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SEMILINEAR PARABOLIC EQUATIONS WITH PRESCRIBED ENERGY

BEI HU - HONG-MING YIN

In this paper we study the reaction-diffusion equation $u_t = \Delta u + f(u, k(t))$ subject to appropriate initial and boundary conditions, where $f(u, k(t)) = u^p - k(t)$ or $k(t)u^p$, with $p > 1$ and $k(t)$ an unknown function. An additional energy type condition is imposed in order to find the solution pair $u(x, t)$ and $k(t)$. This type of problem is frequently encountered in nuclear reaction processes, where the reaction is known to be very strong, but the total energy is controlled. It is shown that the solution blows up in finite time for the first class of functions f, for some initial data. For the second class of functions f, the solution blows up in finite time if $p > n/(n-2)$ while it exists globally in time if $1 < p < n/(n-2)$, no matter how large the initial value is. Partial generalizations are given for the case where $f(u, k(t))$ appears in the boundary conditions.

1. Introduction.

Consider a chemical reaction-diffusion process, where it is known that the reaction is very strong, say like u^p with $p > 1$, but the rate with respect to this power is unknown, say $k(t)$, a function of t. On the other hand, let us assume the total energy is controlled in the system in order to prevent blow-up phenomenon, that is

$$\int_\Omega u(x, t)dx = g(t).$$

This leads to an inverse problem where one needs to find the solution $u(x,t)$ as well as the coefficient of the reaction. Another model arising from nuclear science occurs when the growth of temperature is known to be very fast, like u^p, but some absorption catalytic material is put into the system so that the total mass is conserved. These two models lead us to consider the following parabolic inverse problem: Find $u(x,t)$ and $k(t)$ such that

(1.1) $u_t - \Delta u = f(u, k(t))$ for $(x,t) \in Q_T = \Omega \times (0, T]$,

(1.2) $\frac{\partial u}{\partial \nu}(x,t) = 0$ for $(x,t) \in S_T = \partial\Omega \times (0, T]$,

(1.3) $u(x,0) = u_0(x)$ for $x \in \Omega$,

where Ω is a bounded domain in R^n with smooth boundary $S = \partial\Omega$, and ν is the outward normal on S. An additional energy condition is prescribed by

$$\int_\Omega u(x,t)dx = g(t), \quad t \geq 0. \tag{1.4}$$

The above mathematical problem can also be used to model phenomena in population dynamics and biological sciences where the total mass is often conserved or known, but the growth of a certain cell is known to have some definite form. Research on the well-posedness of parabolic inverse problems is classical and well-known; the reader can find a large number of references in [4]-[6] and the recent proceedings [9]. The essential difference between previous inverse problems and the present one is that solutions of (1.1)-(1.4) may blow up; this fact is well known when $k(t)$ is given. On the other hand, since the function $k(t)$ is given in terms of the solution, the problem may have a global solution because of the stabilizing effect of this function. A natural question then is whether or not the stabilizing factor is strong enough to prevent blow-up. We shall study this problem for two classes of functions $f(u, k(t))$, namely,

$$f(u, k(t)) = u^p - k(t) \quad \text{and} \quad k(t)u^p. \tag{1.5}$$

When the function $k(t) > 0$ is given, there are many papers dealing with various qualitative properties such as finite time blow-up, blow-up rate, blow-up set, etc. (cf. [2], [10], [14], [17] and the references therein). When $f(u, k(t))$ has the form (1.5), with an unknown $k(t)$, it is not difficult to see that condition (1.4) and equation (1.1) imply either

$$k(t) = \frac{1}{|\Omega|}\left(\int_\Omega u^p dx - g'(t)\right) \quad \text{or} \quad \frac{g'(t)}{\int_\Omega u^p dx}.$$

Hence the reaction term in (1.1) can be written in the form

$$f = u^p - \frac{1}{|\Omega|}\left(\int_\Omega u^p dx - g'(t)\right) \quad \text{or} \quad \frac{g'(t)u^p}{\int_\Omega u^p dx} \tag{1.6}$$

