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PROPERTIES OF THE BASIC COHOMOLOGY OF TRANSVERSELY KÄHLER FOLIATIONS

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In this paper we study the complex basic cohomology of transversely Hermitian foliations. We use the methods developed in [7] and prove that for transversely Kähler foliations the foliated version of the Frölicher spectral sequence collapses at the first level and that the minimal model for the complex basic cohomology is formal. To stress that these properties are particular to transversely Kähler foliations we construct examples of transversely Hermitian foliations for which these theorems do not hold.

1. Basic definitions and properties.

Let $\mathcal{F}$ be a folaition on a manifold M. The foliation $\mathcal{F}$ is given by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ modelled on a manifold N_0, i.e.

i) $\{U_i\}$ is an open covering of M,

ii) $f_i : U_i \to N_0$ are submersions with connected fibres defining $\mathcal{F}$,

iii) g_{ij} are local diffeomorphisms of N_0 and $g_{ij} \circ f_j = f_i$ on $U_i \cap U_j$.

The manifold $N = \coprod f_i(U_i)$ we call the transverse manifold of $\mathcal{F}$ associated to the cocycle $\mathcal{U}$ and the pseudogroup H generated by g_{ij}

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the holonomy pseudogroup (representative) on the transverse manifold N.

The foliation is:

a) *transversely holomorphic* if on N there exists a complex structure of which H is a pseudogroup of local holomorphic transformations;

b) *transversely Hermitian* if it is transversely holomorphic and there exists an H-invariant Hermitian metric on N;

c) *transversely Kählerian* if on N there exists an H-invariant Kähler structure.

For the rest of this note we assume that the foliation $\mathcal{F}$ is transversely holomorphic, of complex codimension q and that the manifold M is compact.

Basic forms on the foliated manifold $(M, \mathcal{F})$ are in one-to-one correspondence with H-invariant forms on the transverse manifold N. Moreover, basic k-forms can be considered as foliated sections of $\Lambda^k N(M, \mathcal{F})^*$, the k^{th} exterior product of the conormal bundle of $\mathcal{F}$. If the foliation $\mathcal{F}$ is transversely holomorphic the normal bundle $N(M, \mathcal{F})$ of $\mathcal{F}$ has a complex structure corresponding to the complex structure of N. Therefore any complex valued basic k-form can be represented as a sum of the k-forms of pure type (r, s) corresponding to the decomposition of k-forms on the complex manifold N. We can obtain the same decomposition by looking at the decomposition of sections of the complex bundle $\Lambda^K_{\mathbb{C}} N(M, \mathcal{F})^*$, i.e. a basic k-form α is of pure type (r, s) if for any point of M there exists an adapted chart $(x_1, \dots, x_{n-2q}, z_1, \dots, z_q)$ such that

$$\alpha = \sum f_{IJ} dz_{i_1} \wedge \dots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_s}.$$

where $1 \le i_1 < \dots < i_r \le q$, $1 \le j_1 < \dots < j_s \le q$, $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$.

Let us denote by $\Lambda^k_{\mathbb{C}}(M, \mathcal{F})$ the space of complex valued basic k-forms on the foliated manifold $(M, \mathcal{F})$, and by $\Lambda^{r,s}_{\mathbb{C}}(M, \mathcal{F})$ the space of complex valued basic forms of pure type (r, s). Then

$$\Lambda^k_{\mathbb{C}}(M, \mathcal{F}) = \sum_{r+s=k} \Lambda^{r,s}_{\mathbb{C}}(M, \mathcal{F}),$$

for short $\Lambda^k = \sum_{r+s=k} \Lambda^{r,s}$.

