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A radiating dyon solution

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Abstract. We give a non-static exact solution of the Einstein–Maxwell equations (with null fluid), which is a non-static magnetic charge generalization to the Bonnor–Vaidya solution and describes the gravitational and electromagnetic fields of a non-rotating massive radiating dyon. In addition, using the energy-momentum pseudotensors of Einstein and Landau and Lifshitz we obtain the energy, momentum, and power output of the radiating dyon and find that both prescriptions give the same result.

Keywords. Dyon; Einstein–Maxwell equations; energy-momentum pseudotensor.

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There has been considerable interest in obtaining non-static solutions of Einstein's equations describing the gravitational field of a star radiating null fluid [1–3]. In 1953, Vaidya [2] obtained a nice form of non-static generalization to the Schwarzschild solution and it became well-known in the literature after the discovery of quasars. The quasars are high energy sources and therefore their gravitational field cannot be described by the Schwarzschild metric while the Vaidya metric is relevant to the study of such objects. Vaidya and others gave non-static generalizations to the Kerr solution [3]. Bonnor and Vaidya [4] obtained a non-static generalization to the Reissner–Nordström (RN) solution describing the emission of charged null fluid from a spherically symmetric charged radiating body. Mallett [5] gave an exact solution describing the radiating Vaidya metric [1] in a de Sitter universe. Patino and Rago [6] obtained a solution of the Einstein–Maxwell (EM) equations with null fluid for a spherically symmetric radiating massive charged (electric) object in a de Sitter universe.

The existence of magnetic monopoles is not yet confirmed, but it has been a subject of interest of many physicists (see in ref. 7). The Bonnor–Vaidya solution [4] is not enriched with a magnetic charge parameter. Therefore, it is of interest to obtain a non-static magnetic charge generalization to the Bonnor–Vaidya solution, characterized by three time-dependent parameters: mass, electric charge, and magnetic charge. Further we calculate the energy, momentum and power output of the radiating dyon in prescriptions of Einstein as well as Landau and Lifshitz (LL). Throughout this paper we use geometrized units where the gravitational constant $G = 1$ and the speed of light in vacuum $c = 1$. We follow the convention that Latin indices take values from 0 to 3 (x^0 is the time coordinate) and Greek indices take values from 1 to 3.

The EM equations with null fluid present are [4]

$$R_i^k - \tfrac{1}{2} g_i^k R = 8\pi (E_i^k + N_i^k), \tag{1}$$

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}\,F^{ik})_{,k} = 4\pi J^i_{(e)}, \tag{2}$$

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}\,{}^*F^{ik})_{,k} = 4\pi J^i_{(m)}, \tag{3}$$

where

$$E^k_i = \frac{1}{4\pi}\left[-F_{im}F^{km} + \frac{1}{4}g^k_i F_{mn}F^{mn}\right] \tag{4}$$

is the energy-momentum tensor of the electromagnetic field and

$$N^k_i = V_i\, V^k \tag{5}$$

is the energy-momentum tensor of the null fluid. V^k is the null fluid current vector satisfying

$$g_{ik}\, V^i\, V^k = 0. \tag{6}$$

${}^*F^{ik}$, the dual of the electromagnetic field tensor F^{ik}, is given by

$${}^*F^{ik} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{iklm}F_{lm}. \tag{7}$$

ε^{iklm} is the Levi–Civita tensor density. R^k_i is the Ricci tensor. $J^i_{(e)}(J^i_{(m)})$ stands for the electric (magnetic) current density vector.

An exact solution of the above equations describing the gravitational and electromagnetic fields of a non-rotating radiating dyon is given, in coordinates $x^0 = u, x^1 = r, x^2 = \theta, x^3 = \phi$, by the metric,

$$\mathrm{d}s^2 = B\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{8}$$

where

$$B = 1 - \frac{2M(u)}{r} + \frac{q_e(u)^2 + q_m(u)^2}{r^2} \tag{9}$$

and the non-zero components of the electromagnetic field tensor,

$$F_{10} = \frac{q_e(u)}{r^2},$$

$$F_{23} = q_m(u)\sin\theta. \tag{10}$$

$M(u)$, $q_e(u)$, and $q_m(u)$ are mass, electric and magnetic charge parameters, respectively. These parameters depend on the retarded time coordinate u.

