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The trousers problem revisited

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Abstract. Anderson and DeWitt considered the quantization of a massless scalar field in a spacetime whose spacelike hypersurfaces change topology and concluded that the topology change gives rise to infinite particle and energy production. We show here that their calculations are insufficient and that their propagation rule is unphysical. However, our results using a more general propagation rule support their conclusion.

Keywords. Trousers topology; propagation rules; massless scalar field; energy density.

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1. Introduction

The trousers topology was introduced by Anderson and DeWitt (1986; see also DeWitt 1985) to model the effects of topology change in quantum gravity. Instead of the gravitational field, they consider a massless scalar field on a background spacetime whose spatial cross-sections change topology. By comparing the "natural" Fock vacua before and after the topology change, they conclude that infinite particle and energy production occurs, and that the topology change therefore does not take place.

We re-examine their procedure and show that it is incomplete. Fundamental to the solution of the problem is a choice of propagation rule as well as of "in" and "out" mode functions. We show that their propagation rule is unphysical and furthermore that their choice of mode functions is incomplete.

More specifically, we introduce as a physical constraint on the Bogolubov transformations between in and out modes that they be time-independent and show that this determines the propagation rule completely up to a single free parameter. The shadow rule of Anderson and DeWitt (1986) does *not* satisfy this constraint. Anderson and DeWitt only give a general argument for the presence of an energy density proportional to the square of a delta function; we calculate the energy density explicitly and attempt (unsuccessfully) to use the free parameter to set the coefficient of this term to zero.

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A re-examination of the problem also shows that the set of modes used was incomplete, and that there are in fact new modes associated with the topology change. A satisfactory treatment of this problem would require quantizing these modes as well; we postpone a discussion of this to a future paper.

In § 2 we describe the trousers topology and establish the general framework for the problem. In § 3 we consider the question of which propagation rules are allowed. A complete tabulation of the explicit forms of the propagated mode functions so obtained is deferred to the Appendix, and it is seen that the shadow rule of Anderson and DeWitt (1986) is not acceptable. The inner products (Bogolubov coefficients) between in and out modes are also given in the Appendix. Section 4 contains the calculation of the energy density, and in § 5 we discuss our results.

2. The trousers topology

We consider the massless scalar wave equation propagating on a two-dimensional spacetime whose spatial cross-sections change topology from S^1 to $S^1 + S^1$. At early times the spacetime will be a two-dimensional cylinder with circumference 2λ, while at late times it has split into two disjoint cylinders, each with circumference λ; see figure 1. The spacetime looks like an inverted pair of trousers.

At early and late times the cylinders can be chosen to be flat and Lorentzian. However, if we take the region where the topology change occurs to be smooth, then there must be a coordinate patch at the crotch of the trousers which is Euclidean; there is no global Lorentzian metric on the manifold shown in figure 1. But if the metric changes signature then the determinant of the metric changes sign, and therefore must either be zero somewhere or be complex-valued. In either case, the interpretation of the wave equation is unclear. To avoid these problems we will shrink the Euclidean

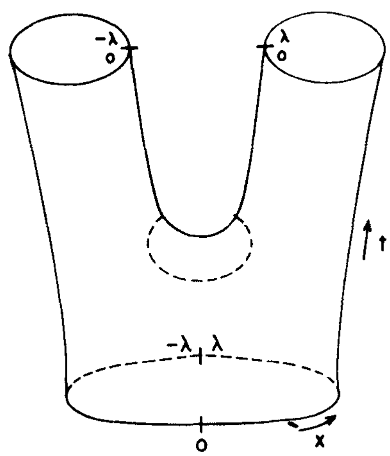

Figure 1. The trousers topology. At early times the cross-section is a circle of circumference 2λ, while at late times the cross-section consists of two, disjoint circles, each of circumference λ. There is necessarily a coordinate patch, indicated by dotted lines, where the manifold is spacelike, i.e. the metric there is positive definite.

patch to a single, extremely singular point and then remove this point from the spacetime. The resulting manifold is Lorentzian everywhere and may be chosen to be flat; see figure 2. We choose coordinates x, t with

$$t \in \mathbb{R}; \quad x \in [-\lambda, \lambda] \tag{1}$$

and where for $t<0$ the line $x=\lambda$ is identified with $x=-\lambda$, while for $t>0$ we identify $x=\pm\lambda$, with $x=0^{\pm}$. We will call the region $t<0$ the trunk or "in" region, and $t>0$ the legs or "out" region. Note that in the legs the coordinate line "$x=0$" is ill-defined and one must specify which leg is being referred to. The point where the topology changes is represented by $t=0$, $x=-\lambda$, 0, λ and is removed from the manifold.