The exterior derivative $d : \Lambda^k_{\mathbb{C}}(M,\mathcal{F}) \to \Lambda^{k+1}_{\mathbb{C}}(M,\mathcal{F})$ decomposes itself into two components $d = \partial + \bar{\partial}$, where ∂ is of bidegree (1,0) and $\bar{\partial}$ is of bidegree (0,1), i.e.

$$\partial : \Lambda^{r,s} \to \Lambda^{r+1,s} \quad \text{and} \quad \bar{\partial} : \Lambda^{r,s} \to \Lambda^{r,s+1}.$$

Now we are going to recall some results of A. El Kacimi (cf. [7]), concerning transversely Hermitian and transversely Kähler foliations.

Let us assume that $\mathcal{F}$ is a transversely Hermitian foliation. The operator

$$* : \Lambda^k(M,\mathcal{F}) \to \Lambda^{2q-k}(M,\mathcal{F})$$

defined in [8] via the transverse part of the bundle-like metric of $\mathcal{F}$ extends to an operator

$$\bar{*} : \Lambda^k_{\mathbb{C}}(M,\mathcal{F}) \to \Lambda^{2q-k}_{\mathbb{C}}(M,\mathcal{F}).$$

Let $B(M,SO(2q),\pi,\mathcal{F})$ be the bundle of transverse orthonormal frames of $\mathcal{F}$. Let κ be the form defining the volume form on each fibre of the bundle $B(M,SO(2q),\pi,F)$. The lifted foliation $\mathcal{F}_1$ on the total space B of $B(M,SO(2q),\pi,\mathcal{F})$ is transversely parallelisable and therefore the closures of the leaves are the fibres of the basic fibration $p : B \to W$, cf. [12].

We define the scalar product on $\Lambda^*_{\mathbb{C}}(M,\mathcal{F}) = \sum_{k=1}^{2q} \Lambda^k_{\mathbb{C}}(M,\mathcal{F})$ as

$$\begin{aligned} &\langle \alpha,\beta\rangle = 0 \text{ if } \alpha \in \Lambda^k_{\mathbb{C}}(M,\mathcal{F}),\ \beta \in \Lambda^l_{\mathbb{C}}(M,\mathcal{F}) \text{ and } k \neq l,\\ (1)\qquad &\langle \alpha,\beta\rangle = \int_W \bar{I}(\pi^*(\alpha \wedge \bar{*}\beta) \wedge \kappa) \text{ for } \alpha,\beta \in \Lambda^r_{\mathbb{C}}(M,\mathcal{F}) \end{aligned}$$

where $\bar{I}$ is the integration along the fibres of the basic fibration.

The operator $\delta : \Lambda^k_{\mathbb{C}}(M,\mathcal{F}) \to \Lambda^{k-1}_{\mathbb{C}}(M,\mathcal{F})$ defined as $\delta = \bar{*}^{-1} d \bar{*}$ is the adjoint operator of d relative to the scalar product $\langle , \rangle$.

Similarly, the «foliated» Laplacian operator is defined by $\Delta = d\delta + \delta d$; it is, in this case, an auto-adjoint foliated (transversely) elliptic operator.

Let us consider the following differential complex

$$0 \to \Lambda^{r,0} \xrightarrow{\bar\partial} \Lambda^{r,1} \longrightarrow \ldots \xrightarrow{\bar\partial} \Lambda^{r,q} \to 0. \tag{2}$$

We denote its cohomology $H^{r,*}(M, \mathcal{F}) = \sum_{s=0}^{q} H^{r,s}(M, \mathcal{F})$ and we call it the basic Dolbeault cohomology of the foliation $\mathcal{F}$.

The operator $\bar{*}$ induces isomorphisms $\bar{*} : \Lambda^{r,s} \to \Lambda^{q-r,q-s}$. Let us put $\bar\delta = -\bar{*}\bar\partial\bar{*}$. Then the operator $\bar\delta$ is the adjoint $\bar\partial$ relative to the inner product $\langle , \rangle$ defined in (1). Moreover, the operator $\Delta'' = \bar\partial\bar\delta + \bar\delta\bar\partial$ is an auto-adjoint foliated (transversely) elliptic operator.