In general, it is not clear which terms will dominate the reaction. As there are non-local integral terms in (1.6), a comparison principle is invalid. Nevertheless, by applying the energy method, we prove for the first class of functions that solutions will blow up in finite time for a class of initial data. For the second case, one might also believe that the solution will blow up in finite time. Surprisingly, it turns out that both finite time blow-up or global existence can occur, depending upon the exponent p and the space dimension n. It will be seen in Section 3 that the solution exists globally if $p < n/(n-2)$, no matter how large the initial datum is. On the other hand, the solution will blow up in finite time if $p > n/(n-2)$, provided the initial data satisfy appropriate conditions. This is quite different from the case of regular reaction-diffusion equations (cf. [2], [10], [14], etc.).

We mention that diffusion equations with non-local reactions have been considered by a number of authors (cf. [1], [7]-[8], [17], etc.). However, none of these papers deals with problems of the type (1.1)-(1.4), since the energy in the previous problems blows up in finite time. More recently, problem (1.1)-(1.4) with

$f(u, k(t)) = u^2 - \int_0^1 u^2 dx$ was studied in [3] for the case of one space dimension. They proved blow-up of solution for some special initial values and also discussed blow-up rate. On the other hand, their argument is not suitable for the present situation.

The paper is organized in the following way. In Section 2 we study the problem with $f(u, k(t)) = u^p - k(t)$. In Section 3 we prove global existence for the case where $f(u, k(t)) = k(t)u^p$ with $p < n/(n-2)$, and also prove global existence for $p = n/(n-2)$ when the initial value is large enough. In Section 4 we consider the case $p > n/(n-2)$ and prove that the solution blows up in finite time for suitable radially symmetric initial data. Section 5 deals with the case where the function $f(u, k(t))$ occurs in the boundary conditions.

2. Blowup for $f(u, k(t)) = u^p - k(t)$.

Throughout the paper, the letter C denotes various generic constants, unless otherwise indicated. Since we shall not require that $u(x, t)$ is nonnegative, we write $|u|^{p-1}$ instead of u^p. We assume moreover that $g(t) = constant$ (say 1), as is the case in some applications. The equation (1.1)-(1.4) can now be rewritten as follows (with $g(t) = 1$):

$$
(2.1) \qquad \begin{aligned}
&u_t = \Delta u + |u|^{p-1}u - k(t) && \text{for } x \in \Omega,\ t > 0, \\
&\frac{\partial u}{\partial \nu} = 0 && \text{for } x \in \partial\Omega,\ t > 0, \\
&u(x, 0) = u_0(x) && \text{for } x \in \Omega,
\end{aligned}
$$

where

$$
k(t) = \frac{1}{|\Omega|}\left(\int_\Omega |u|^{p-1} u dx\right).
$$

The following conditions on the data are assumed throughout this section:

$$u_0(x) \in C^3(\bar{\Omega}) \text{ and } \int_\Omega u_0(x)dx = 1. \tag{2.2}$$

The existence and uniqueness of this system for small t is clear, by the standard theory of parabolic estimates and the contraction mapping principle. It is also clear that the solution can be extended in the variable t, as long as the L^∞ norm of the solution remains finite. On the other hand, we have the following

THEOREM 2.1. *The solution of (2.1) blows up in finite time if* $p > 1$ *and*

$$-\frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \frac{1}{p+1}\int_\Omega |u_0|^{p+1} dx.$$

is suitably large.

Remark 2.1. The condition on $u_0(x)$ in Theorem 2.1 can be satisfied if we choose

$$u_0(x) = 1 + \lambda u_1(x), \quad \int_\Omega u_1(x)dx = 0, \quad u_1(x) \not\equiv 0,$$

where $u_1(x)$ is smooth and $\lambda \gg 1$. It is clear that in this case the expression

$$\begin{aligned} &-\frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \frac{1}{p+1}\int_\Omega |u_0|^{p+1} dx \\ &= -\frac{\lambda^2}{2}\int_\Omega |\nabla u_1|^2 dx + \lambda^{p+1}\int_\Omega \left|\frac{1}{\lambda^{p+1}} + u_1\right|^{p+1} dx \end{aligned}$$

is dominated by the term involving λ^{p+1}, and therefore is large when λ is large.

