PRAMANA
—journal of
physics

Vol. 49, No. 6,
December 1997
pp. 609–615



# Einstein pseudotensor and total energy of the universe

N BANERJEE and S SEN
Relativity and Cosmology Research Centre, Department of Physics, Jadavpur University, Calcutta 700 032, India

MS received 17 September 1996; revised 3 May 1997

**Abstract.** The total energy of the universe has been calculated assuming that it is the sum of the contributions from the matter part and gravitational part. The calculations involve the use of Einstein pseudotensor. Calculations have been carried out in some specific examples of spacetime geometries. In some cases the total energy is indeed zero confirming previous results but in other cases the total energy is nonzero. So Rosen's idea that the pseudotensorial calculations will lead to the result that the total energy of the universe is zero, is very much model dependent.

**Keywords.** Energy of the universe; pseudotensors.

**PACS No. 04.20**

## 1. Introduction

It is well known that quite a few descriptions of the origin of the universe led to the conjecture that the total energy of the universe should be zero. Investigations of Albrow [1] and Tyron [2] assume that the universe might have arisen as a quantum fluctuation of the vacuum. After the success of the inflationary cosmological models in solving some outstanding problems in standard cosmology, the idea of vacuum fluctuations has been developed further. Inspired by these results, there have been attempts to show that the total energy of the universe is zero even from a purely classical point of view. The total energy of the universe, however, is not really zero if it is calculated on the basis of the matter part alone. Recently, there have been several attempts to include the gravitational energy along with the matter part so that the total energy is zero. We refer to the works of Cooperstock [3] and Johri *et al* [4] for brief but excellent reviews.

Cooperstock [3] expressed the covariant conservation law

$$T^{\mu\nu}_{;\nu} = 0 \tag{1}$$

(where $\mu$, $\nu = 0, 1, 2, 3$) in the form of an ordinary divergence

$$\left[\sqrt{-g}\left(T^0_0 - \frac{3}{8\pi}\left(\frac{\dot{R}^2}{R^2} + \frac{k^2}{R^2}\right)\right)\right]_{,0} = 0, \tag{2}$$

where the spacetime is described by conformal Friedman–Robertson–Walker (FRW) metric

$$ds^2 = R^2(t)\left[dt^2 - \frac{dr^2}{(1-kr^2)} - r^2 d\Omega^2\right]. \tag{3}$$

He arrived at (2) by making use of some calculations involving Killing vectors and concluded that the total density of the universe is zero.

Rosen [5], on the other hand, used the Einstein pseudotensor formalism and showed that the total energy of the closed FRW universe is zero. Very recently Johri *et al* [4] also considered the pseudotensorial calculations to include the contribution from the gravitational part and arrived at the result that the total energy of the spatially closed FRW universe is zero and so also the total energy enclosed within any finite volume of the spatially flat FRW universe.

In this connection, it deserves mention that Raychaudhuri and Banerji [6] investigated the possibility of calculation of the energy of the universe using a different approach. They considered a fluid sphere embedded in an otherwise empty closed FRW universe and found that when the radius of the sphere goes to infinity, the mass of the sphere tends to zero. The advantage of this work is that it does not involve any pseudotensor and thus the results obtained are independent of the coordinate frame. But the problem of this approach is that when the radius tends to infinity, nothing can be said about the exterior spacetime in which the fluid sphere is embedded.

In the present work, we examine the calculation of the total energy on the basis of pseudotensor a bit critically. Pseudotensorial calculations are always dangerous as they are very much coordinate dependent and thus may lead to ambiguous results except, however, in asymptotically flat spacetimes if one is using quasi-Minkowskian coordinates [7]. Rosen's idea was that the Einstein pseudotensor gives the gravitational energy and together with the matter part yields the result that the total energy of the universe is zero. In what follows, we test Rosen's idea against some examples. Some anisotropic cosmological models are considered, in which the nonzero components of the pseudotensor are used to calculate the total energy of the universe following Rosen's idea. It is interesting to note that although in some cases the total energy indeed comes out to be zero, in some other cases the energy integral is in fact nonzero. In § 2 we discuss specific examples and in § 3 we discuss the results.