The surviving components of the Einstein tensor, $G^k_i \equiv R^k_i - \frac{1}{2}g^k_i R$, the energy-momentum tensor of the electromagnetic field, E^k_i, and the energy-momentum tensor of the null fluid, N^k_i, are

$$G^0_0 = G^1_1 = -G^2_2 = -G^3_3 = \frac{q_e^2 + q_m^2}{r^4},$$

$$G^1_0 = K^2, \tag{11}$$

$$E_0^0 = E_1^1 = -E_2^2 = -E_3^3 = \frac{q_e^2 + q_m^2}{8\pi r^4}, \tag{12}$$

$$N_0^1 = \frac{K^2}{8\pi}, \tag{13}$$

where

$$K^2 = \frac{2(q_e\dot{q}_e + q_m\dot{q}_m - \dot{M}r)}{r^3}. \tag{14}$$

The dot denotes the derivative with respect to the retarded time coordinate u. The null fluid current vector, the electric and magnetic current density vectors are

$$V^i = g_1^i \frac{K}{\sqrt{8\pi}}, \tag{15}$$

$$J_{(e)}^i = -\frac{\dot{q}_e}{4\pi r^2} g_1^i,$$

$$J_{(m)}^i = -\frac{\dot{q}_m}{4\pi r^2} g_1^i. \tag{16}$$

The radiating dyon solution given by us yields: (a) the Bonnor–Vaidya solution when $q_m = 0$, (b) the Vaidya radiating star solution [2] when $q_e = q_m = 0$, (c) the Reissner–Nordström solution when $q_m = 0$ and q_e and M are constants, (d) the Schwarzschild solution when $q_e = q_m = 0$ and M is constant and (e) the static dyon solution [8] when M, q_e, and q_m are constants.

Using the Tolman pseudotensor, Vaidya [9] calculated the total energy of a spherically symmetric radiating star and got $E = M$. Further Lindquist *et al* (LSM) [10], using the LL pseudotensor, obtained the energy, momentum, and power output for the Vaidya spacetime and found that the total energy and momentum components are $p^i = M; 0, 0, 0$ and the power output is $-\mathrm{d}M/\mathrm{d}u$. One of the present authors [11], using several energy-momentum pseudotensors, calculated the energy and momentum components for the Vaidya metric and found the same result as obtained by LSM. Now we obtain the energy, momentum and power output for the radiating dyon in prescriptions of Einstein as well as LL and show that both give the same result. The energy-momentum pseudotensors of Einstein [12] and LL [13] are

$$\Theta_i^k = \frac{1}{16\pi} H_{i\ ,l}^{kl}, \tag{17}$$

where

$$H_i^{kl} = -H_i^{lk} = \frac{g_{in}}{\sqrt{-g}} [-g(g^{kn}g^{lm} - g^{ln}g^{km})]_{,m}, \tag{18}$$

and

$$L^{mn} = L^{nm} = \frac{1}{16\pi} S^{mjnk}{}_{,jk}, \tag{19}$$

where

$$S^{mjnk} = -g(g^{mn}g^{jk} - g^{mk}g^{jn}). \tag{20}$$

They satisfy the local conservation laws:

$$\frac{\partial \Theta_i^k}{\partial x^k} = 0, \tag{21}$$

$$\frac{\partial L^{mn}}{\partial x^m} = 0. \tag{22}$$

Θ_0^0 is the energy density. Θ_α^0 and Θ_0^α are the momentum and energy current density components, respectively. L^{00} and $L^{\alpha 0}$ give, respectively, the energy density and momentum (energy current) density components in LL prescription. The energy and momentum components in the prescription of Einstein are

$$P_i = \iiint \Theta_i^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3, \tag{23}$$

whereas in the LL prescription are

$$P^i = \iiint L^{0i} \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3. \tag{24}$$

