infty}K_n=G$. By Lemma 2.2, K_n consists of C_{2n}^n intervals whose length is 2^{-2n}, so we get

$$\dim_B G=\lim_{n\to\infty}\frac{\log C_{2n}^n}{-\log 2^{-2n}}=\lim_{n\to\infty}\frac{\log\dfrac{2^{2n}}{\sqrt{\pi n}}}{\log 2^{2n}}=1,$$

since $\lim_{n\to\infty}\dfrac{C_{2n}^n}{\left(\dfrac{2^{2n}}{\sqrt{\pi n}}\right)}=1$ by Stirling formula $n!=n^ne^{-n}\sqrt{2\pi n}(1+\varepsilon_n)$, where $\lim_{n\to\infty}\varepsilon_n=0$ and by the equality $C_{2n}^n=\dfrac{(2n)!}{(n!)^2}$.

Theorem 2.4. $\dim_P G=1$.

Proof. Let $H=G\cap[0,\frac{1}{2})$, $H'=G\cap[\frac{1}{2},1)$, then $G=H\cup H'$ and $H'=1-H$, so we have $\dim_B H'=\dim_B H$, $\dim_P H'=\dim_P H$, thus $\dim_P G=\dim_P H$, $\dim_B G=\dim_B H$.

For $k\in\mathcal{N}$, define

$$h_k=\{x=0.x_1x_2\cdots x_{2k},\ x(0)>x(1)\},$$

$$H_k=\{[x,x+2^{-2k}),\ x\in h_k\},$$

then $H_1\supset H_2\supset\cdots$, and $\bigcap_{k=1}^{\infty}H_k=H$.

Define $h_k^m=\{x\in h_k, x(0)-x(1)=2m\}$, $m=1,2,\cdots,k$, we call $[x,x+2^{-2k})$ m-type interval if $x\in h_k^m$, it is easy to see if I_1,I_2 are both m-type intervals, then $I_1\cap H$, $I_2\cap H$ have the similar structure with ratio $\dfrac{|I_1|}{|I_2|}$, thus $\dim_P(I_1\cap H)=\dim_P(I_2\cap H)$.

For any open set V that intersects H, suppose there exists $e\in H_k$ with $e\subset V$ for large enough k, without loss of generality, suppose $e=[x,x+2^{-2k})$ where $x\in h_k^m$ $(1\leqslant m\leqslant k)$ and $x=0.x_1x_2\cdots x_{2k}$, then $e'=[y,y+2^{-2(k+m-1)})$ is of type 1 and a subinterval of e where $y=0.x_1x_2\cdots x_{2k}\underbrace{11\cdots1}_{2m-2}$, i.e. e contains a subinterval e' of type 1. Also we have $H=H\cap[0,0.01)$ and $[0,0.01)$ is of type 1, i.e. $e'\cap H$ and H have the similar structure, thus

$$\dim_B(H\cap V)\geqslant\dim_B(H\cap e)\geqslant\dim_B(H\cap e')=\dim_B H,$$

so we have $\dim_B(H\cap V)=\dim_B H$. By Corollary 3.9 of ref. [2], we have $\dim_P H=\dim_B H$, hence $\dim_P G=\dim_P H=\dim_B H=\dim_B G=1$ by Corollary 2.3.

Corollary 2.5. $\dim_B F=1$, $\dim_P F=1$.

Proof. Let $F_1=\{0.01x_1x_2\cdots; 0.x_1x_2\cdots\in G\}$, then $F_1\subset F$ and $F_1=\frac{1}{4}G+\frac{1}{4}$, thus $\dim_B F_1=\dim_B G=1$, $\dim_P F_1=\dim_P G=1$, so we have

$$\dim_B F\geqslant 1,\quad \dim_P F\geqslant 1,$$

the opposite inequality is obvious since $F\subset[0,1)$.

Lemma 2.6. $N_k^*=4C_{2k-2}^{k-1}-C_{2k}^{k}$, $k\geqslant 1$.

Proof. It is obvious to get $N_k^*+g(k)=4\cdot g(k-1)$, then by Lemma 2.2,

$$N_k^*=4\cdot g(k-1)-g(k)=4\cdot C_{2k-2}^{k-1}-C_{2k}^{k}.$$

Theorem 2.7. $\dim_H \bar{E}=1$.

Proof. By Stirling formula, it is easy to check that $N_k^*=4C_{2k-2}^{k-1}-C_{2k}^{k}=O(\frac{2^{2k}}{k^{\frac{3}{2}}})$ when $k\to\infty$, hence $N_k^*(2^{-2k})^t=O(\frac{2^{2k(1-t)}}{k^{\frac{3}{2}}})$, so we have

$$s=\inf_t\{t:\sum_{k=1}^{\infty}N_k^*(2^{-2k})^t\leqslant 1\}\geqslant 1,$$

and $s\leqslant 1$ is obvious since $\bar{E}\subset[0,1)$.

Corollary 2.8. $\dim_B\bar{E}=1$, $\dim_P\bar{E}=1$.

Proof. This is immediately from the fact that

$$\dim_B\bar{E}\geqslant\dim_P\bar{E}\geqslant\dim_H\bar{E}=1.$$

Theorem 2.9. $\dim_H \underline{E}=\frac{1}{2}$.

Proof. It is easy to see that $\underline{E}=\bigcup_{k=1}^{\infty}E_k$, where $E_k=\{x: f_n(x)=\frac{1}{2},\ n\geqslant k\}$, thus

$$\{2^{2k}x-[2^{2k}x]: x\in E_k\}=E_1,$$

hence

$$\dim_H E_k=\dim_H E_1\quad\text{for any } k,$$

so we have

$$\dim_H\underline{E}=\sup_k(\dim_H E_k)=\dim_H E_1.$$

Let $S_1:[0,1]\to\left[\frac{1}{4},\frac{1}{2}\right]$, $S_2:[0,1]\to\left[\frac{1}{2},\frac{3}{4}\right]$, then E_1 is the invariant set of $\{S_1, S_2\}$, so it follows

$$\dim_H E_1=\frac{\log 2}{-\log\frac{1}{4}}=\frac{1}{2},$$

hence

$$\dim_H \underline{E} = \frac{1}{2}.$$

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