## 2. Calculation of the total energy

In order to write down the covariant conservation equation (1) in the form of an ordinary divergence equation, Einstein introduced a pseudotensor [8] $t^\mu_\nu$ defined by the relation

$$t^\mu_\nu = \frac{1}{16\pi}\left[\mathcal{L}\,\delta^\mu_\nu - g_{\alpha\beta,\nu}\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu}}\right], \tag{4}$$

where

$$\mathcal{L} = g^{\alpha\beta}(\Gamma^\nu_{\alpha\mu}\Gamma^\mu_{\beta\nu} - \Gamma^\mu_{\mu\nu}\Gamma^\nu_{\alpha\beta}). \tag{5}$$

With this definition, Einstein wrote the ordinary divergence equation in the form

$$[\sqrt{-g}(T^\mu_\nu + t^\mu_\nu)]_{,\mu} = 0. \tag{6}$$

In fact, the quantity $\sqrt{-g}(T^\mu_\nu + t^\mu_\nu)$ can be expressed as an ordinary divergence of a quantity $\mathcal{H}^{\mu\alpha}_\nu$,

$$\sqrt{-g}(T^\mu_\nu + t^\mu_\nu) = \frac{1}{16\pi}\mathcal{H}^{\mu\alpha}_\nu, \alpha, \tag{7}$$

where

$$\mathcal{H}^{\mu\alpha}_\nu = \frac{1}{\sqrt{-g}} g_{\nu\beta}[-g(g^{\mu\beta}g^{\alpha\sigma} - g^{\alpha\beta}g^{\mu\sigma})]_{,\sigma}. \tag{8}$$

It should, however, be noted that as $\mathcal{L}$ is not a scalar, the quantities $t^\mu_\nu$ do not form a tensor. For details of the calculations we refer to Rosen [5] and references therein.

Rosen argued that $t^\mu_\nu$ accounts for the contribution from the gravitational energy towards the total energy density and the integral of the quantity $\sqrt{-g}(T^0_0 + t^0_0)$ over all space has the significance of the total energy of the universe. Thus $E$, the total energy of the universe, is given by

$$E = \int \sqrt{-g}(T^0_0 + t^0_0)\mathrm{d}^3x, \tag{9}$$

where the integral extends over the whole spatial volume.

With the spacetime geometry described by the closed FRW metric of the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{R^2(t)}{1 + r^2/4}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{10}$$

where $r^2 = x^2 + y^2 + z^2$, Rosen proved that $E$, given by (9), is indeed equal to zero.

A few examples from the anisotropic Bianchi type line elements has been picked up for similar kind of investigation.

I. *Bianchi type I*

The line element in this case is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2l}\mathrm{d}x^2 - \mathrm{e}^{2m}\mathrm{d}y^2 - \mathrm{e}^{2n}\mathrm{d}z^2, \tag{11}$$

where $l, m, n$ are functions of $t$ alone. From (7), we get

$$\sqrt{-g}(T^0_0 + t^0_0) = \frac{1}{16\pi}\mathcal{H}^{0\alpha}_0, \alpha, \tag{12}$$

and by using the metric given by (11) one can obtain from eq. (8),

$$\mathcal{H}^{0\alpha}_0 = 0 \tag{13}$$

for all $\alpha$. Therefore, in this case

$$\sqrt{-g}(T^0_0 + t^0_0) = \frac{1}{16\pi}\mathcal{H}^{0\alpha}_0, \alpha = 0$$

and hence the total energy of the universe

$$E = \int \sqrt{-g}(T^0_0 + t^0_0)\mathrm{d}^3x = 0 \tag{14}$$

for any finite volume.