Proof. We use convexity argument (see Levine and Payne [15]). Multiplying the equation by u and u_t, respectively, and then integrating over Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega u^2 dx\right)+\int_\Omega |\nabla u|^2 dx =$$

$$= \int_\Omega |u|^{p+1}dx - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u dx; \tag{2.3}$$

$$\int_\Omega u_t^2 dx + \frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla u|^2 dx = \frac{d}{dt}\left(\frac{1}{p+1}\int_\Omega |u|^{p+1}dx\right).$$

Let

$$J(t) = -\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{p+1}\int_\Omega |u|^{p+1}dx.$$

The second identity of (2.3) gives

$$J'(t) = \int_\Omega u_t^2 dx \geq 0, \tag{2.4}$$

and it follows that

$$J(t) = J(0) + \int_0^t \int_\Omega u_t^2 dx dt. \tag{2.5}$$

Introduce a new function

$$I(t) = \int_0^t \int_\Omega u^2 dx dt + A + Bt^2,$$

where A and B are two constants to be specified later. Clearly

$$I'(t) = \int_\Omega u^2 dx + 2Bt, \quad I''(t) = \frac{d}{dt}\int_\Omega u^2 dx + 2B.$$

By the first identity (2.3),

$$I''(t) = \frac{d}{dt}\int_\Omega u^2 dx + 2B$$

$$= -2\int_\Omega |\nabla u|^2 dx + 2\int_\Omega |u|^{p+1}dx - \frac{2}{|\Omega|}\int_\Omega |u|^{p-1}u dx + 2B.$$

We claim that there exists a constant $\delta > 0$ such that

$$I''(t) \geq 4(1+\delta)J(t). \tag{2.6}$$

Indeed, the desired inequality is equivalent to

$$-2\int_\Omega |\nabla u|^2 dx + 2\int_\Omega |u|^{p+1} dx - \frac{2}{|\Omega|}\int_\Omega |u|^{p-1} u dx + 2B \geq 4(1+\delta)\left[-\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{p+1}\int_\Omega |u|^{p+1} dx\right].$$

Now Hölder's and Young's inequalities imply that for any $\delta_1 > 0$

$$\int_\Omega |u|^p dx \leq \delta_1 \int_\Omega |u|^{p+1} dx + C(\delta_1, |\Omega|).$$

Since $p > 1$, we can choose δ and δ_1 small enough such that

$$2 - \frac{4(1+\delta)}{1+p} - \delta_1 > 0.$$

Then we have the desired claim if we simply take $B \geq C(\delta_1, |\Omega|)$. Clearly

$$I'(t) = \int_\Omega u^2 dx + 2Bt = 2\int_0^t \int_\Omega u u_t dx dt + \int_\Omega u_0^2 dx + 2Bt. \tag{2.7}$$

It follows that, for any $\varepsilon > 0$,

$$I'(t)^2 \leq 4(1+\varepsilon)\int_0^t \int_\Omega u^2 dx dt \int_0^t \int_\Omega u_t^2 dx dt + \left(1 + \frac{1}{\varepsilon}\right)\left[\int_\Omega u_0^2 dx + 2Bt\right]^2. \tag{2.8}$$

Combining the above estimates, we find that, for $\alpha > 0$,

$$
\begin{aligned}
&I''(t)I(t) - (1+\alpha)I'(t)^2 \\
&\geq 4(1+\delta)\left[J(0) + \int_0^t \int_\Omega u_t^2 dx dt\right]\left[\int_0^t \int_\Omega u^2 dx dt + A + Bt^2\right] \\
&\quad - (1+\alpha)\left[4(1+\varepsilon)\int_0^t \int_\Omega u^2 dx dt \int_0^t \int_\Omega u_t^2 dx dt\right] \\
&\quad - (1+\alpha)\left(1+\frac{1}{\varepsilon}\right)\left[\int_\Omega u_0^2 dx + 2Bt\right]^2 .
\end{aligned}
$$