Now, let $\mathcal{F}$ be transversely Kähler. The Kähler form of N defines a basic (1,1)-form ω on $(M, \mathcal{F})$ which we call the transverse Kähler form of the foliation $\mathcal{F}$. This form allows to define the following operator:

$$L : \Lambda^k_{\mathbb{C}}(M, \mathcal{F}) \to \Lambda^{k+2}_{\mathbb{C}}(M, \mathcal{F}),\ L\alpha = \alpha \wedge \omega,$$

and its adjoint $\Lambda = -\bar{*}L\bar{*}$.

Then for a transversely Kähler foliation $\mathcal{F}$ on a compact manifold, the following relations hold:

$$\begin{cases} \Lambda\partial - \partial\Lambda = -\sqrt{-1}\bar\delta, \\ \Lambda\bar\partial - \bar\partial\Lambda = \sqrt{-1}\delta, \\ \partial\bar\delta + \bar\delta\partial = \bar\partial\delta + \delta\bar\partial = 0, \\ \Delta = 2\Delta'', \\ \Delta L = L\Delta,\ \Delta\Lambda = \Lambda\Delta. \end{cases} \tag{3}$$

These identities lead to the following theorem:

THEOREM 1. ([7]) *Let $\mathcal{F}$ be a transversely Kähler foliation on a compact manifold M. If $\mathcal{F}$ is homologically oriented, then*

i) a basic k-form $\alpha = \sum_{r+s=k} \alpha_{r,s}$, $\alpha_{r,s} \in \Lambda^{r,s}$, is harmonic if and only if the forms $\alpha_{r,s}$ are harmonic; thus

$$H^k_{\mathbb{C}}(M, \mathcal{F}) \simeq \sum_{r+s=k} H^{r,s}(M, \mathcal{F}).$$

ii) the conjugation induces isomorphisms $H^{r,s}(M,\mathcal{F}) \simeq H^{s,r}(M,\mathcal{F})$.

iii) for any $0 \leq r \leq q$, *the form* ω^r *is harmonic, thus* $H^{r,r}(M,\mathcal{F}) \neq 0$.

2. The basic Frölicher spectral sequence.

Let us considere the complex $\left(\Lambda = \sum_{r,s} \Lambda^{r,s}, d\right)$ of complex valued basic forms of the foliated manifold $(M,\mathcal{F})$. We can filtrate it as follows

$$F^k\Lambda = \sum_{r \geq k} \Lambda^{r,*}.$$

This filtration is compatible with the bigradation of the complex. The spectral sequence associated to this filtration is called the basic Frölicher spectral sequence of the transversely holomorphic foliation $\mathcal{F}$, cf. [9]. It can be easily shown that it converges to the complex basic cohomology of $(M,\mathcal{F})$.

The terms $E_1^{r,s}$ of this spectral sequence are the cohomology groups of the differential complex (2), i.e. $E_1^{r,s} = H^{r,s}(M,\mathcal{F})$, the (r,s)-th basic Dolbeault cohomology group.

When the foliation $\mathcal{F}$ is homologically oriented and transversely Kähler, Theorem 1 ensures that $E_1^{r,s} \simeq \mathcal{H}^{r,s}, \mathcal{H}^{*,*}$ - the complex of complex valued basic harmonic forms; therefore the differential operator $d_1 : E_1^{r,s} \to E_1^{r+1,s}$ vanishes and the spectral sequence collapses at the level E_1.

THEOREM 2. *Let* $\mathcal{F}$ *be a homologically oriented transversely Kähler foliation on a compact manifold* M. *The basic Frölicher spectral sequence of* $\mathcal{F}$ *collapses at the first term, i.e.* $E_1 \simeq E_2 \simeq \ldots \simeq E_\infty$.

