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Relativistic fluid sphere on pseudo-spheroidal space-time

RAMESH TIKEKAR and V O THOMAS*
Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India
* Department of Mathematics, The Faculty of Science, The Maharaja Sayajirao University of Baroda, Baroda 390 002, India

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Abstract. A new exact closed form solution of Einstein's field equations is reported describing the space-time in the interior of a fluid sphere in equilibrium. The physical 3-space, $t=$ constant of its space-time has the geometry of a 3-pseudo spheroid. The suitability of this solution for describing the model of a relativistic superdense star is discussed and the stability of the model under radial pulsations is examined.

Keywords. General relativity; static fluid sphere; superdense stars.

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1. Introduction

The non-linear nature of the Einstein's field equations is a consequence of the self-interaction of the gravitational field. This makes it difficult to obtain relativistic models of spherical stars based on exact solution of Einstein's field equations describing spherical distributions of matter. The actual properties in the central region of a relativistic compact star are not precisely known and so assumptions of general nature to obtain exact solutions of Einstein's field equations become necessary. It is desired that solutions should be physically plausible and at the same time simple in form. This problem has been considered by several authors [1–6].

Here we have investigated the gravitational significance of space-times whose physical space obtained as $t=$ constant section has the geometry of 3-pseudo spheroid and it is shown that it can describe spherical compact distributions of matter. The form of the space-time metric and its general features are discussed in § 2. It is shown that the geometry of the space-time is governed by two parameters R and K. The space-time metric corresponding to a new exact closed form solution of Einstein's field equations is written in § 3. The suitability of this solution for describing the model of a superdense star is explored following the approach of [4, 5, 9] in the subsequent sections.

2. Static pseudo spheroidal space-time

A 3-pseudo spheroid immersed in the 4-dimensional Euclidean space with metric

$$d\sigma^2 = dx^2 + dy^2 + dz^2 + dw^2, \tag{1}$$

will have the cartesian equation

$$\frac{w^2}{b^2} - \frac{x^2 + y^2 + z^2}{R^2} = 1, \tag{2}$$

where b and R are constants. The section w = constant of the 3-pseudo spheroid are pseudo spheres, while sections x = constant, y = constant and z = constant represent, respectively hyperboloids of two sheets.

The parametrization

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \quad w = b(1 + r^2/R^2)^{1/2} \tag{3}$$

of the 3-pseudo spheroid leads to

$$d\sigma^2 = \frac{1 + Kr^2/R^2}{1 + r^2/R^2}\, dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

where

$$K = 1 + b^2/R^2. \tag{5}$$

The pseudo spheroidal 3-space given by eq. (4) is spherically symmetric and regular for $K > 1$. It is flat when $K = 1$ and generates into open hyperboloid when $K = 0$.

Following the Vaidya–Tikekar [4] approach we consider the space-time with metric

$$ds^2 = e^{\nu(r)}dt^2 - \frac{1 + Kr^2/R^2}{1 + r^2/R^2}\, dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6}$$

Various aspects of spherical distributions of perfect fluid in equilibrium described by space-times with metric in a form similar to (6) have been investigated by Buchdahl [1]. Vaidya and Tikekar [4] have examined the geometrical features of the 3-dimensional physical spaces obtained as t = constant hypersurfaces of such space-times. The space-time of the specific solution discussed by Buchdahl has the geometry of a 3-spheroid immersed in a 4-dimensional Euclidean space of Vaidya–Tikekar type. An extensive study of such solutions has also been done by Maharaj and Leach [6]. The 3-space of the metric (6) is characterized by pseudo spheroidal geometry and therefore the specific class of solutions in this set up is clearly distinct from the specific class of solutions reported by Buchdahl, Vaidya and Tikekar [4] and Maharaj and Leach [6].

