SOME REMARKS ON THE OPTIMAL SUBSPACES OF A CONVOLUTION CLASS WITH A NCVD KERNEL*

In Memory of Professor M. T. Cheng

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Abstract

In this note a new generalized version of the classical Landau-Kolmogorov and Stein inequalities is established on a convolution class of periodic functions with a NCVD kernel. On this basis some sets of optimal subspaces for the 2n-dimensional Kolmogorov width of such function class are identified.

In [1] V. M. Tikhomirov proved that for a convolution class of 2π-periodic functions

$$W_\infty^K(T) := \{x(\cdot) = (K * u)(\cdot) : \| u(\cdot) \|_{L_\infty} \leqslant 1\}, \tag{1}$$

with a NCVD kernel $K(\cdot) \in C(T)$ the $2n$-dimensional Kolmogorov width $d_{2n}(W_\infty^K, C(T))$ has an infinite sequence of optimal subspaces as follows:

$$S_{2n}^{K_m} := \text{span}\{K_m(\cdot - \frac{j\pi}{n}); j = 0, \cdots, 2n-1\}, \tag{2}$$

where $m \in Z_+, K_1 = K$, when $m \geqslant 2, K * \cdots * K := K_m$. In this note we consider the same question. Some further results will be proved in the following two sections.

1 Landau-Kolmogorov type and Stein type inequalities

Theorem 1 *Suppose $K_1(t), K_2(t) \in C(T)$, and both of them are NCVD functions. Denote $K = K_1 * K_2$. If $\varphi(\cdot) \in L_p(T), 1 \leqslant p \leqslant \infty$, $\| \varphi \|_{L_p(T)} \leqslant 1$, and*

$$\| K * \varphi \|_{L_p(T)} \leqslant \| \Phi_n(K; \cdot) \|_{C(T)}, \tag{3}$$

* Project supported by NSFC(19671012)

where $\Phi_n(K;x):=(K*\text{sgn}\sin n(\cdot))(x)$ is the generalized Euler spline relating to K, then

$$\|K_2*\varphi\|_{L_p(T)}\leqslant\|\Phi_n(K_2;\cdot)\|_{C(T)}. \tag{4}$$

Proof. (1) $p=\infty$ (Landau-Kolmogorov case). Without loss of generality we may assume $K_1,K_2\in C^\infty(T)$, and that they possess STP_{2n+1} property for every $n\in Z_+$ (for the STP property, see Pinkus [2], p. 44, 60). Now suppose the assertion is false, then for some $\varphi\in L_\infty(T)$, $\|\varphi\|_{L_\infty}\leqslant 1, n\in Z_+$ we have

$$\|K*\varphi\|_{L_\infty(T)}\leqslant\|\Phi_n(K;\cdot)\|_{C(T)},$$

and

$$\|\Phi_n(K_2;\cdot)\|_{C(T)}<\|K_2*\varphi\|_{L_\infty(T)}.$$

Take a number $\beta>1$ such that

$$\|\Phi_n(K_2;\cdot)\|_{C(T)}=\beta^{-1}\|K_2*\varphi\|_{L_\infty(T)}.$$

Choose points x_0,x_1 such that $\Phi_n(K_2;x-x_1)$ and $(K_2*\varphi)(x)$ attain their maximal values at the same point x_0. We may assume $\text{sgn}(K_2*\varphi)(x_0)=\text{sgn}\Phi_n(K_2;x_0-x_1)>0$. Set

$$\varphi_n(t)=\text{sgn}\sin n(x-x_1)-\beta^{-1}\varphi(t).$$

$\varphi_n\in L_\infty(T)$, and $S_c(\varphi_n)=2n$. Put

$$g_1(\cdot)=(K_2*\varphi)(\cdot),\qquad g_2(\cdot)=(K_1*g_1)(\cdot).$$

We have

$$g_2(\cdot)=\Phi_n(K;\cdot-x_1)-\beta^{-1}(G*\varphi)(\cdot).$$

On account of

$$\|\Phi_n(K;\cdot-x_1)\|_{C(T)}=\|\Phi_n(K;\cdot)\|_{C(T)}>\beta^{-1}\|K*\varphi\|_{C(T)},$$

and there exists $\alpha\in R$ such that

$$\Phi_n(K;\alpha+\frac{j\pi}{n})=(-1)^j\Phi_n(K;\alpha)=(-1)^j\|\Phi_n(K;\cdot)\|_{C(T)}$$