Now, we shall present examples of homologically oriented transversely Hermitian foliations on compact nilmanifolds whose basic Frölicher spectral sequence behaves in a markedly different manner. The examples are based on the examples of [3] of compact complex nilmanifolds whose Frölicher spectral sequences have similar properties. The following is the general construction of which our examples are particular cases.

Let N be a simply connected nilpotent Lie group and Γ a torsionfree, finitely generated subgroup of N. Then according to [11], or [14], there exist a simply connected nilpotent Lie group U containing Γ as a uniform subgroup and a homomorphism $u : U \to N$ such that u is the identity on Γ (if we identify the subgroups of U and N isomorphic to Γ). The homomorphism u is a surjective submersion with connected fibres since both manifolds U and N are contractible. The foliation defined by the submersion u is Γ-invariant and therefore it projects to a foliation $\mathcal{F}(U,\Gamma,u)$ on the compact manifold $M(\Gamma) = \Gamma\backslash U$. The foliation $\mathcal{F}(U,\Gamma,u)$ is an (N,Γ)-structure, a developable one, and the submersion u is its developing mapping. Therefore, foliated geometric structures on $(M(\Gamma),\mathcal{F}(U,\Gamma,u))$ correspond bijectively to Γ-invariant ones on N. In fact, any foliated geometric structure on $(M(\Gamma),\mathcal{F}(U,\Gamma,u))$ lifts to a Γ-invariant foliated structure on U. This one, in its turn, defines a Γ-invariant structure on N which projects to a geometric structure on $E(\Gamma_0) = \Gamma_0\backslash N$.

All our examples are homologically oriented. In fact the basic cohomology $H^*(M,\mathcal{F})$ is the cohomology $H^*(N,\Gamma)$ of the Γ-invariant forms on the nilpotent Lie group N. The holonomy groups Γ of the examples presented in this paper admit cocompact subgroup Γ_0 of N. Thus we have the following:

$$A^*(N,N) \subset A^*(N,\Gamma) \subset A^*(N,\Gamma_0).$$

The Nomizu theorem, cf. [13], ensures that $H^*(N,\Gamma_0) \simeq H^*(N,N)$. The cohomology $H^*(N,\Gamma_0)$ is the cohomology ring of the oriented compact manifold $\Gamma_0\backslash N$, thus $H^q(N,\Gamma_0)\neq 0$. Therefore, as the natural inclusion

$$A^*(N,\Gamma) \hookrightarrow A^*(N,\Gamma_0)$$

induces a surjective mapping on the level of cohomology $H^q(N,\Gamma)\neq 0$, which means that the foliation is homologically oriented, cf. Corollary 1 of [15].

EXAMPLE 1. A transversely Hermitian foliation for which $E_1 \not\simeq E_2$.

Let us consider the 3-dimensional complex Heisenberg group $N = (\mathbb{C}^3, *)$ where

$$(a_1,a_2,a_3) * (b_1,b_2,b_3) = (a_1+b_1, a_2+b_2, a_3+b_3+a_1b_2).$$

The compact complex nilmanifold $G^3\backslash\mathbb{C}^3$, where G^3 is the lattice of Gauss integers, is the well known Iwasawa manifold for which $E_1 \not\simeq E_2$, cf. [3,10].

Let Γ_3 be the following finitely generated subgroup of $(\mathbb{C}^3, *)$:

$$\{(n_1 + sm_1, n_2, n_3 + sm_3) | n_i, m_i \text{ Gauss integers}, s \notin \mathbb{Q}\}.$$

The subgroup Γ_3 can be considered as a uniform subgroup of $U = (\mathbb{C}^5, \square)$ with the following group operation:

$$(a_1, \dots, a_5)\square(b_1, \dots, b_5) = (a_i + b_i, a_4 + b_4 + a_1 b_3, a_5 + b_5 + a_2 b_3).$$