3. Matter distribution on pseudo spheroidal space-time

In this section the gravitational significance of a static, spherical distribution of matter in the form of perfect fluid is explored on the background of the space-time of the metirc (6) with energy-momentum tensor

$$T_{ij} = (\rho + p/c^2)u_i u_j - (p/c^2)g_{ij}. \tag{7}$$

Here ρ, p and u^i respectively denote the matter density, fluid pressure and the unit 4-velocity field of the fluid. Since the fluid distribution is at rest,

$$u^i = (0, 0, 0, e^{-\nu/2}). \tag{8}$$

Einstein's field equations

$$\mathcal{R}_{ij} - \frac{1}{2}\mathcal{R}g_{ij} = -\frac{8\pi G}{c^2}T_{ij} \tag{9}$$

for a specific choice of the curvature parameter $K = 2$ leads to the space-time metric written explicitly as

$$\begin{aligned} ds^2 = & \left[A\sqrt{1+r^2/R^2} + B\left(X(r) - \frac{1}{\sqrt{2}}\sqrt{1+2r^2/R^2}\right)\right]^2 dt^2 \\ & - \frac{1+2r^2/R^2}{1+r^2/R^2}\, dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{10}$$

where

$$X(r) = \sqrt{1+r^2/R^2}\ln[\sqrt{2}\sqrt{1+r^2/R^2} + \sqrt{1+2r^2/R^2}] \tag{11}$$

and A, B are arbitrary constants.

The matter density and fluid pressure for the distribution (10) are expressed as

$$\frac{8\pi G}{c^2}\rho = \frac{3}{R^2}\left(1+\frac{2r^2}{3R^2}\right)\left(1+\frac{2r^2}{R^2}\right)^{-2}, \tag{12}$$

$$\frac{8\pi G}{c^4}p = \frac{A\sqrt{1+r^2/R^2} + B[X(r) + (1/\sqrt{2})\sqrt{1+2r^2/R^2}]}{R^2(1+2r^2/R^2)(A\sqrt{1+r^2/R^2} + B[X(r) - (1/\sqrt{2})\sqrt{1+2r^2/R^2}])}. \tag{13}$$

It is evident from eq. (12) that density is positive throughout the distribution. The gradient of density $d\rho/dr$ is found to be negative indicating that ρ is decreasing radially outward.

4. Size of the fluid sphere

The total mass and size of the configuration can be estimated using the scheme given by Vaidya and Tikekar [4], as follows.

Equation (12) determine ρ at the boundary $r = a$ of the distribution as

$$\frac{8\pi G}{c^2}\rho(a) = \frac{3(1+2a^2/3R^2)}{R^2(1+2a^2/R^2)^2}. \tag{14}$$

We introduce the density variation parameter

$$\lambda = \frac{\rho(a)}{\rho(0)} = \frac{1+2a^2/3R^2}{(1+2a^2/R^2)^2}, \tag{15}$$

where $\rho(0)$ is the density at the centre. Since ρ is a decreasing function of $r, \lambda < 1$. Solving (15) as quadratic in (a^2/R^2) one finds

$$\frac{a^2}{R^2} = \frac{1 - 6\lambda + \sqrt{1+24\lambda}}{12\lambda}. \tag{16}$$

The algebraic root assigning negative values to a^2/R^2 is rejected to ensure that a/R is real.

Equation (12) implies that the matter density at the centre is explicitly related with curvature parameter R as

$$\frac{8\pi G}{c^2}\rho(0) = \frac{3}{R^2}. \tag{17}$$

Equation (17) therefore determines R in terms of $\rho(a)$ and λ. Equation (16) then determines the boundary radius a of the distribution. Thus the size of the configuration is determined in terms of the surface density $\rho(a)$ and density variation parameter λ.

5. Physical plausibility

Any physically acceptable solution must comply with the following conditions:

(i) The matter density ρ and fluid pressure p should be non-negative throughout the distribution.
(ii) The gradients $\mathrm{d}\rho/\mathrm{d}r$ and $\mathrm{d}p/\mathrm{d}r$ should be negative.
(iii) The speed of sound should not exceed the speed of light as implication of causality fulfillment.
(iv) The interior metric should match continuously with the Schwarzschild exterior solution

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \tag{18}$$

at the boundary surface $r = a$ of the distribution where $p(a) = 0$.