$(\alpha=x_0-x_1), j=0,1,\cdots,2n-1$, we derive

$$S_c(g_2)\geqslant 2n.$$

So that $Z_c(g_2)\geqslant 2n$. By the CVD property, from $S_c(g_2)=S_c(K_1*g_1)\leqslant S_c(g_1)\Rightarrow S_c(g_1)\geqslant 2n$. So $Z_c(g_1)\geqslant 2n$. In addition, by assumption, we have $g_1(x_0)=0$, and also $g'_1(x_0)=0$. Thus x_0 is at least a double zero of g_1. To sum up, we get

$$2n+2\leqslant\tilde{Z}_c(g_1)\leqslant S_c(\varphi_n)=2n, \tag{5}$$

where $\tilde{Z}_c$ counts cyclically the number of zeros of a function, where zeros which are not sign changes are counted twice. The contradiction (5) completes our proof.

Note 1. The above result was essentially contained in Nguen Thi Thieu Hoa [3].

(2) $1\leqslant p<\infty$ (Stein case).

We use Stein's technique (see [4]). Suppose $\varphi \in L_p(T)$,

$$\| K * \varphi \|_{L_p(T)} \leqslant \| \Phi_n(K; \cdot) \|_{C(T)}, \qquad 1 \leqslant p < \infty.$$

Set

$$h(x) = \operatorname{sgn}((K_2 * \varphi)(x)) \frac{|(K_2 * \varphi)(x)|^{p-1}}{\| |(K_2 * \varphi)(\cdot)|^{p-1} \|_{L_{p'}(T)}},$$

where $p>1, \frac{1}{p}+\frac{1}{p'}=1$; $h(x)=\operatorname{sgn}((K_2 * \varphi)(x))$, when $p=1$. Then $\| h \|_{L_{p'}(T)}=1$, and

$$\int_0^{2\pi} (K_2 * \varphi)(x) h(x) dx = \| (K_2 * \varphi)(\cdot) \|_{L_p(T)}.$$

Let

$$F(x) = \int_0^{2\pi} (K * \varphi)(x-t) h(t) dt.$$

A simple calculation yields

$$F(x) = -\int_0^{2\pi} K(x-v) \{ \int_0^{2\pi} \varphi(t-v) h(t) dt \} dv,$$

where the function

$$\Psi(v) := \int_0^{2\pi} \varphi(t-v) h(t) dt \in L_\infty(T),$$

because $|\Psi(v)| \leqslant \| \varphi \|_{L_p} \cdot \| h \|_{L_{p'}} \leqslant 1$ by Hölder's inequality. Besides this, we have

$$|F(x)| \leqslant \| K * \varphi \|_{L_p(T)} \cdot \| h \|_{L_{p'}} \leqslant \| \Phi_n(K; \cdot) \|_{C(T)}.$$

Thus, if we apply the above obtained result to $F(x)$, we get

$$\| K_2 * \Psi \|_{L_\infty(T)} \leqslant \| \Phi_n(K_2; \cdot) \|_{C(T)}.$$

But it is easy to see

$$\| K_2 * \varphi \|_{L_p(T)} = \left| \int_0^{2\pi} (K_2 * \varphi)(x) h(x) dx \right| \leqslant \| K_2 * \Psi \|_{L_\infty(T)}.$$

So we obtain

$$\| K_2 * \varphi \|_{L_p(T)} \leqslant \| \Phi_n(K_2; \cdot) \|_{C(T)}.$$

The case $1 \leqslant p < \infty$ is over.

Note 2. The Stein type inequality (3) is sharp when $p=1$.

2 Optimal subspaces of $d_{2n}(W_p^K(T), L_p(T)), p=1,\infty$

Denote

$$W_p^K := \{ x(\cdot) = (K * u)(\cdot); \| u(\cdot) \|_{L_p(T)} \leqslant 1 \}. \tag{6}$$

Theorem 2 *Let K be a NCVD kernel and $K \in C(T)$. Then for any 2π-periodic continuous function G which satisfies NCVD condition, the $2n$-dimensional Kolmogorov width $d_{2n}(W_p^K$*

$(T), L_p(T))$ has an infinite set of optimal subspaces $\{S_{2n}^{\overline{K}_m}: m \in Z_+\}$, where $p=1,+\infty$; $\overline{K}_m = G_m * K$, $G_1 = G$, $G_m = G * \cdots * G$ m times when $m \geqslant 2$.