As Bianchi type I line element reduces to the spatially flat FRW metric for $l = m = n$, (14) confirms the results obtained by Johri *et al* [4] that the energy contained in any finite volume of flat FRW universe is zero.

II. *Bianchi type II*

Bianchi type II line element can be written in the form

$$ds^2 = dt^2 - S^2(t)(dx + \tfrac{1}{2}z^2 dy)^2 - R^2(t)(z^2 dy^2 + dz^2) \tag{15}$$

which once again yields from (8) that $\mathcal{H}_0^{0\alpha} = 0$ for all $\alpha$, so that in this case also

$$\sqrt{-g}(T_0^0 + t_0^0) = \frac{1}{16\pi}\mathcal{H}_{0,\alpha}^{0\alpha} = 0,$$

and hence

$$E = \int \sqrt{-g}(T_0^0 + t_0^0)d^3x \tag{16}$$

for any finite spatial volume.

III. *Bianchi type III*

The spacetime is described by the metric

$$ds^2 = dt^2 - a_1 e^{-2mz}dx^2 - a_2 dy^2 - a_3 dz^2, \tag{17}$$

where $a_1$, $a_2$, $a_3$ are functions of time while $m$ is a constant. The only nonvanishing component of $\mathcal{H}_0^{0\alpha}$ is

$$\mathcal{H}_0^{03} = 2m\sqrt{\left(\frac{a_1 a_2}{a_3}\right)}e^{-mz}. \tag{18}$$

Thus,

$$\begin{aligned}\sqrt{-g}(T_0^0 + t_0^0) &= \frac{1}{16\pi}\mathcal{H}_{0,\alpha}^{0\alpha} \\ &= \frac{1}{16\pi}\mathcal{H}_{0,3}^{03} \\ &= -\frac{m^2}{8\pi}\sqrt{\left(\frac{a_1 a_2}{a_3}\right)}e^{-mz}\end{aligned} \tag{19}$$

and

$$\begin{aligned}E &= \int \sqrt{-g}(T_0^0 + t_0^0)d^3x \\ &= -\frac{m^2}{8\pi}\sqrt{\left(\frac{a_1 a_2}{a_3}\right)}\int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} e^{-mz}dz,\end{aligned} \tag{20}$$

which is readily integrable and evidently yields $E \neq 0$ for a finite spatial volume. Furthermore, $E$ diverges to infinitely large value if the limits of integration are extended to infinity ($x, y, z$ going from $-\infty$ to $+\infty$).

IV. *Bianchi type V*

In this case the general metric is

$$ds^2 = dt^2 - e^{2z}(b_1^2 dx^2 + b_2^2 dy^2) - b_3^2 dz^2 - 2b_{13} dx\, dz, \tag{21}$$

where $b_1$, $b_2$, $b_3$ and $b_{13}$ are all functions of time alone. In order to make the calculations simpler, we choose $b_{13}$ to be zero, so that the metric becomes diagonal as

$$ds^2 = dt^2 - e^{2z}(b_1^2 dx^2 + b_2^2 dy^2) - b_3^2 dz^2. \tag{22}$$

The relevant nonvanishing component of $\mathcal{H}_0^{0\alpha}$ is

$$\mathcal{H}_0^{03} = -\frac{4b_1 b_2}{b_3}\, e^{2z} \tag{23}$$

and so,

$$\begin{aligned}\sqrt{-g}(T_0^0 + t_0^0) &= \frac{1}{16\pi}\, \mathcal{H}_{0,\alpha}^{0\alpha} \\ &= \frac{1}{16\pi}\, \mathcal{H}_{0,3}^{03} \\ &= -\frac{e^{2z}}{2\pi}\frac{b_1 b_2}{b_3}.\end{aligned} \tag{24}$$

The total energy $E$ is therefore given by

$$\begin{aligned}E &= \int \sqrt{-g}(T_0^0 + t_0^0) d^3x \\ &= -\frac{b_1 b_2}{2\pi b_3} \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} e^{2z} dz.\end{aligned} \tag{25}$$

This is also readily integrable and again, for a finite spatial volume, does not vanish. Here also $E$ attains an infinitely large value if the limits of integration are extended to infinity. One should note that Bianchi type V model is the anisotropic generalization of an open FRW model.