The continuity of the metric coefficients give

$$m = \frac{a^3}{2R^2(1 + 2a^2/R^2)} \tag{19}$$

and

$$\sqrt{1 - 2m/a} = A\sqrt{1 + a^2/R^2} + B\,(X(a) - (1/\sqrt{2})\sqrt{1 + 2a^2/R^2}), \tag{20}$$

where

$$X(a) = \sqrt{1 + a^2/R^2}\ln(\sqrt{2}\sqrt{1 + a^2/R^2} + \sqrt{1 + 2a^2/R^2}). \tag{21}$$

The continuity of pressure across $r = a$ requires that pressure to vanish on the boundary implying that

$$A\sqrt{1 + a^2/R^2} = -B(X(a) + (1/\sqrt{2})\sqrt{1 + 2a^2/R^2}). \tag{22}$$

The constants A and B are determined from eqs (19), (20) and (22) as

$$A = \frac{X(a) + (1/\sqrt{2})\sqrt{1 + 2a^2/R^2}}{\sqrt{2}(1 + 2a^2/R^2)}, \tag{23}$$

$$B = -\frac{\sqrt{1+a^2/R^2}}{\sqrt{2}(1+2a^2/R^2)}, \tag{24}$$

while the total mass m is determined by (19).

The expressions (23) and (24) for A and B when substituted in (13), one can find after a lengthy but straightforward computation that $p \geq 0$ throughout the sphere. Further using the TOV equation we found that the pressure is decreasing radially outward.

Using the expressions (12) and (13) for density and pressure and the values of A and B in the TOV equation it can be readily seen that the speed of sound will not exceed the speed of light in the central region and boundary of the distribution, ensuring the fulfilment of causality requirements.

The expression for $\mathrm{d}p/\mathrm{d}\rho$, which represents the speed of sound in isentropic fluids, takes the form

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{\pi G R^2(\rho + p/c^2)(1+2r^2/R^2)^3\,[1+(8\pi G R^2 p/c^4)(1+2r^2/R^2)]}{(5+2r^2/R^2)(1+r^2/R^2)}. \tag{25}$$

At the centre, $\mathrm{d}p/\mathrm{d}\rho$ has the value

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0 = \frac{\pi G R^2[\rho(0)+p(0)/c^2](1+8\pi G p(0)R^2/c^4)}{5}. \tag{26}$$

From eqs (12), (13), (23) and (24) we can show that

$$\rho(0) - 3p(0)/c^2 > 0 \tag{27}$$

at the centre and using (17) and (27) it readily follows from (26) that

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0 < 0.2c^2. \tag{28}$$

At the boundary, using (14), we have the expression

$$\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_s = \frac{3(1+2a^2/R^2)(1+2a^2/3R^2)}{8(5+2a^2/R^2)(1+a^2/R^2)}\,c^2 < c^2. \tag{29}$$

The variation of $\mathrm{d}p/\mathrm{d}\rho$, which represents the speed of sound in isentropic fluids, is examined using numerical procedures for certain specific models in this set up. It is found that p/ρ is decreasing radially outward indicating that $\mathrm{d}(p/\rho)/\mathrm{d}\rho > 0$. However the speed of sound is found to be increasing radially outward for a number of models, which is an unsatisfactory feature of these solutions in view of the expectation that it should be decreasing radially outward which follows from equations of state found in literature. Since definite information about the equation of state for matter in nuclear density ranges is lacking, as argued by Knutsen [9], one must be careful in this respect.

6. Superdense star model

When all thermonuclear sources of energy are exhausted, a star will gradually cool down and in this process it will collapse gravitationally and form a compact star – white dwarf,

Table 1. Masses and equilibrium radii of superdense star models corresponding to $K = 2$ and $\rho(a) = 2 \times 10^{14}$ gm/cm^3.