The submersion $u : U \to N$ is given by the correspondence

$$(a_1, a_2, a_3, a_4, a_5) \mapsto (a_1 + sa_2, a_3, a_4 + sa_5).$$

The foliation $\mathcal{F}(U, \Gamma_3, u)$ is a holomophic and transversely symplectic foliation of complex codimension 3 on a compact complex nilmanifold $\Gamma_3\backslash U$ of complex dimension 5, and it cannot be made transversely Kähler either, cf. [5]. Its basic forms are in one-to-one correspondence with the Γ_3-invariant forms on $(\mathbb{C}^3, *)$. The same considerations as in [3,10] ensure that the basic Frölicher spectral sequence of $\mathcal{F}(U, \Gamma_3, u)$ has the property of being $E_1 \not\simeq E_2$.

EXAMPLE 2. A transversely Hermitian foliation for which $E_2 \not\simeq E_3$.

Let us consider the group $N = (\mathbb{C}^4, *)$ with the following group operation:

$$(a_1, a_2, a_3, a_4)*(b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2,$$
$$a_3 + b_3 + (a_2 + \bar{a}_2)b_1, a_4 + b_4 - \bar{a}_1 b_2).$$

This is a real nilpotent Lie group with a left invariant complex structure. The compact complex nilmanifold $G^4\backslash\mathbb{C}^4$, where G^4 is the lattice of Gauss integers, has the required property, cf. [3].

Now, let us consider the following finitely generated subgroup Γ_4 of $(\mathbb{C}^4, *)$:

$$\{(n_1, n_2 + sm_2, n_3 + sm_3, n_4 + sm_4) | n_i, m_i \text{ Gauss integers}, s \notin \mathbb{Q}\}.$$

Γ_4 can be embedded as a uniform subgroup of the group $U = (\mathbb{C}^7, \square)$ with the following group operation:

$$\begin{aligned}(a_1, \ldots, a_7)\square(b_1, \ldots, b_7) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + (a_2 + \bar{a}_2)b_1, \\ &\quad a_5 + b_5 + (a_3 + \bar{a}_3)b_1, a_6 + b_6 - \bar{a}_1 b_2, a_7 + b_7 - \bar{a}_1 b_3).\end{aligned}$$

It is a real nilpotent Lie group with a left invariant complex structure. The Γ-equivariant submersion $u : U \to N$ is given by the formula:

$$(a_1, \ldots, a_7) \mapsto (a_1, a_2 + sa_3, a_4 + sa_5, a_6 + sa_7)$$

The foliation $\mathcal{F}(U, \Gamma_4, u)$ is tranversely Hermitian. Its basic forms are in one-to-one correspondence with the Γ_4-invariant forms on $(\mathbb{C}^4, *)$. The same considerations as in [3] ensure that the basic Frölicher spectral sequence of $\mathcal{F}(U, \Gamma_4, u)$ has the property $E_2 \not\simeq E_3$.

EXAMPLE 3. A transversely Hermitian foliation for which $E_3 \not\simeq E_4$.

Let us consider the group $N = (\mathbb{C}^6, *)$ with the following group operation:

$$\begin{aligned}(a_1, \ldots, a_6) * (b_1, \ldots, b_6) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4 + (a_2 + \bar{a}_2)b_1, \\ &\quad a_5 + b_5 - \bar{a}_1 b_2, a_6 + b_6 + (1/2)(a_2 + \bar{a}_2)b_1^2 + a_4 b_1 + \bar{a}_3 b_1).\end{aligned}$$

The manifold $G^6 \backslash \mathbb{C}^6$, where G^6 is the lattice of Gauss integers, has the required property, cf. [3].