Proof. (1) First case: $p=\infty$. $\forall f \in C(T)$, by duality theorem [5]

$$E(f, S_{2n}^{\overline{K}_m}, C(T)) := \min_{g \in S_{2n}^{\overline{K}_m}} \| f - g \|_{C(T)}$$

$$= \min_{g \in S_{2n}^{\overline{K}_m}} \| f - g \|_{L_\infty(T)} = \sup_{\|h\|_{L_1(T)} \leqslant 1, h \perp S_{2n}^{\overline{K}_m}} \left| \int_0^{2\pi} f(t) h(t) dt \right|.$$

Notice that

$$h \perp S_{2n}^{\overline{K}_m} \Leftrightarrow \int_0^{2\pi} h(t) \{(G_m * K)(t - \frac{j\pi}{n})\} dt = 0, \quad j = 0, \cdots, 2n-1.$$

Set

$$H(x) := \int_0^{2\pi} (G_m * K)(t - x) h(t) dt.$$

Then

$$H(\frac{j\pi}{n}) = 0, \quad j = 0, \cdots, 2n-1.$$

Inasmuch as $(G_m * K)^*$ (the conjugate kernel) is also a NCVD functioin, so by Pinkus ([2]p. 182, Th. 4.13),

$$\| H(t) \|_{L_1(T)} \leqslant \| \Phi_n((G_m * K)^*, \cdot) \|_{C(T)} = \| \Phi_n((G_m * K), \cdot) \|_{C(T)}. \tag{7}$$

Thus from Theorem 1($p=1$ case) we derive

$$\left\| \int_0^{2\pi} K(t - x) h(t) dt \right\|_{L_1} \leqslant \| \Phi_n(K^*, \cdot) \|_{C(T)} = \| \Phi_n(K, \cdot) \|_{C(T)}, \tag{8}$$

where $h(t)$ is any function such that $\| h \|_{L_1(T)} \leqslant 1$, $h \perp S_{2n}^{\overline{K}_m}$.

Consider

$$E(W_\infty^K(T); S_{2n}^{\overline{K}_m}; C(T)) := \sup_{f \in W_\infty^K(T)} E(f, S_{2n}^{\overline{K}_m}, C(T)))$$

$$= \sup_{f \in W_\infty^K} \sup_{\|h\|_{L_1} \leqslant 1, h \perp S_{2n}^{\overline{K}_m}} \left| \int_0^{2\pi} f(t) h(t) dt \right|$$

$$= \sup_{\|h\|_{L_1} \leqslant 1, h \perp S_{2n}^{\overline{K}_m}} \sup_{\|u\|_{C_\infty} \leqslant 1} \left| \int_0^{2\pi} \{ \int_0^{2\pi} K(x - t) h(x) dx \} u(t) dt \right|$$

$$= \sup_{\|h\|_{L_1} \leqslant 1, h \perp S_{2n}^{\overline{K}_m}} \left\| \int_0^{2\pi} K(x - t) h(x) dx \right\|_{L_1(T)} \leqslant \| \Phi_n(K, \cdot) \|_{C(T)}$$

by (8).

Thus we have proved

$$E(W_\infty^K(T); S_{2n}^{\overline{K}_m}; C(T)) = \| \Phi_n(K; \cdot) \|_{C(T)} = d_{2n}(W_\infty^K(T), C(T)), \; m \in Z_+.$$

(2) $p=1$ case.

In the same manner as above we may obtain the required result. The details are omitted.

References

[1] Tikhomirov, V. M., On the Optimal Subspaces of Classes of Functions with Cyclic Variation-Diminishing Kernel, Vestnik Moskov, Univ. ser. 1, Mat. Meh., 4(1997), 16—19(In Russian).

[2] Pinkus, A., n-widths in Approximation Theory, Springer-Verlag, 1985.

[3] Nguen Thi Thieu Hoa, On Extremal Problem for Variation-Diminishing Transformations, Vestnik Moskov, Univ. Ser. I, Mat. Meh., 5(1982), 3—7(In Russian).

[4] Stein, E. M., Functions of Exponential Type, Ann. Math., 65, 3(1957), 582—592.

[5] Korneichuk, N. P., Extremal Problems in Approximation Theory, Nauka, Noscow, 1976, (In Russian).

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