λ	R (km)	a (km)	m (km)	m/Mo	A	B
0.900	26.928	04.875	0.075	0.050	1.094	−0.674
0.800	25.388	06.828	0.215	0.146	1.062	−0.639
0.700	23.748	08.271	0.403	0.273	1.027	−0.602
0.600	21.987	09.429	0.634	0.429	0.987	−0.562
0.500	20.071	10.382	0.904	0.613	0.943	−0.518
0.400	17.952	11.162	1.216	0.825	0.890	−0.469
0.300	15.547	11.771	1.571	1.065	0.826	−0.413
0.200	12.694	12.176	1.972	1.337	0.742	−0.345
0.125	10.035	12.291	2.304	1.562	0.653	−0.279
0.100	08.976	12.274	2.421	1.641	0.613	−0.252

Note: 1 Mo = 1.475 km.

neutron star, or black hole. Generally a star keeps its equilibrium with outward pressure against the self gravitational force. The model that we have presented here can describe the hydrostatic equilibrium conditions in such a superdense star with densities in the range 10^{14}–10^{16} gm/cm^3. We take the matter density on the boundary $r = a$ of the star as $\rho(a) = 2 \times 10^{14}$ gm/cm^3. Choosing different values for λ, we determine the boundary radius a of the star and its total mass m in accordance with the scheme of § 4 and § 5. The value of m obtained is in kilometers. The mass of the star in grams is obtained using $M = mc^2/G$. The results of these computations together with the values of constants A and B as determined by eqs (23) and (24) are given in table 1.

For $\lambda \leq 0.7$ in table 1, we get a set of physically viable models of relativistic compact stars. The models with $\lambda > 0.7$ in the table have their equilibrium radii much smaller than that of a neutron star. The equilibrium models presented here take lesser values for radii a and total mass m, than the corresponding values given by Vaidya and Tikekar [4].

7. Dynamic stability

A sufficient condition for the dynamic stability of a spherically symmetric distribution of matter under small radial adiabatic perturbations has been developed by Chandrasekhar [7]. A normal mode of radial oscillation for an equilibrium configuration i.e.

$$\delta r = \xi(r) \exp(i\omega t) \tag{30}$$

is stable when the frequency of oscillation ω is real and unstable when ω is imaginary.

Chandrasekhar's [7] pulsation equation for the line element (6) is given by

$$\begin{aligned} &\omega^2 \int_{\text{centre}}^{\text{boundary}} \exp\left(\frac{3\alpha+\nu}{2}\right) \frac{(\rho+p)u^2}{r^2}\,\mathrm{d}r \\ &= \int_{\text{centre}}^{\text{boundary}} \exp\left(\frac{3\nu+\alpha}{2}\right)\left(\frac{\rho+p}{r^2}\right) \\ &\times \left\{\left[\frac{-2}{r}\frac{\mathrm{d}\nu}{\mathrm{d}r} - \frac{1}{4}\left(\frac{\mathrm{d}\nu}{\mathrm{d}r}\right)^2 + 8\pi p\,\mathrm{e}^{\alpha}\right]u^2 + \frac{\mathrm{d}p}{\mathrm{d}\rho}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2\right\}\mathrm{d}r, \end{aligned} \tag{31}$$

where

$$u = \xi r^2 e^{-\nu/2} \quad \text{and } e^{\alpha} = \frac{1+2r^2/R^2}{1+r^2/R^2} \quad \text{with } K=2. \tag{32}$$

The boundary condition to be satisfied at $r=a$ is that the Lagrangian change in pressure

$$\Delta p = -e^{\nu/2}\left(\frac{\gamma p}{r^2}\right)\left(\frac{du}{dr}\right) = 0 \quad \text{at } r=a,$$

where γ is the adiabatic index. We must have

$$\frac{du}{dr} = 0 \quad \text{at } r=a. \tag{33}$$

Following the method of Bardeen *et al* [8] and used by Knutsen [9] to investigate the stability of Vaidya–Tikekar models, we choose

$$u = R^3 x^{3/2}(1 + a_1 x + b_1 x^2 + \cdots) \tag{34}$$

as the trial function where the new variable x is taken as $x = 2r^2/R^2$.