Let us consider the following finitely generated subgroup Γ_6 of $(\mathbb{C}^6, *)$:

$$\{(n_1, n_2 + sm_2, n_3, n_4 + sm_4, n_5 + sm_5, n_6 + sm_6) | n_i, m_i \text{ Gauss integers, } s \notin \mathbb{Q}\}.$$

As in the previous examples, we can find a simply connected nilpotent Lie group U containing Γ_6 as a uniform subgroup and a surjective homomorphism of Lie groups $u : U \to (\mathbb{C}^6, *)$ which is the identity on Γ_6. The resulting foliation $\mathcal{F}(U, \Gamma_6, u)$ of the manifold

$\Gamma_6\backslash U$ is transversely Hermitian. The same considerations as in [3] ensure that the basic Frölicher spectral sequence of $\mathcal{F}(U, \Gamma_6, u)$ has the property $E_3 \not\cong E_4$.

The group U can be represented as $(\mathbb{C}^{10}, \square)$ with the following group operation:

$$
\begin{aligned}
&(a_1, \dots, a_{10})\square(b_1, \dots, b_{10}) \\
&\quad = (a_i + b_i, a_5 + b_5 + (a_2 + \bar{a}_2)b_1, a_6 + b_6 + (a_3 + \bar{a}_3)b_1, a_7 + b_7 - \bar{a}_1 b_2, \\
&\qquad a_8 + b_8 - \bar{a}_1 b_3, a_9 + b_9 + (1/2)(a_2 + \bar{a}_2)b_1^2 + a_5 b_1 + \bar{a}_4 b_1, \\
&\qquad\qquad a_{10} + b_{10} + (1/2)(a_3 + \bar{a}_3)b_1^2 + a_6 b_1).
\end{aligned}
$$

The submersion u is the following:

$$u(a_1, \dots, a_{10}) = (a_1, a_2 + sa_3, a_4, a_5 + sa_6, a_7 + sa_8, a_9 + sa_{10}).$$

3. Complex conjugation.

Theorem 1 asserts that for homologically oriented transversely Kähler foliations the complex conjugation induces an isomorphism of the basic Dolbeault cohomology. We are going to give a simple example of a homologically oriented transversely Hermitian foliation for which this is not true.

EXAMPLE 4. Let us consider the group $N = (\mathbb{C}^2, *)$ with

$$(a_1, a_2) * (z_1, z_2) = (a_1 + z_1, a_2 + z_2 + \bar{a}_1 z_1)$$

and a subgroup Γ_2 of $\mathbb{C}^2$:

$$\{(n_1 + sm_1, n_2 + sm_2 + s^2 p_2) | n_i, m_i, p_2 \text{ Gauss integers}, s \notin \mathbb{Q}\};$$

Γ_2 contains the lattice G of Gauss integers. The manifold $G\backslash N$ is the Kodaira-Thurston manifold, cf. [1,4,16]. The group Γ_2 can be considered as a uniform subgroup of the group $U = (\mathbb{C}^5, \square)$ with the group operation:

$$
\begin{aligned}
(a_1, \dots, a_5)\square(b_1, \dots, b_5) = &(a_i + b_i, a_3 + b_3 + \bar{a}_1 b_1, \\
&a_4 + b_4 + \bar{a}_1 b_2 + \bar{a}_2 b_1, a_5 + b_5 + \bar{a}_2 b_2).
\end{aligned}
$$

The Γ_2-equivariant submersion $u : U \to N$ is given by the correspondence

$$(a_1, a_2, a_3, a_4, a_5) \mapsto (a_1 + sa_2, a_3 + sa_4 + s^2 a_5).$$

The basic forms of $\mathcal{F}(U, \Gamma_2, u)$ are in one-to-one correspondence with Γ_2-invariant forms on N. Since Γ_2 is dense in N the basic forms can be identified with left invariant forms. As Γ_2 contains G the basic cohomology of $\mathcal{F}(U, \Gamma_2, u)$ is isomorphic to the cohomology of the Kodaira-Thurston manifold. The computations of [1] show that $\dim H^{1,0}(G\backslash N) = 1$ and $\dim H^{0,1}(G\backslash N) = 2$, which means that in this case the complex conjugation does not induce an isomorphism in the Dolbeaut cohomology. However its basic Frölicher spectral sequence collapses at the first level.