Since the metric is chosen to be flat everywhere, the massless scalar wave equation takes the familiar form

$$(-\partial_t^2 + \partial_x^2)\phi = 0. \tag{2}$$

The Klein-Gordon inner product between any two solutions ϕ, $\tilde{\phi}$ of (2) is

$$(\phi, \tilde{\phi}) = i \int_{-\lambda}^{\lambda} (\phi^* \dot{\tilde{\phi}} - \dot{\phi}^* \tilde{\phi}) \mathrm{d}x \tag{3}$$

where the integral is over any surface $\Sigma = \{t = \text{constant}\}$ and where $*$ denotes complex conjugation and dot denotes derivatives with respect to t. We assume that ϕ satisfies the appropriate periodic boundary conditions, namely

$$\begin{aligned} &\phi(-\lambda, t) = \phi(+\lambda, t) \quad (t<0), \\ &\phi(\pm\lambda, t) = \phi(0^{\pm}, t) \quad (t>0). \end{aligned} \tag{4}$$

Note that we do not require ϕ to solve (2) at the singularity; we will return to this fundamental issue in § 3.

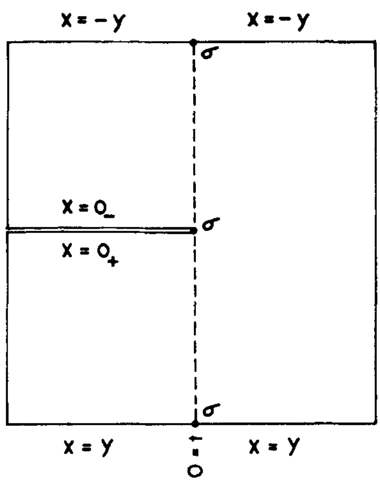

Figure 2. The unrolled trousers topology. For $t<0$ the line $x=\lambda$ is to be identified with $x=-\lambda$, while for $t>0$ the line $x=+\lambda$ is to be identified with $x=0^{\pm}$. The point p where the topology changes is removed from the manifold.

The quantization of ϕ on a cylinder of radius 2λ proceeds as follows. The modes

$$u_k = \frac{1}{\sqrt{4|k|\lambda}} \exp[i(kx - |k|t)] \quad \left(k = \frac{\pi n}{\lambda}, 0 \neq n \in \mathbb{Z}\right) \tag{5}$$

are orthonormal with respect to (3), i.e.

$$(u_k, u_{k'}) = \delta_{kk'} = -(u_k^*, u_{k'}^*) \tag{6}$$
$$(u_k, u_{k'}^*) = 0.$$

However, the set $\{u_k, u_k^*\}$ is *not* complete. Unlike the case of unbounded Minkowski space, on the cylinder the constant mode and the mode proportional to t have finite Klein-Gordon product with each other and are orthogonal to u_k and u_k^* and therefore must be included in the complete set of modes. Note that these "zero-frequency" modes have zero norm.**

The quantum field ϕ can be expanded as

$$\phi = \sum_{k \neq 0} (a_k u_k + a_k^\dagger u_k^*) + Q\alpha_0 + P\beta_0, \tag{7}$$

where the operators Q and P are Hermitian. Demanding the usual equal time commutation relations

$$[\phi(x,t), \dot{\phi}(x',t)] = i\delta(x - x') \tag{8}$$

leads to the commutation relations

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}; \quad [Q, P] = i \tag{9}$$

(all others zero). We define the vacuum state $|0_{\text{in}}\rangle$ by

$$a_k|0_{\text{in}}\rangle = 0 = P|0_{\text{in}}\rangle \tag{10}$$

so that it will be the state of lowest energy. (The Hamiltonian has a term proportional to P^2.)

The apparent contradiction between (9) and (10) (try evaluating $\langle 0|[Q,P]|0\rangle$) is resolved by noticing that Q is *not* a well defined operator on eigenstates of P and furthermore the vacuum $|0\rangle$ is not normalizable. Vacuum expectation values of powers of Q are divergent; we will return to this point in § 5.

The procedure used by Anderson and DeWitt (1986) is to assume that at early times (i.e. in the trunk) ϕ can be expanded as in (7) in terms of the trunk modes u_k while at late times (i.e. in the legs) ϕ possesses a similar expansion in terms of leg modes. More specifically, we introduce leg modes $u_{lL}, \alpha_L, \beta_L$ ($u_{lR}, \alpha_R, \beta_R$) for $t > 0$ having support only in the left (right) leg, where they satisfy

**One can in fact introduce orthonormal modes u_0, u_0^* given by e.g. $u_0 = \frac{1}{(2\lambda)^{\frac{1}{2}}}(1 - it)$. However, we will choose the modes $\alpha_0 = N \in \mathbb{R}$ and $\beta_0 = t(1/2\lambda N)$.