The boundary condition $du/dr = 0$ at $r=a$ implies

$$3 + 5a_1 b + 7b_1 b^2 + \cdots = 0, \tag{35}$$

where $b = 2a^2/R^2$.

The pulsation equation (31) for the metric (6) now takes the form

$$\omega^2 \int_{\text{centre}}^{\text{boundary}} \exp\left(\frac{3\alpha+\nu}{2}\right)\frac{(\rho+p)u^2}{r^2}\, dr = \int_0^b R_1 R_2((R_3+R_4+R_5)R_6+R_7R_8)dx, \tag{36}$$

where

$$R_1 = \frac{2(x+1)}{\sqrt{x+2}}(A_1\sqrt{x+2} + B_1[\sqrt{x+2}\,l(x) - \sqrt{x+1}])^3,$$

$$R_2 = \frac{L(x)(x+2)+p(x)(x+2)+q(x)}{4\pi R^2(x+1)^2[L(x)+p(x)+q(x)]},$$

$$R_3 = \frac{-8[L(x)+p(x)]}{R^2(x+2)[L(x)+p(x)+q(x)]},$$

$$R_4 = \frac{-2x[L(x)+p(x)]^2}{R^2(x+2)^2[L(x)+p(x)+q(x)]^2},$$

$$R_5 = \frac{2(x+1)[L(x)+p(x)-q(x)]}{R^2(x+1)(x+2)[L(x)+p(x)+q(x)]},$$

$$R_6 = 2R^4x^2(1+a_1x+b_1x^2+\cdots)^2,$$

$$R_7 = \frac{(x+1)[L(x)+p(x)][L(x)(x+2)+p(x)(x+2)+q(x)]}{(x+5)(x+2)[L(x)+p(x)+q(x)]^2},$$

Table 2. The values of the integral on the right hand side of the pulsation equation (36) for some specific choices of the constants a_1 and b_1 and $\lambda = 0.4$.

a_1	b_1	Integral
0.000	−0.717	1.476
−0.776	0.000	0.536
−0.858	1.000	0.084
1.000	−1.641	3.447
5.000×10^2	-4.627×10^2	1.076×10^5
-5.410×10^2	5.000×10^2	1.246×10^5
1.000×10^5	-9.240×10^4	4.272×10^9
-1.000×10^5	9.240×10^4	4.272×10^9

$$R_8 = 4R^2x(3 + 5a_1x + 7b_1x^2 + \cdots)^2,$$

$$L(x) = r(x)[l(b) - l(x)],$$

$$l(x) = \ln[\sqrt{x+1} + \sqrt{x+2}],$$

$$l(b) = \ln[\sqrt{b+1} + \sqrt{b+2}],$$

$$p(x) = \sqrt{x+2}\,\sqrt{b+1},$$

$$q(x) = \sqrt{x+1}\,\sqrt{b+2},$$

$$r(x) = \sqrt{x+2}\,\sqrt{b+2},$$

$$A_1 = \frac{\sqrt{b+2}\,l(b) + \sqrt{b+1}}{b+1},$$

$$B_1 = -\frac{\sqrt{b+2}}{(b+1)}.$$

We have evaluated the integral on the right of the equation (36) numerically for different values of b. It is found that the integral admits positive values for $0.243 \leq b \leq 1.146$ (i.e. $0.3 \leq \lambda \leq 0.7$) and for different choices small, large, postive or negative values of the constants a_1, b_1.

We have reported in table 2 the values of the integral on the right side of the equation (36) evaluated numerically for certain specific choices of a_1 and b_1 for the model with $\lambda = 0.4$ of table 1.

This analysis indicates that these models with $0.3 \leq \lambda \leq 0.7$ will be stable. The space-time with pseudo spheroidal geometry for its spatial sections t = constant thus may admit the possibilities of describing interiors of superdense fluid stars in equilibrium.

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