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A QUANTIFIED VERSION OF THE DIRICHLET-JORDAN TEST IN L^1-NORM

F. MÓRICZ(*) - A. H. SIDDIQI

We introduce the notion of bounded variation in the sense of L^1-norm for periodic functions and prove a version of the classical Dirichlet-Jordan test for the convergence of Fourier series in L^1-norm. We also give an estimate of the rate of convergence.

1. Introduction.

The classical Dirichlet-Jordan test asserts that the Fourier series of a 2π-periodic function f of bounded variation on $[-\pi, \pi]$ converges at each point. Bojanic [1] gave a quantified version of this by estimating the rate of convergence.

It is well known that the Dirichlet-Jordan test may be generalized by weakening the requirement that f is of bounded variation. (See, e.g., [2,3,4].)

In this note, we prove a sufficient condition for the convergence of Fourier series in L^1-norm. In particular, we consider functions that are of bounded variation in the sense of L^1-norm and obtain a version of the Dirichlet-Jordan test for the convergence in L^1-norm.

2. Definitions.

Let f be a real-valued 2π-periodic function, integrable on $[-\pi, \pi]$ in Lebesgue's sense, in sign: $f \in L^1_{2\pi}$. We say that f is of

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bounded variation (on $[-\pi, \pi]$) in L^1-norm if there exists a positive constant C such that

$$\sum_{k=1}^{n} \int_a^b |f(x+t_k) - f(x+t_{k-1})| dx \leq C$$

for all subdivisions $-\pi = t_0 < t_1 < t_2 < \ldots < t_n = \pi$, $n \geq 1$. The smallest such C is called the total variation of f in L^1-norm, denoted by $V_1(f)$.

Set

$$\phi_x(t) := f(x+t) + f(x-t) - 2f(x). \tag{2.1}$$

Given a subinterval $I \subset [-\pi, \pi]$, define

$$\Omega(\phi, I) := \sup \left\{ \int_{-\pi}^{\pi} |\phi_x(t) - \phi_x(t')| dx : t, t' \in I \right\}. \tag{2.2}$$

It is plain that

$$\Omega(\phi, I) \leq 2\omega_1(f, |I|), \tag{2.3}$$

where $|I|$ denotes the length of the interval I and

$$\omega_1(f, \delta) := \sup \left\{ \int_{-\pi}^{\pi} |f(x+t) - f(x)| dx : |t| \leq \delta \right\}, \quad \delta > 0,$$

is the integral modulus of continuity of f.

We note that $\Omega(\phi, I)$ may be considered as a kind of local integral modulus of continuity of $\phi_x(t)$, or perhaps as a kind of oscillation of $\phi_x(t)$ in L^1-norm restricted to the interval I in a certain sense. Later on, we shall see that this quantity taken over appropriate subintervals I_{kn} of $[-\pi, \pi]$ controls the rate of convergence of the Fourier series of f in L^1-norm.

3. Results.

Denote by $s_n(f, x)$ the nth partial sum of the Fourier series of a function $f \in L^1_{2\pi}$. Set

$$I_{kn} := \left[\frac{k\pi}{n+1}, \frac{(k+1)\pi}{n+1} \right], \quad k = 0, 1, \ldots, n \text{ and } n \geq 1. \tag{3.1}$$

Our main result gives a sufficient condition for the convergence of the Fourier series of f in L^1-norm.

THEOREM *If $f \in L^1_{2\pi}$ and n is a positive integer, then*

$$\int_{-\pi}^{\pi} |s_n(f,x) - f(x)| dx \le \left(1 + \frac{1}{\pi}\right) \sum_{k=0}^{n} \frac{1}{k+1} \Omega(\phi, I_{kn}), \tag{3.2}$$

where $\Omega(\phi, I)$ is defined in (2.1) and (2.2).

COROLLARY 1. (Dini-Lipschitz test in L^1-norm). *If $f \in L^1_{2\pi}$ and $n \ge 1$, then*

$$\int_{-\pi}^{\pi} |s_n(f,x) - f(x)| dx \le 2\left(1 + \frac{1}{\pi}\right) \omega_1\left(f, \frac{\pi}{n+1}\right) \ln(n+2), \tag{3.3}$$

where $\omega_1(f, \delta)$ is the integral modulus of continuity of f.

COROLLARY 2. (Dirichlet-Jordan test in L^1-norm). *If f is of bounded variation in L^1-norm, then*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |s_n(f,x) - f(x)| dx = 0. \tag{3.4}$$

4. Proofs.

We start with the well-known representation formula

$$s_n(f,x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \phi_x(t) D_n(t) dt, \tag{4.1}$$

where $\phi_x(t)$ is defined in (2.1) and

$$D_n(t) := \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin(n+1/2)t}{2\sin t/2}, \quad n \ge 0,$$

is the Dirichlet kernel. We shall make use of the following inequality (see [1]):

$$\left| \int_x^{\pi} D_n(t) dt \right| \le \frac{\pi}{(n+1)x}, \quad 0 < x \le \pi \text{ and } n \ge 0. \tag{4.2}$$