$$u_{lL} = 1/(2|l|\lambda)^{\frac{1}{2}} \exp[i(lx - |l|t)] \quad \left(l = \frac{2\pi m}{\lambda}, 0 \neq m \in \mathbb{Z}\right),$$
$$\alpha_L = N_L \in \mathbb{R}, \tag{11}$$
$$\beta_L = \frac{1}{N_L \lambda} t$$

in the left leg and zero in the right leg, and

$$u_{lR} = 1/(2|l|\lambda)^{\frac{1}{2}} \exp[i(lx - l|t)] \quad \left(l = \frac{2\pi m}{\lambda}, 0 \neq m \in \mathbb{Z}\right),$$
$$\alpha_R = N_R \in \mathbb{R}, \tag{12}$$
$$\beta_R = \frac{1}{N_R \lambda} t$$

in the right leg and zero in the left leg. ϕ is now assumed to satisfy the expansion (7) in the trunk, and the expansion

$$\begin{aligned} \phi = &\sum_{l \neq 0} (a_{lL} u_{lL} + a_{lL}^{\dagger} u_{lL}^{*}) + Q_L \alpha_L + P_L \beta_L \\ &+ \sum_{l \neq 0} (a_{lR} u_{lR} + a_{lR}^{\dagger} u_{lR}^{*}) + Q_R \alpha_R + P_R \beta_R \end{aligned} \tag{13}$$

in the legs.

The energy density is given by

$$\begin{aligned} T_{tt} &= \tfrac{1}{2}(\dot{\phi}^2 + \phi'^2) \\ &= \phi_{,u}^2 + \phi_{,v}^2, \end{aligned} \tag{14}$$

where prime denotes differentiation with respect to x and $u = t - x$, $v = t + x$.

The idea is now to treat T_{tt} as an operator $T_{tt}{}^{\mathrm{out}}$ by inserting (13) into (14), and to evaluate the expectation value

$$E = \frac{\langle 0_{\mathrm{in}} | T_{tt}{}^{\mathrm{out}} | 0_{\mathrm{in}} \rangle}{\langle 0_{\mathrm{in}} | 0_{\mathrm{in}} \rangle} \tag{15}$$

of the "out" energy density in the "in" vacuum $|0_{\mathrm{in}}\rangle$ defined by (10). The way to do this is to expand the leg operators a_{lL}, etc., in terms of the trunk operators a_k, etc., using Bogolubov transformations. However, in order to do this we must be able to compare the two expansions (7) and (13), and to do so we must have a propagation rule which tells us how to propagate trunk modes into the legs and vice versa.

3. Propagation rules

Consider a smooth solution ϕ of (2) in the trunk satisfying the periodic boundary condition

$$\phi(x+2\lambda, t) = \phi(x,t) \quad (t<0). \tag{16}$$

Just prior to the topology change, at $t = 0^-$, we can represent ϕ in terms of its Cauchy data

$$\phi \leftrightarrow \begin{pmatrix} \varphi \\ \pi \end{pmatrix} = \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}\Bigg|_{t=0^-} \tag{17}$$

A propagation rule is a mapping

$$\begin{pmatrix} \varphi \\ \pi \end{pmatrix} \mapsto \begin{pmatrix} \varphi_L \\ \pi_L \end{pmatrix} \oplus \begin{pmatrix} \varphi_R \\ \pi_R \end{pmatrix} \tag{18}$$

from Cauchy data in the trunk ($t = 0^-$) to Cauchy data in the legs ($t = 0^+$). After the singularity, the Cauchy data (at $t = 0^+$) will again propagate according to (2).

Away from the singularity there is no problem—causality requires that until one crosses the future light cone emanating from the singularity the propagation is uniquely determined by (2). Thus, at first it seems natural to assume a propagation rule of the form

$$\begin{aligned} \varphi_R(x) &= [1-\theta(x)]\varphi(x+\lambda) + \theta(x)\varphi(x) \quad (0 \leqslant x < \lambda), \\ \varphi_L(x) &= [1-\theta(x)]\varphi(x) + \theta(x)\varphi(x-\lambda) \quad (-\lambda < x \leqslant 0), \end{aligned} \tag{19}$$

$$\begin{aligned} \tilde{\pi}_R(x) &= [1-\theta(x)]\pi(x+\lambda) + \theta(x)\pi(x) \quad (0 \leqslant x < \lambda), \\ \tilde{\pi}_L(x) &= [1-\theta(x)]\pi(x) + \theta(x)\pi(x-\lambda) \quad (-\lambda < x \leqslant 0), \end{aligned} \tag{20}$$