Now we choose ε and α small enough such that

$$1+\delta \geq (1+\alpha)(1+\varepsilon). \tag{2.9}$$

If

$$J(0) \geq 4(1+\alpha)\left(1+\frac{1}{\varepsilon}\right)B \tag{2.10}$$

and A is chosen large enough, then

$$I''(t)I(t) - (1+\alpha)I'(t)^2 \geq 0.$$

It follows (cf. [15]) that $\int_0^t \int_\Omega u^2 dx dt$ will blow up in a finite time T^*. □

Remark 2.2. When $u_0(x) = M$ is a constant, then $u(x,t) = M$ is always a solution. In this case, the proof is invalid since in the inequality (2.6) the constant B depends on $|\Omega|$, but the condition (2.2) implies $M|\Omega| = 1$. Consequently, B depends on M, as do the other constants in the proof. Therefore, it is impossible to choose $J(0)$ to satisfy the desired inequality (2.10). Of course in this case the condition of Theorem 2.1 also fails, since

$$-\frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \frac{1}{p+1}\int_\Omega |u_0|^{p+1} dx = \frac{1}{(p+1)|\Omega|^p}$$

cannot be made arbitrary large.

3. Global Existence for $f(u, k(t)) = k(t)u^p$.

For simplicity, we shall assume that $g'(t) = 1$ in Sections 3 and 4. It is not difficult to see that the results can be immediately carried over for a general function $g(t) > 0$ with $g'(t) > 0$. In this case the problem (1.1)-(1.4) is equivalent to

$$(3.1)\qquad \begin{aligned} &u_t = \Delta u + k(t)u^p && \text{for}\quad x \in \Omega,\ t > 0, \\ &\frac{\partial u}{\partial \nu} = 0 && \text{for}\quad x \in \partial\Omega,\ t > 0, \\ &u(x, 0) = u_0(x) && \text{for}\quad x \in \Omega, \end{aligned}$$

where

$$(3.1)\qquad k(t) = \frac{1}{\int_\Omega u^p(x, t)dx}.$$

We shall assume throughout this section that $u_0(x)$ is smooth, $u_0(x) > 0$ on $\bar{\Omega}$, and satisfies the compatibility conditions $\frac{\partial u_0}{\partial \nu} = 0$ on $\partial\Omega$. (In (1.1)-(1.4), we also require that $\int_\Omega u_0(x)dx = g(0) \equiv m > 0$; this is not an assumption since we do not specify $g(0)$ here).

By the standard theory of parabolic estimates and the contraction mapping principle, the existence and uniqueness of a solution of this system for small t is guaranteed. The solution can be extended in t as long as it remains finite. Since we assume that the initial data are positive, the solution $u(x, t)$ is positive, by the maximum principle.

THEOREM 3.1. *Suppose that $1 < p < n/(n-2)$ for $n \geq 3$, and $1 < p < \infty$ when $n = 1$ or 2. Then there exists a unique global solution to the system for all $t \in [0, \infty)$.*

Proof. We give proof only for $n \geq 3$. The proof for $n = 1$ and $n = 2$ can be obtained with obvious modifications. Fix a large number β so that $(\beta + 1)/p > (n + 2)/2$, and without loss of generality assume that $|\Omega| = 1$. For any $0 < a < 1$, and $\dfrac{1}{q} + \dfrac{1}{q^*} = 1$, we have

$$(3.2) \qquad \int_\Omega u^{p+\beta} dx \leq \left(\int_\Omega u^{(p+\beta)aq} dx \right)^{1/q} \left(\int_\Omega u^{(p+\beta)(1-a)q^*} dx \right)^{1/q^*}.$$