Proof of the Theorem. The idea of the proof goes back to that of [2, Lemma 1]. Let $n \geq 1$ be fixed. Set

$$\theta_{kn} := \frac{k\pi}{n+1}, \ k = 0, 1, \ldots, n+1.$$

Then $I_{kn} = [\theta_{kn}, \theta_{k+1,n}]$ (cf.(3.1)). By (4.1), we estimate as follows

$$\begin{aligned}
\int_{-\pi}^{\pi} |s_n(f,x) - f(x)| dx &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{I_{0n}} |\phi_x(t) D_n(t)| dt dx \\
&+ \sum_{k=1}^{n} \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{I_{kn}} |\{\phi_x(t) - \phi_x(\theta_{kn})\} D_n(t)| dt dx \\
&+ \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \sum_{k=1}^{n} \phi_x(\theta_{kn}) \int_{I_{kn}} D_n(t) dt \right| dx \\
&= A_n + B_n + C_n,
\end{aligned} \tag{4.3}$$

say.

First, we use the obvious inequality

$$|D_n(t)| \leq n + 1/2$$

and the fact that $\phi_x(0) = 0$ in order to get

$$A_n \leq \frac{1}{\pi} \int_{I_{0n}} |D_n(t)| dt \int_{-\pi}^{\pi} |\phi_x(t) - \phi_x(0)| dx \leq \Omega(\phi, I_{0n}). \tag{4.4}$$

Second, by the inequality

$$\sin t/2 \geq t/\pi, \ 0 \leq t \leq \pi,$$

we obtain

$$\begin{aligned}
B_n &\leq \sum_{k=1}^{n} \frac{1}{\pi} \int_{I_{kn}} |D_n(t)| dt \int_{-\pi}^{\pi} |\phi_x(t) - \phi_x(\theta_{kn})| dx \\
&\leq \sum_{k=1}^{n} \Omega(\phi, I_{kn}) \int_{I_{kn}} \frac{dt}{2t} \\
&\leq \sum_{k=1}^{N} \frac{1}{2k} \Omega(\phi, I_{kn}).
\end{aligned} \tag{4.5}$$

Third, to estimate C_n we set

$$R_{kn} := \int_{\theta_{kn}}^{\pi} D_n(t)dt, \; k = 0, 1, \dots, n+1.$$

Clearly, $R_{n+1,n} = 0$, while by (4.2),

$$|R_{kn}| \leq \frac{1}{k}, \qquad k = 0, 1, \dots, n.$$

A summation by parts gives

$$\begin{aligned}\sum_{k=1}^{n} \phi_x(\theta_{kn}) \int_{I_{kn}} D_n(t)dt &= \sum_{k=1}^{n} \phi_x(\theta_{kn})(R_{kn} - R_{k+1,n}) \\ &= \sum_{k=1}^{n} \{\phi_x(\theta_{kn}) - \phi_x(\theta_{k-1,n})\} R_{kn}.\end{aligned}$$

Thus, we conclude

$$\begin{aligned}C_n &\leq \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{k} \int_{-\pi}^{\pi} |\phi_x(\theta_{kn}) - \phi_x(\theta_{k-1,n})| dx \\ &\leq \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{k} \Omega(\phi, I_{k-1,n}).\end{aligned} \tag{4.6}$$

Combining (4.3) - (4.6) yields (3.2).

Proof of Corollary 1. Inequality (3.3) is an obvious consequence of (3.2) and (2.3).

Proof of Corollary 2. As is well known, for any function $f \in L^1_{2\pi}$

$$\lim_{n \to \infty} \omega_1 \left(f, \frac{\pi}{n+1} \right) = 0.$$

By (2.3), for a fixed $k_0 \geq 1$ we also have

$$\lim_{n \to \infty} \sum_{k=1}^{k_0} \frac{1}{k+1} \Omega(\phi, I_{kn}) = 0.$$

On the other hand, by definition,

$$\sum_{k=k_0+1}^{n} \frac{1}{k+1} \Omega(\phi, I_{kn}) \leq \frac{2}{k_0+2} V_1(f).$$

The right-hand side here can be made as small as we want by choosing k_0 large enough. This completes the proof of (3.4).

REFERENCES

[1] Bojanic R., *An estimate of the rate of convergence for Fourier series of functions of bounded variation,* Publ. Inst. Math. (Beograd) (N.S) **26**, No. **40** (1979), 57-60.

[2] Bonjanic R., Waterman D., *On the rate of convergence of Fourier series of functions of generalized bounded variation,* Akad. Nauka Umjet. Bosne Hercegov. Rad. Odjelj. Prirod. Mat. Nauka **22** (1983), 5-11.

[3] Salem R., *Essais sur les séries trigonométriques,* Actualité Sci. Ind. No. **862**, Paris, Herman, 1940.

[4] Waterman D., *On convergence of Fourier series of functions of generalized bounded variation,* Studia Math. **44** (1972), 107-117.

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F. Móricz
University of Szeged
Bolyai Institute
Aradi vértanúk tere
Szeged 6720, UNGARY

A.H. Siddiqi
Aligarh Muslim University
Department of Mathematics
Aligarh 202002
Uttar Pradesh, INDIA