with $\varphi_R, \tilde{\pi}_R, \varphi_L, \tilde{\pi}_L$ periodic with period λ and where $\theta(x)$ denotes the step function.** But note that, except in the special case $\phi(x+\lambda) = \phi(x)$, the data (19) is discontinuous. This suggests that one must allow for the possibility of delta functions in π_R and π_L. On the other hand, there is no way to determine the coefficient of these delta functions uniquely—one has freedom to add "data" at the singularity. We therefore assume a propagation rule of the form (19) but replace (20) by

$$\begin{aligned} \pi_R(x) &= \tilde{\pi}_R(x) + a\delta(x) \quad (0 \leqslant x < \lambda), \\ \pi_L(x) &= \tilde{\pi}_L(x) + b\delta(x) \quad (-\lambda < x \leqslant 0), \end{aligned} \tag{21}$$

with π_R, π_L periodic with period λ and a and b arbitrary (ϕ-dependent) constants.

Analogously, in the other direction, starting from ϕ_R, ϕ_L which are smooth in the legs, we assume a propagation rule of the form**

$$\begin{aligned} \varphi(x) &= [1-\theta(x)]\varphi_L(x) + \theta(x)\varphi_R(x) + \theta(x-\lambda)[\varphi_L(x-2\lambda) - \varphi_R(x)] \\ \pi(x) &= [1-\theta(x)]\pi_L(x) + \theta(x)\pi_R(x) + \theta(x-\lambda)[\pi_L(x-2\lambda) - \pi_R(x)] \\ &\quad - c\delta(x) - d\delta(x-\lambda) \\ &\qquad (-\lambda < x \leqslant +\lambda) \end{aligned} \tag{22}$$

**To avoid having to deal with step functions and delta functions defined at the boundary of the given range of x, one could replace the range of x throughout (19), (20), (21) with $x \in [-\lambda/2, \lambda/2]$ using periodicity. A similar comment applies to (22) with the range of x replaced by $x \in [-\lambda/2, 3\lambda/2]$.

with φ, π periodic with period 2λ and where c, d are arbitrary (ϕ_L, ϕ_R dependent) constants.

What restrictions can we impose on the constants a, b, c, d? Although the inner product (3) is independent of t both in the trunk and in the legs, it is *not* necessarily conserved between the trunk and the legs since ϕ does *not* need to solve the wave equation at the singularity. It does, however, seem physically reasonable to *impose* the condition that (3) be conserved. We now show that this reduces the countably infinite number of degrees of freedom inherent in the choice of a, b, c, d for a given basis to a single degree of freedom!

Consider two solutions $\phi, \tilde{\phi}$ which are smooth *in the trunk* and propagate them to the legs using (19) and (21). Then direct calculation at $t = 0^{\pm}$ yields

$$(\phi, \tilde{\phi})_{\text{legs}} = (\phi, \tilde{\phi})_{\text{trunk}} + i[\tilde{a}\varphi_R^*(0) - a^*\tilde{\varphi}_R(0)] + i[\tilde{b}\varphi_L^*(0) - b^*\tilde{\varphi}_L(0)]. \tag{23}$$

But from (19) we have

$$\varphi_R(0) = \varphi_L(0) = \tfrac{1}{2}[\varphi(0) + \varphi(\lambda)] \tag{24}$$

since smoothness of the trunk solution implies that $\varphi(-\lambda) = \varphi(\lambda)$. We therefore conclude that

$$a + b = 0. \tag{25}$$

A similar calculation assuming that $\phi_L, \phi_R, \tilde{\phi}_L, \tilde{\phi}_R$ are smooth *in the legs* and are propagated to the trunk using (22) yields

$$c + d = 0. \tag{26}$$

Finally, assuming that ϕ is smooth in the trunk and $\tilde{\phi}_L$, $\tilde{\phi}_R$ are smooth in the legs (note that now (24) does not hold), and equating the values of $(\phi, \tilde{\phi})$ in the trunk and in the legs yields

$$\tilde{c}\varphi^*(0) + \tilde{d}\varphi^*(\lambda) = a^*\tilde{\varphi}_R(0) + b^*\tilde{\varphi}_L(0). \tag{27}$$

Using (25), (26) we get

$$\frac{\varphi^*(0) - \varphi^*(\lambda)}{a^*} = \frac{\tilde{\varphi}_R(0) - \tilde{\varphi}_L(0)}{\tilde{c}}, \tag{28}$$

which finally yields (since (28) must hold for *all* $\phi, \tilde{\phi}_R, \tilde{\phi}_L$)

$$\begin{aligned} a[\phi] &= -b[\phi] = A[\varphi(0) - \varphi(\lambda)] \\ c[\phi_L, \phi_R] &= -d[\phi_L, \phi_R] = A^*[\varphi_R(0) - \varphi_L(0)], \end{aligned} \tag{29}$$

where A is an arbitrary constant, i.e. the amount of delta function added in each case to the (propagated) π data is proportional to the discontinuity in the (propogated) φ data. If we now further require that ϕ, ϕ^* should propagate to the complex conjugates of each other then A must be real.

