Pramāṇa - J. Phys., Vol. 28, No. 4, April 1987, pp. L429–L432. © Printed in India.

A Lagrangian basis for Ashtekar's reformulation of canonical gravity

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MS received 30 January 1987

Abstract. A manifestly covariant Lagrangian is presented which leads to the reformulation of canonical general relativity using new variables recently discovered by Ashtekar.

Keywords. General relativity; canonical gravity; new variables; Lagrangian formulation.

PACS No. 04·60

In the constrained Hamiltonian formulation of general relativity, as developed by Dirac and others, the basic variables are the three-metric on a spatial slice and the extrinsic curvature. Both of these have a direct geometrical interpretation, but this choice of variables leads to complicated, non-polynomial constraints, which no one has been able to consistently carry over to the quantum theory. Recently, starting from the Hamiltonian formulation of gravity in the triad formalism, Ashtekar (1986) has performed a canonical transformation to a new set of variables, which leads to a dramatic simplification of the constraints of the theory. The new variables consist of a soldering form and the Ashtekar-Sen-Witten connection. This reformulation of canonical gravity is explained in the references quoted below. It raises hopes of arriving at a consistent quantum theory of gravity and is of considerable interest in its own right.

The basic variables in Ashtekar's formulation are a canonically conjugate pair $(\tilde{\sigma}^a, A_a)$:

$$\{\tilde{\sigma}^a, \tilde{\sigma}^b\} = \{A_a, A_b\} = 0$$

$$\{A_a, \tilde{\sigma}^b\} = \delta^b_a, \tag{1}$$

$\tilde{\sigma}^a$ is a densitized soldering form and has information about the three-metric on a spatial slice. A_a is a complex SU(2) connection which has information about the extrinsic curvature of a spatial slice. With the SU(2) indices in, these objects are $\tilde{\sigma}^a_{AB}$ and A_a^{AB}. $\tilde{\sigma}^a_{AB}$ is a Hermitean, traceless 2×2 matrix. In terms of these variables, the constraints of general relativity assume a strikingly simple form. They are

(i) The "Gauss law" constraint

$$\mathscr{D}_a \tilde{\sigma}^a \approx 0, \tag{2}$$

where $\mathcal{D}_a$ is the covariant derivative operator corresponding to the Ashtekar-Sen-Witten connection A_a,
(ii) the vector constraint

$$\mathrm{Tr}(\tilde{\sigma}^a F_{ab}) \approx 0, \tag{3}$$

where F_{ab} is the curvature of the connection A_a and
(iii) the scalar constraint

$$\mathrm{Tr}(\tilde{\sigma}^a \tilde{\sigma}^b F_{ab}) \approx 0. \tag{4}$$

The Hamiltonian is (as in all reparametrization-invariant theories) a linear combination of the constraints. The classical evolution of the system is then determined by Hamilton's equations. Thus, equations (1)–(4) completely describe Ashtekar's theory.

As is clear from the last paragraph, the new formulation is cast in Hamiltonian language. While this is ideally suited for purposes of quantization, it lacks manifest four-dimensional covariance. It would be of some interest to derive the formulation from a covariant Lagrangian. This is the purpose of this note. A set of basic variables is postulated and a manifestly covariant Lagrangian formed out of these. A straightforward application of Dirac's procedure for singular Lagrangians leads to a constrained Hamiltonian theory identical in content to Ashtekar's. Details will appear elsewhere. For background and notation, see the references at the end of the paper.

The basic variables used are a real SL$(2, C)$ valued connection 1-form A_μ and a Hermitean, matrix valued 1-form γ_μ. With the SL$(2, C)$ indices in, these objects are A_μ^{AB} and $\gamma_\mu^{AA'}$. [A, B are Weyl spinor indices and take values 1, 2. Primed indices transform according to the complex conjugate representation of SL$(2, C)$]. A_μ^{AB} is a connection and so transforms inhomogeneously under local SL$(2, C)$ rotations, whereas $\gamma_\mu^{AA'}$ transforms homogeneously. From these objects one can construct the curvature 2-form

$$F_{\mu\nu}^{AB} \equiv \partial_\mu A_\nu^{AB} - \partial_\nu A_\mu^{AB} + [A_\mu, A_\nu]^{AB}$$

and the antisymmetric tensor

$$\Sigma_{\mu\nu}^{AB} \equiv i(\gamma_\mu{}^{AA'}\gamma_{\nu A'}{}^{B} - \gamma_\nu{}^{AA'}\gamma_{\mu A'}{}^{B})$$

The trace (over internal indices) of the wedge product of Σ and F is a 4-form and can be integrated over space-time to produce a general coordinate scalar. We postulate the Action

$$I = \int \Sigma_A{}^B \wedge F_B{}^A \tag{5}$$

which is manifestly generally covariant and invariant under local SL$(2, C)$ transformations.