V. *Bianchi type VI*

$$ds^2 = dt^2 - c_1 e^{-nz} dx^2 - c_2 e^z dy^2 - c_3 dz^2, \tag{26}$$

where $c_1$, $c_2$, $c_3$ are functions of time alone and $n$ is a constant. The relevant nonzero component of $\mathcal{H}_0^{0\alpha}$ is

$$\mathcal{H}_0^{03} = -(1-n)\sqrt{\frac{c_1 c_2}{c_3}}\, e^{[(1-n)/2]z}$$

and

$$\begin{aligned}E &= \int \sqrt{-g}(T_0^0 + t_0^0) d^3x \\ &= -\frac{1}{16\pi}\frac{(1-n)^2}{2}\sqrt{\left(\frac{c_1 c_2}{c_3}\right)} \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy \int_{z_1}^{z_2} e^{[(1-n)/2]z} dz.\end{aligned} \tag{27}$$

This also yields a nonzero value for $E$, the total energy of the universe.

VI. *Bianchi type IX*

The most well-known form of the Bianchi type IX model, which is the anisotropic generalization of spatially closed FRW model, is given by

$$ds^2 = dt^2 - l^2(d\psi + \cos\theta\, d\phi)^2 - m^2(\sin\psi\, d\theta - \cos\psi \sin\theta\, d\phi)^2 \\ - n^2(\cos\psi\, d\theta - \sin\psi \sin\theta\, d\phi)^2, \tag{28}$$

where $l, m, n$ are functions of time alone and $\psi$, $\theta$, $\phi$ can vary from 0 to $4\pi$, $\pi$ and $2\pi$ respectively. But pseudotensorial calculations are notoriously dangerous in polar coordinates. The results are believed to be more trustworthy in cartesian coordinates. For this reason, it is worthwhile to carry out the calculations in the Bianchi IX model in cartesian coordinate system. We take the line element as

$$ds^2 = dt^2 - S^2 dx^2 - R^2\, dy^2 - [R^2 \sin^2 y + S^2 \cos^2 y] dz^2 + 2S^2 \cos y\, dz\, dx, \tag{29}$$

where $S$ and $R$ are functions of $t$ alone.

Calculations following the same prescription as before, the total energy comes out to be

$$E = \frac{S}{8\pi} \int \sin y\, dx\, dy\, dz \tag{30}$$

which is not equal to zero for a finite universe.

## 3. Discussion

Following the investigations of Rosen [5], the total energy $E$ of the universe has been computed in the present work in some anisotropic cosmological models on the basis of Einstein pseudotensor. It has been found that although in Bianchi type I and II, $E$ is zero, i.e. same as that obtained by Rosen [5] and also by Johri *et al* [4] but in some other examples, namely Bianchi type III, V, VI, and IX, the total energy is nonzero. So the speculation that the total energy of the universe is zero cannot in fact be proved by this type of calculations.

The present work neither intends to contest nor defend the speculation. It only shows that the pseudotensorial calculations lead to different results for different spacetime symmetries. Moreover, these objects, $\sqrt{-g}(T^i_j + t^i_j)$ do not form a tensor and thus the calculations based on it are coordinate dependent. This limitation has been noticed long back by Weyl and Pauli. For an excellent discussion on these issues, we refer to the work of Chandrashekhar and Ferrari [9]. The choice of the pseudotensor, so as to write the covariant divergence as an ordinary one, is not at all unique. Landau–Lifshitz [10] pseudotensor for instance, is another choice, which is at least symmetric unlike Einstein pseudotensor, but it has its share of problems as well. Sorkin [11] developed the idea of a conserved current in the form of Noether operator and Burnett and Wald [12] in the form of a symplectic current. These conserved currents seem to work well when the spacetime is static and thus cannot be successfully applied in a cosmological situation where the spacetime evolves with time.