One additional requirement on a propagation rule is that it should be invertible—if a propagated solution is propagated back again, it should agree with the original solution. Although the mappings (19), (21), (22) have only been defined for smooth functions, they can be extended to discontinuous functions in such a way that this condition is satisfied.

4. Energy density

We now turn to the evaluation of the energy density (15). Consider first (cf. (14)) the term

$$E_u = \frac{\langle 0_{\text{in}} | \phi_{,u}^{\text{out}} \phi_{,u}^{\text{out}} | 0_{\text{in}} \rangle}{\langle 0_{\text{in}} | 0_{\text{in}} \rangle} \tag{30}$$

evaluated in the right leg. Using (13) and the fact that each leg mode has support in one leg only, we obtain**

$$\phi_{,u}^{\text{out}}\Big|_{\substack{\text{right}\\\text{leg}}} = \sum_{l>0} (-il)[a_{lR} u_{lR} - a_{lR}^{\dagger} u_{lR}^{*}] + \frac{P_R}{2N_R \lambda}. \tag{31}$$

From (13) we obtain

$$\begin{aligned} a_{lR} &= (u_{lR}, \phi), \\ a_{lR}^{\dagger} &= -(u_{lR}^{*}, \phi), \\ P_R &= -i(\alpha_R, \phi). \end{aligned} \tag{32}$$

We use (7) and (32) to rewrite the expansion (31) in terms of trunk operators (whose expectation values in the trunk vacuum $|0_{\text{in}}\rangle$ are known) and Bogolubov coefficients which are given in the Appendix. A messy but straightforward calculation then yields

$$\begin{aligned} E_u = \sum_{\substack{n>0\\ n\,\text{odd}}} \sum_{m,m'=-\infty}^{\infty} \frac{1}{4\pi\lambda^2 n} \exp[-(2\pi i/\lambda)(m+m')u] \\ \times \left[\left(A + 1 + \frac{4m}{n-2m} \right)\left(A + 1 - \frac{4m'}{n+2m'} \right) + (A-1)^2 \right] + \frac{\pi}{2} \sum_{m>0} m. \end{aligned} \tag{33}$$

It turns out that this expression is valid in both the left and right legs. A similar calculation for E_v, defined as in (30) with u replaced by v, yields

$$\begin{aligned} E_v = \sum_{\substack{n>0\\ n\,\text{odd}}} \sum_{m,m'=-\infty}^{\infty} \frac{1}{4\pi\lambda^2 n} \exp[-(2\pi i/\lambda)(m+m')v] \\ \times \left[(A+1)^2 + \left(A - 1 + \frac{4m}{n+2m} \right)\left(A - 1 - \frac{4m'}{n-2m'} \right) \right] + \frac{\pi}{2} \sum_{m>0} m. \end{aligned} \tag{34}$$

**Note that for $k>0$, $u_k = u_k(u)$, while for $k<0$, $u_k = u_k(v)$. Compare footnote in Appendix A.

The expressions (33) and (34) are of course divergent; we have not yet implemented any renormalization procedure. Normal ordering the expression (30) with respect to leg operators will, among other things, remove the last sum in (33), (34) which has support everywhere and which is proportional to $\delta^2(0)$ (i.e. it is of the same order as is ordinarily removed by normal ordering in Minkowski space). However, the leading order divergence in (33) goes like

$$(A^2+1)\sum_{\substack{n>0\\ n\,\text{odd}}}\frac{1}{n}\sum_{m=-\infty}^{\infty}\exp(-2\pi imu/\lambda)\sum_{m'=-\infty}^{\infty}\exp(-2\pi im'u/\lambda). \tag{35}$$

The last two sums are each proportional to $\delta(u)$ and the first sum contributes a logarithmically divergent factor as well. This term is not altered by normal ordering. (34) has exactly the same divergence, with u replaced by v. Since (35) has support only along $u=0$, the divergences in E_u and E_v cannot cancel each other. The only way to remove them is therefore to choose the coefficient in front to be zero. But this is manifestly impossible as this coefficient is strictly positive. It is worth mentioning that this would still be true if the calculation were repeated with A allowed to be complex.

5. Discussion

The general argument given by Anderson and DeWitt (1986) for the presence of δ^2 terms (i.e. $\delta^2(u)$ and $\delta^2(v)$) in the energy density, leading to infinite total energy, does not take into account the possibility of the coefficients of these terms being zero. Furthermore, as the Minkowski divergence is proportional to $\delta^2(0)$ one could hope that renormalization might remove the δ^2 divergences anyway. Also we have shown that the shadow rule used by Anderson and DeWitt (1986) is unphysical because it does not lead to well-defined inner products, and that there exists only a one-parameter family of physically acceptable propagation rules.