By using the Levi-Civita tensor density $\tilde{\eta}^{\mu\nu\alpha\beta}$ $(\tilde{\eta}^{\mu\nu\alpha\beta} = \tilde{\eta}^{[\mu\nu\alpha\beta]}$, $\tilde{\eta}^{0123} = 1)$, one can write the Action (5) as the integral of Lagrangian density

$$I = \int \mathrm{d}^4x\, \tilde{\mathcal{L}} = \tfrac{1}{2}\int \mathrm{d}^4x\, \tilde{\eta}^{\mu\nu\alpha\beta}\,\mathrm{Tr}(\Sigma_{\mu\nu}F_{\alpha\beta}). \tag{6}$$

(In future, we drop the internal indices and use more compact matrix notation). We now take (6) as the starting point and in a straightforward manner apply Dirac's analysis for singular Lagrangians. The Lagrangian can be written in $3+1$ form as

$$L = \int \mathrm{d}^3x\, \tilde{\eta}^{abc}\{\mathrm{Tr}(\Sigma_{bc}F_{0a}) + \mathrm{Tr}(\Sigma_{0a}F_{bc})\}. \tag{7}$$

Defining the canonical momenta as usual

$$\pi^0 \equiv \partial L/\partial \dot{A}_0 = 0, \quad \psi^0 \equiv \partial L/\partial \dot{\gamma}_0 = 0,$$

$$\pi^a \equiv \partial L/\partial \dot{A}_a = \tilde{\eta}^{abc}\Sigma_{bc}, \quad \psi^a \equiv \partial L/\partial \dot{\gamma}_a = 0,$$

we find the following primary constraints

$$\pi^0 \approx 0, \tag{8}$$

$$\psi^0 \approx 0, \tag{9}$$

$$\psi^a \approx 0, \tag{10}$$

$$\pi^a - \tilde{\eta}^{abc}\Sigma_{bc} \approx 0. \tag{11}$$

Let us define

$$\tilde{\sigma} \equiv \tilde{\eta}^{abc}\Sigma_{bc}. \tag{12}$$

Since the γ's are supposed Hermitean, it follows from the definition that $\tilde{\sigma}^a$ is traceless and Hermitean. The constraint (11) now reads

$$\varphi^a \equiv \pi^a - \tilde{\sigma}^a \approx 0. \tag{13}$$

With the usual Poisson bracket structure between the canonically conjugate pairs (A_μ, π^μ), (γ_μ, ψ^μ), we find that ψ^0 and π^0 commute with all the constraints. φ^a depends on γ_b through $\tilde{\sigma}^a$ and so has a non-vanishing bracket with some of the ψ^a's. The remaining ψ^a's commute with all constraints and are so first class.

The Hamiltonian of the theory, given by $H = p\dot{q} - L$, is

$$H = \int \mathrm{d}^3x\, \mathrm{Tr}\{\tilde{\sigma}^a(\partial_a A_0 + [A_0, A_a]) - \Sigma_{0a}F_{bc}\tilde{\eta}^{abc}\} + \text{constraints}. \tag{14}$$

We now demand the preservation in time of the first class constraint (8). This leads to the Gauss law constraint (2). (Throughout this paper we drop surface terms; they are important, but not for this calculation.) The Hamiltonian now is

$$H = -\int \mathrm{d}^3x\, \tilde{\eta}^{abc}\,\mathrm{Tr}(\Sigma_{0a}F_{bc}) + \text{constraints}. \tag{15}$$

Next we demand that the first class constraint (9) is preserved in time. This leads to the constraint

$$i\tilde{\eta}^{abc}[\gamma_a, F_{bc}] \approx 0 \tag{16}$$

and modulo (16) the Hamiltonian vanishes. Since the Hamiltonian is now zero (and so certainly first class!), the constraint analysis ends. No new constraints emerge. The Hamiltonian is a linear combination of constraints. The content of (16) can be extracted by multiplying with γ's and taking the trace. Multiplying (16) by γ_b and tracing yields the vector constraint (3). Multiplying (16) by $i\tilde{\eta}^{def}\gamma_d\gamma_e\gamma_f$ and tracing yields the scalar constraint (4).

The constraints of the theory now are (8, 9, 10, 11) and (2, 3, 4). Let us denote by Γ_0 the initial phase space spanned by $(A_\mu, \pi^\mu, \gamma_\mu, \psi^\mu)$ endowed with the natural sympletic form:

$$\omega_0 = \mathrm{d}\pi^\mu \wedge \mathrm{d}A_\mu + \mathrm{d}\psi^\mu \wedge \mathrm{d}\gamma_\mu. \tag{17}$$

[An integral over space is understood in (17) and (18).] We now impose constraints (8, 9, 10, 11). These define a submanifold $\mathcal{M} \subset \Gamma_0$. The pull-back of the two form (17) to $\mathcal{M}$ is

$$\omega = \mathrm{d}\tilde{\sigma}^a \wedge \mathrm{d}A_a. \tag{18}$$

This form on $\mathcal{M}$ is, of course, degenerate. Vector fields in its kernel are generators of "gauge" transformations. Let us quotient $\mathcal{M}$ with respect to these "gauge" transformations. The quotient space Γ is a symplectic manifold spanned by the coordinates $(\tilde{\sigma}^a, A_a)$ and endowed with the *non-degenerate* symplectic form (18) (now regarded as a form on Γ).

From (18) we see that Γ is spanned by the variables σ^a and A_a which are canonically conjugate to each other. $\tilde{\sigma}^a$ is a Hermitean, traceless matrix and the real SL(2, C) connection A_a can also be viewed as a complex SU(2) connection. The theory now consists of canonically conjugate (1) variables $(\tilde{\sigma}^a, A_a)$ and the constraints (2, 3, 4). Thus the manifestly covariant Lagrangian (5) completely reproduces Ashtekar's theory. A remarkable feature of the Lagrangian (5) is that it is polynomial.

Acknowledgements

It is a pleasure to thank Abhay Ashtekar from whom I learnt about the new formulation of canonical gravity. I have also had the benefit of some unpublished notes by Paul Renteln, a collection of appendices from his Ph.D. thesis. I thank Rajaram Nityananda for reading through the manuscript and making a number of suggestions.

I understand that Ted Jacobson and Lee Smolin have also addressed and solved the problem discussed here.

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