If, however, the metric components depend solely on time, then it was shown by Prasanna [13] that the total energy will be zero in a finite proper volume. This result is consistent with our calculation as only in Bianchi type I metric all nonzero components of $g_{ij}$ are functions of time alone and energy is also zero.

Very recently Aguirregabiria, Chamorro and Virbhadra [7] showed that pseudotensors of Einstein, Landau–Lifshitz and many others like Tolman, Papapetrou and Weinberg give essentially the same result for the energy and energy current density components for any Kerr–Schild class metric. In fact, most of these formalisms yields consistent results for a Kerr–Newman spacetime, was known even earlier by the investigations of Virbhadra [14, 15]. As the Kerr–Schild class includes a wide variety of spacetimes, including the exterior field of a radiating object, these studies are indeed very important. But this type of results are not in the literature for a cosmological scenario. The coordinate independent definition of energy, given by Komar [16] and later used by many workers such as Cohen [17], also involves a stationary spacetime. The generalization of Komar's work by Kulkarni, Chellathurai and Dadhich [18] also cannot be used for a nonstatic cosmological metric. For a very brief but excellent review of different pseudotensorial formalisms, we refer to the introduction of ref. [7].

To conclude, one can suggest that in order to prove (or disprove) classically, the speculation that the total energy of the universe is zero, one should resort to calculations based on tensorial objects so that these can be applied to different models with equal comfort and confidence. The method suggested by Rosen proves the conjecture ($E = 0$) only in some models and gives rise to counter examples as well.

## Acknowledgements

The authors would like to thank N K Dadich, B R Iyer and A R Prasanna for discussions and also the anonymous referee for some helpful suggestions. One of them (SS) would like to thank the University Grants Commission, India for financial support.

## References

[1] M G Albrow, *Nature (London)* **241**, 56 (1973)
[2] E P Tyron, *Nature (London)* **246**, 396 (1973)
[3] F I Cooperstock, *Gen. Relativ. Gravit.* **26**, 323 (1994)
[4] V B Johri, D Kalligas, G P Singh and C W F Everitt, *Gen. Relativ. Gravit.* **27**, 313 (1995)
[5] N Rosen, *Gen. Relativ. Gravit.* **26**, 319 (1994)
[6] A K Raychaudhuri and S Banerji, *Pramana – J. Phys.* **34**, 383 (1990)
[7] J M Aguirregabiria, A Chamorro and K S Virbhadra, *Gen. Relativ. Gravit.* **28**, 1393 (1996)
[8] A Einstein, *Ann. Phys.* **49**, 769 (1916)
[9] S Chandrashekhar and V Ferrari, *Proc. R. Soc. London* **A435**, 645 (1991)
[10] L D Landau and E M Lifshitz, *Classical theory of fields* (Pergamon, Oxford, 1962)
[11] R D Sorkin, *Proc. R. Soc. London* **A435**, 635 (1991)
[12] G A Burnett and R M Wald, *J. Math. Phys.* **31**, 725 (1990)
[13] A R Prasanna, *Prog. Theor. Phys.* **45**, 1330 (1971)
[14] K S Virbhadra, *Phys. Rev.* **D41**, 1086 (1990)
[15] K S Virbhadra, *Phys. Rev.* **D42**, 2919 (1990)
[16] A Komar, *Phys. Rev.* **113**, 934 (1959)
[17] J M Cohen, *J. Math. Phys.* **9**, 905 (1968)
[18] R Kulkarni, V Chellathurai and N Dadhich, *Class. Quant. Gravit.* **5**, 1443 (1988)