However, calculating the energy density explicitly using these propagation rules shows that the leading divergence is worse then δ^2 due to the presence of an additional logarithmically divergent factor. This term cannot therefore be removed by normal ordering. Furthermore, the coefficient of this term is strictly positive, independent of the choice of parameter. Our calculations therefore tend to support the conclusions of Anderson and DeWitt (1986) that a change in the topology of spacetime would require an infinite amount of energy.

It is also of interest to consider the time-reversed problem, i.e. the expectation value of the "in" energy density in the "out" vacuum. Although the calculation is formally identical, the result turns out to involve terms like

$$\frac{\langle 0_{\text{out}}|Q_R^2|0_{\text{out}}\rangle}{\langle 0_{\text{out}}|0_{\text{out}}\rangle} \tag{36}$$

which are divergent. That these terms are absent in the calculation considered above can be traced directly to the fact that the trunk modes α_0, β_0 propagate smoothly to the legs, whereas the leg modes $\alpha_R, \beta_R, \alpha_L, \beta_L$ do *not* propagate smoothly to the trunk.

This raises the intriguing possibility that the divergences (36) might cancel the others, leading to an "entropy" law reminiscent of black hole physics: one cylinder could not split into two, but two could join together to form one. While we are pursuing this possibility, it seems so far that the Q^2 divergence also enters the energy density in a positive-definite way.

Bosonic string interactions can be thought of as massless scalar fields X^μ propagating on the trousers topology. Why, then, doesn't the conclusion of Anderson and DeWitt (1986) apply to string theory and prevent strings from interacting? The answer is that in string theory the scalar fields represent the physical coordinates embedding the string in a higher dimensional spacetime. This embedding is continuous. Physical strings only interact when they can, i.e. when they touch in the embedding space; our "strings" are forced to interact regardless of whether or not they "touch". In fact, if the conclusion of Anderson and DeWitt (1986) were shown to be false, it would probably imply the existence of *nonlocal* string interactions!

There is however one more possibility to be considered before accepting the conclusions of Anderson and DeWitt (1986). Throughout the calculation it has been *assumed* that the expansions (7) and (13) are complete (or at least equivalent) so that the Bogolubov transformations make sense. Furthermore, one can verify by direct calculation that the Bogolubov coefficients satisfy the appropriate relations, e.g. the analogue for a non-orthonormal basis of the usual relations (Birrell & Davies, 1982)

$$\alpha\alpha^\dagger - \beta\beta^\dagger = I; \quad \beta\alpha^T - \alpha\beta^T = 0,$$
$$\alpha^\dagger\alpha - \beta^T\beta^* = I; \quad \alpha^\dagger\beta - \beta^T\alpha^* = 0. \tag{37}$$

Contrary to popular belief, the fact that the Bogolubov coefficients satisfy (37) does *not* imply that the two sets of modes are complete (Dray and Manogue 1988). One must then ask if there are additional modes which have been overlooked.

Surprisingly, the answer appears to be yes. Consider the functions γ_0, γ defined by (A3), (A6), (A15) and assumed to be zero elsewhere. These functions are illustrated in figure 3. Since they are a linear combination of purely right- and left-moving functions, they satisfy the wave equation everywhere (except at the singularity). But γ_0 is orthogonal to all trunk modes, while γ is orthogonal to all leg modes! It thus appears that one must include γ_0 in the expansion (7), and γ in (13), and that they constitute extra degrees of freedom which must be quantized. The point is that unlike standard problems our modes are fundamentally discontinuous objects, and so it is *not* clear

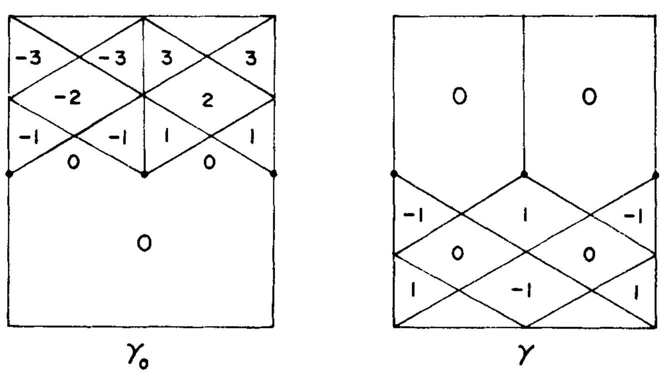

Figure 3. The new mode functions γ_0, γ defined by (A3), (A6), (A15). γ_0 is orthogonal to all trunk modes, while γ is orthogonal to all leg modes.

which space of functions the field ϕ belongs to. We are actively pursuing these questions.