$i = 0$ gives the energy and $i = 1, 2, 3$ give the momentum components. One knows that the energy-momentum pseudotensors give the correct result if calculations are carried out in quasi-cartesian coordinates [9–14]. The quasi-cartesian coordinates are those in which the metric g_{ik} approaches the Minkowski metric η_{ik} at great distances. Therefore, one transforms the line element (8), given in u, r, θ, ϕ coordinates, to quasi-cartesian coordinates t, x, y, z according to

$$\begin{aligned} u &= t - r, \\ x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta, \end{aligned} \tag{25}$$

and gets

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 - \left(\frac{2M(u)}{r} - \frac{q_e(u)^2 + q_m(u)^2}{r^2}\right) \times \left[\mathrm{d}t - \frac{x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z}{r}\right]^2, \tag{26}$$

where,

$$r = (x^2 + y^2 + z^2)^{1/2}. \tag{27}$$

We are interested in calculating the energy-momentum components and the power output of the radiating dyon. Therefore, the required components of H_i^{kl} are

$$H_0^{01} = \frac{2x\Lambda}{r^4},$$

$$H_0^{02} = \frac{2y\Lambda}{r^4},$$

$$H_0^{03} = \frac{2z\Lambda}{r^4},$$

$$H_1^{01} = \frac{x^2\Omega - Q^2 r^2}{r^5},$$

$$H_2^{02} = \frac{y^2\Omega - Q^2 r^2}{r^5},$$

$$H_3^{03} = \frac{z^2\Omega - Q^2 r^2}{r^5},$$

$$H_1^{02} = H_2^{01} = \frac{xy\Omega}{r^5},$$

$$H_2^{03} = H_3^{02} = \frac{yz\Omega}{r^5},$$

$$H_3^{01} = H_1^{03} = \frac{zx\Omega}{r^5},$$

$$H_0^{12} = H_0^{23} = H_0^{31} = 0, \tag{28}$$

and those of S^{mjnk} are

$$S^{0101} = \frac{\Lambda(x^2 - r^2) - r^4}{r^4},$$

$$S^{0202} = \frac{\Lambda(y^2 - r^2) - r^4}{r^4},$$

$$S^{0303} = \frac{\Lambda(z^2 - r^2) - r^4}{r^4},$$

$$S^{0102} = \frac{\Lambda xy}{r^4},$$

$$S^{0203} = \frac{\Lambda yz}{r^4},$$

$$S^{0301} = \frac{\Lambda zx}{r^4},$$

$$S^{0221} = S^{0331} = \frac{\Lambda x}{r^3},$$

$$S^{0112} = S^{0332} = \frac{\Lambda y}{r^3},$$

$$S^{0113} = S^{0223} = \frac{\Lambda z}{r^3},$$

$$S^{0123} = S^{0231} = S^{0312} = 0, \tag{29}$$

where

$$\Lambda = 2Mr - Q^2,$$
$$\Omega = 3Q^2 - 4Mr,$$
$$Q^2 = q_e^2 + q_m^2. \tag{30}$$

We use (17) and (28) in (23), and (19) and (29) in (24), apply the Gauss theorem and evaluate the integrals over the surface of two-sphere of radius r_0. The energy and momentum components in both prescriptions (Einstein as well as LL) are

$$E(r_0) = M - \frac{q_e^2 + q_m^2}{2r_0} \tag{31}$$

and

$$P_x = P_y = P_z = 0. \tag{32}$$

Thus the total energy and momentum of the radiating dyon is $P^i = M;0,0,0$ as expected. Now using (17)–(20) and (28)–(30) we calculate the energy current density components in both prescriptions and get

$$\Theta_0^1 = L^{01} = x\Delta, \tag{33}$$
$$\Theta_0^2 = L^{02} = y\Delta, \tag{34}$$
$$\Theta_0^3 = L^{03} = z\Delta, \tag{35}$$

where

$$\Delta = \frac{q_e \dot{q}_e + q_m \dot{q}_m - r\dot{M}}{4\pi r^4}. \tag{36}$$

Again we get the same result in both prescriptions. Switching off the electric and magnetic charge parameters we get the result for the Vaidya metric which were earlier obtained by one of the present authors [11]. Using the energy current density components, given by (33–36), we calculate the power output across a 2-sphere of radius r_0 and get,

$$W(r_0) = -\frac{\mathrm{d}}{\mathrm{d}u}\left[M - \frac{q_e^2 + q_m^2}{2r_0}\right]. \tag{37}$$

$q_e = q_m = 0$ in (37) gives the power output for the Vaidya spacetime [10]. The total power output (r_0 approaching infinity in (37)) is $-\mathrm{d}M/\mathrm{d}u$. Without using any energy-momentum pseudotensor, Bonnor and Vaidya [3] obtained the power output for the metric given by them. They got the same result as obtained by us.

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