4. Formality of the minimal model.

We are going to look at the minimal model of the complex basic cohomology of a homologically oriented transversely Kähler foliation. In fact, as for compact Kähler manifolds, the minimal model for the complex basic cohomology is formal, and hence all Massey products must vanish, cf. [6].

LEMMA 1. *The $dd^{\mathbb{C}}$-lemma is true in the algebra of complex valued basic forms of a homologically oriented transversely Kähler foliation on a compact manifold.*

Proof. The identities (3) and Theorem 1 ensure taht we can repeat the proof of the $dd^{\mathbb{C}}$-lemma for compact Kähler manifolds cf. [6, 5.11].□

As the theorem on the formality of the minimal model is a purely «formal» consequence of the $dd^{\mathbb{C}}$-lemma, cf. [6, Sect. 6], we have the the following theorem:

THEOREM 3. *Let $\mathcal{F}$ be a transversely Kähler foliation on a compact manifold M. If $\mathcal{F}$ is homologically oriented then the minimal model of the complex basic cohomology of $\mathcal{F}$ is formal and thus Massey products of complex valued basic forms vanish.*

To show the non-triviality of this result we present a homologically oriented transversely Hermitian foliation whose complex cohomology possesses non-vanishing Massey products, thus whose minimal model of the basic cohomology cannot be formal. The following proposition for Lie foliations asserts that Examples 1, 2 and 3 have this property.

PROPOSITION 1. *Let $\mathcal{F}$ be a Lie foliation on a compact manifold modelled on a nilpotent Lie group N with a left invariant complex structure, $dim_{\mathbb{C}} N = q$. If the holonomy group Γ of $\mathcal{F}$ contains a uniform subgroup Γ_0 of N then the following conditions are equivalent:*

i) $\mathcal{F}$ is transversely Kähler;

ii) the complex basic cohomology of $(M, \mathcal{F})$ has no non-trivial Massey products;

iii) the group N is commutative.

Proof. We can assume that the group N is simply connected. As the manifold M is compact, the developing mapping is surjective and has connected fibres. Therefore basic forms on $(M, \mathcal{F})$ are in one-to-one correspondence with Γ-invariant forms on N, cf. [17]. On the manifold N we can consider three complexes of complex valued forms:

$\Lambda^*_{\mathbb{C}}(N, \Gamma_0)$-the complex of Γ_0-invariant forms,

$\Lambda^*_{\mathbb{C}}(N, \Gamma)$-the complex of Γ-invariant forms,

$\Lambda^*_{\mathbb{C}}(N, N)$-the complex of N-invariant forms.

Of couse, $\Lambda^*_{\mathbb{C}}(N, N) \subset \Lambda^*_{\mathbb{C}}(N, \Gamma) \subset \Lambda^*_{\mathbb{C}}(N, \Gamma_0)$. Then Nomizu's theorem, cf. [13], ensures that in cohomology we have

$$H^*_{\mathbb{C}}(N, N) \hookrightarrow H^*_{\mathbb{C}}(N, \Gamma) \twoheadrightarrow H^*_{\mathbb{C}}(N, \Gamma_0) \tag{4}$$

as $H^*_{\mathbb{C}}(N, N) \simeq H^*_{\mathbb{C}}(N, \Gamma_0)$.

Theorem 3 ensures that i)$\Rightarrow$ii).

Let us look at the second implication ii)$\Rightarrow$iii). Assume that the group N is not commutative. In [4], the authors proved that in this case there exist non-trivial Massey products in $H^*_{\mathbb{C}}(N, N)$. In view of (4), the Massey product of the same cohomology classes considered in $H^*_{\mathbb{C}}(N, \Gamma)$ must be also non-trivial. Contradiction.

The third implication is trivial as N is just $\mathbb{C}^q$.□

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