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Appendix

We first give the results of applying the propagation rules discussed in § 3 to the trunk and leg modes discussed in § 2, thus obtaining two sets of mode functions which are defined on all of spacetime. We then tabulate the inner products between the two sets of mode functions. *Assuming* that the two sets are equivalent these inner products are just the Bogolubov transformations. (This assumption is discussed in § 5).

Applying the propagation rule given by (19), (21), (29) to the trunk modes U_k, V_k for $k=(\pi n/\lambda)$, n even and to α_0, β_0 shows that these modes are unchanged.** However, applied to U_k, V_k for $k=(\pi n/\lambda)$, *n odd* yields in the right leg ($t \geqslant 0$, $x \in [0,\lambda]$)

$$U_k = \frac{1}{(4|k|\lambda)^{\frac{1}{2}}}\exp(-iku)\sum_{n=-1}^{\infty}(-1)^n[\theta(u-n\lambda+\lambda)-\theta(u-n\lambda)] + \frac{(A+1)}{(4|k|\lambda)^{\frac{1}{2}}}\gamma_0, \tag{A1}$$

$$V_k = \frac{1}{(4|k|\lambda)^{\frac{1}{2}}}\exp(-ikv)\sum_{n=-1}^{\infty}(-1)^n[\theta(v-n\lambda)-\theta(v-n\lambda-\lambda)] + \frac{(A-1)}{(4|k|\lambda)^{\frac{1}{2}}}\gamma_0, \tag{A2}$$

where

$$\gamma_0 = \sum_{n=0}^{\infty}[\theta(u-n\lambda)+\theta(v-n\lambda-\lambda)]. \tag{A3}$$

and in the left leg ($t \geqslant 0$, $x \in [-\lambda, 0]$)

$$U_k = \frac{1}{(4|k|\lambda)^{\frac{1}{2}}}\exp(-iku)\sum_{n=-1}^{\infty}(-1)^n[\theta(u-n\lambda)-\theta(u-n\lambda-\lambda)] + \frac{(A+1)}{(4|k|\lambda)^{\frac{1}{2}}}\gamma_0, \tag{A4}$$

**For ease of calculation we have introduced the right- and left-moving trunk modes $U_k(u)=u_k$, $V_k(v)=u_{-k}$ for $k>0$. We will also use the notation $U_{-k}=U_k^*, V_{-k}=V_k^*$. Analogous definitions hold in the legs.

$$V_k = \frac{1}{(4|k|\lambda)^{\frac{1}{2}}} \exp(-ikv) \sum_{n=-1}^{\infty} (-1)^n [\theta(v - n\lambda + \lambda) - \theta(v - n\lambda)] + \frac{(A-1)}{(4|k|\lambda)^{\frac{1}{2}}} \gamma_0, \tag{A5}$$

where here

$$\gamma_0 = -\sum_{n=0}^{\infty} [\theta(u - n\lambda - \lambda) + \theta(v - n\lambda)]. \tag{A6}$$

The first (γ_0 independent) part of each of (A1), (A2), (A4), (A5) corresponds to the shadow rule of Anderson and DeWitt and takes right (left) moving solutions in the trunk to right (left) moving solutions in the legs. Note that no value of A yields the shadow rule for *both* right and left moving modes. This means that the shadow rule does *not* have the form required by (29) and therefore does not conserve inner products and is unphysical. Physically propagated modes are no longer purely right (left) moving.

Applying the propagation rule given by (22), (29) to the leg modes yields in the trunk ($t < 0$, $x \in [-\lambda, \lambda]$)

$$U_{lR} = \frac{1}{(2|l|\lambda)^{\frac{1}{2}}} \exp(-ilu) \sum_{n=-1}^{\infty} [\theta(-u - 2n\lambda) - \theta(-u - 2n\lambda - \lambda)] + \frac{(A-1)}{2(2|l|\lambda)^{\frac{1}{2}}} \gamma, \tag{A7}$$

$$U_{lL} = \frac{1}{(2|l|\lambda)^{\frac{1}{2}}} \exp(-ilu) \sum_{n=-1}^{\infty} [\theta(-u - 2n\lambda + \lambda) - \theta(-u - 2n\lambda)] - \frac{(A-1)}{2(2|l|\lambda)^{\frac{1}{2}}} \gamma, \tag{A8}$$

$$V_{lR} = \frac{1}{(2|l|\lambda)^{\frac{1}{2}}} \exp(-ilv) \sum_{n=-1}^{\infty} [\theta(-v - 2n\lambda + \lambda) - \theta(-v - 2n\lambda)] + \frac{(A+1)}{2(2|l|\lambda)^{\frac{1}{2}}} \gamma, \tag{A9}$$

$$V_{lL} = \frac{1}{(2|l|\lambda)^{\frac{1}{2}}} \exp(-ilv) \sum_{n=-1}^{\infty} [\theta(-v - 2n\lambda) - \theta(-v - 2n\lambda - \lambda)] - \frac{(A+1)}{2(2|l|\lambda)^{\frac{1}{2}}} \gamma, \tag{A10}$$

$$\alpha_R = \frac{N_R}{2} \sum_{n=-1}^{\infty} \left[\begin{matrix} \theta(-u - 2n\lambda) - \theta(-u - 2n\lambda - \lambda) \\ + \theta(-v - 2n\lambda + \lambda) - \theta(-v - 2n\lambda) \end{matrix} \right] + \frac{1}{2} A N_R \gamma, \tag{A11}$$

$$\alpha_L = \frac{N_L}{2}\sum_{n=-1}^{\infty}\begin{bmatrix} \theta(-u-2n\lambda+\lambda)-\theta(-u-2n\lambda) \\ +\theta(-v-2n\lambda)-\theta(-v-2n\lambda-\lambda) \end{bmatrix} - \frac{1}{2}AN_L\gamma, \quad \text{(A12)}$$

$$\beta_R = \frac{v}{2N_R\lambda} + \frac{1}{2N_R\lambda}\sum_{n=0}^{\infty}(-1)^n\begin{bmatrix} (u+n\lambda)\theta(-u-n\lambda) \\ -(v+n\lambda)\theta(-v-n\lambda) \end{bmatrix} \quad \text{(A13)}$$

$$\beta_L = \frac{u}{2N_R\lambda} + \frac{1}{2N_R\lambda}\sum_{n=0}^{\infty}(-1)^n\begin{bmatrix} (v+n\lambda)\theta(-v-n\lambda) \\ -(u+n\lambda)\theta(-u-n\lambda) \end{bmatrix} \quad \text{(A14)}$$

where

$$\gamma = \sum_{n=0}^{\infty}\begin{bmatrix} \theta(-u-2n\lambda)-\theta(-u-2n\lambda-\lambda) \\ +\theta(-v-2n\lambda)-\theta(-v-2n\lambda-\lambda) \end{bmatrix} - 1. \quad \text{(A15)}$$

Again the first (γ independent) part of (A7)–(A14) corresponds to the shadow rule of Anderson and DeWitt. No choice of A yields the shadow rule for all leg modes.

Now we give the inner products between the trunk and leg modes. By construction, these do not depend on where they are evaluated. First consider the case $k = n\pi/\lambda$ with n even. We have

$$\left.\begin{aligned} (U_k, U_{lR}) &= \frac{1}{\sqrt{2}}\delta_{kl} = (V_k, V_{lR}) \\ (U_{-k}, U_{-lR}) &= -\frac{1}{\sqrt{2}}\delta_{kl} = (V_{-k}, V_{-lR}), \end{aligned}\right\} \quad (k > 0) \quad \text{(A16)}$$

$$(\alpha_0, \beta_R) = \frac{N}{N_R}i$$

$$(\beta_0, \alpha_R) = -\frac{N_R}{2N}i \quad \text{(A17)}$$

with all others zero; the same results hold for the left leg modes with R replaced by L. Now consider the case $k = \pi n/\lambda$ with n odd. We have

$$(U_k, U_{lR}) = \frac{-i}{(2|kl|\lambda^2)^{\frac{1}{2}}}\left(A + \frac{k+l}{k-l}\right),$$

$$(V_k, V_{lR}) = \frac{-i}{(2|kl|\lambda^2)^{\frac{1}{2}}}\left(A - \frac{k+l}{k-l}\right), \quad \text{(A18)}$$

$$(U_k, V_{lR}) = \frac{-i}{(2|kl|\lambda^2)^{\frac{1}{2}}}(A+1),$$

$$(V_k, U_{lR}) = \frac{-i}{(2|kl|\lambda^2)^{\frac{1}{2}}}(A-1),$$

$$
\begin{aligned}
(U_k, \alpha_R) &= \frac{-iN_R}{(|k|\lambda)^{\frac{1}{2}}}(A+1) \\
(V_k, \alpha_R) &= \frac{-iN_R}{(|k|\lambda)^{\frac{1}{2}}}(A-1), \\
(U_k, \beta_R) &= \frac{1}{N_R k(|k|\lambda^3)^{\frac{1}{2}}}, \\
(V_k, \beta_R) &= \frac{-1}{N_R k(|k|\lambda^3)^{\frac{1}{2}}},
\end{aligned}
\tag{A19}
$$

where k, l take on both positive and negative values.** For the corresponding results in the left leg, replace R by L and multiply by -1.

References

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