There are two free parameters a and q to be determined. We first let

$$(3.3) \qquad (p + \beta)aq = (\beta + 1)\frac{n}{n-2}.$$

If $q = n/(n-2)$ (it is clear that $0 < a < 1$ with this choice of q, since $p > 1$), then

$$(3.4) \qquad (p + \beta)(1 - a)q^* = \frac{n}{2}(p - 1) < p.$$

However, the choice $q = n/(n-2)$ will not be good enough in the following proof. Since we have a strict inequality in (3.4) when $q = n/(n-2)$, we can take $q > n/(n-2)$ and $q - n/(n-2) \ll 1$ so that (3.4) is still valid for this particular choice of q. Using Hölder's inequality, we obtain, (recalling that $|\Omega| = 1$),

$$(3.5) \qquad \begin{aligned} \int_\Omega u^{p+\beta} dx &\leq \left(\int_\Omega u^{(\beta+1)\frac{n}{n-2}} dx \right)^{1/q} \left(\int_\Omega u^{(p+\beta)(1-a)q^*} dx \right)^{1/q^*} \\ &\leq \left(\int_\Omega u^{(\beta+1)\frac{n}{n-2}} dx \right)^{1/q} \left(\int_\Omega u^p dx \right)^{(p+\beta)(1-a)q^*}. \end{aligned}$$

Next, multiplying the equation (3.1) by u^β and integrating over

Ω, we obtain

$$(3.6)\qquad \left(\int_\Omega \frac{1}{\beta+1}u^{\beta+1}dx\right)_t + \beta\int_\Omega u^{\beta-1}|\nabla u|^2dx = \frac{\int_\Omega u^{p+\beta}dx}{\int_\Omega u^{p+\beta}dx}.$$

Integrating the equation (3.1) over Ω, we immediately get

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx + t.$$

Hence by Hölder's inequality ($|\Omega| = 1$),

$$(3.7)\qquad \int_\Omega u^p dt \geq \left(\int_\Omega u_0(x)dx + t\right)^p \geq \left(\int_\Omega u_0(x)dx\right)^p = m^p > 0.$$

Combining (3.4), (3.5), (3.6) and (3.7), we obtain

$$(3.8)\qquad \begin{aligned}\left(\int_\Omega \frac{1}{\beta+1}u^{\beta+1}dx\right)_t &+ \frac{4\beta}{(\beta+1)^2}\int_\Omega |\nabla u^{(\beta+1)/2}|^2dx \\ &\leq C\left(\int_\Omega u^{(\beta+1)\frac{n}{n-2}}dx\right)^{1/q}.\end{aligned}$$

Let $v = u^{(\beta+1)/2}$. Then

$$\left(\frac{1}{\beta+1}\int_\Omega v^2dx\right)_t + \frac{4\beta}{4(\beta+1)^2}\int_\Omega |\nabla v|^2dx \leq C\left(\int_\Omega v^{\frac{2n}{n-2)}}dx\right)^{1/q}.$$

Now we use the following (elliptic) Sobolev embedding theorem (cf. [16]) in the above inequality,yielding

$$\|w\|_{L^r(\Omega)} \leq C(\|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}),\ \left(r = \frac{2n}{n-2}\right).$$

Since $q > n/(n-2)$, we obtain

$$(3.9)\qquad \left(\int_\Omega v^2dx\right)_t + \int_\Omega |\nabla v|^2dx \leq C_\beta\left(\int_\Omega v^2dx + 1\right).$$

By Gronwall's inequality, we conclude that the right-hand-side of the inequality (3.9) is bounded. Therefore

$$
\text{(3.10)}\qquad
\begin{aligned}
&\left(\int_\Omega u^{\beta+1}dx\right)_t + \left(\int_\Omega u^{(\beta+1)\frac{n}{n-2}}dx\right)^{\frac{n-2}{n}} \le C_\beta,\\
&\sup_{0\le t\le T}\left(\int_\Omega u^{\beta+1}dx\right) + \int_0^t \left(\int_\Omega u^{(\beta+1)\frac{n}{n-2}}dx\right)^{\frac{n-2}{n}} \le C_\beta,
\end{aligned}
$$

It follows that u is bounded in $L^{\beta+1}$. Thus the L^p-estimate (cf. [13]) implies that u is in $W^{2,1}_{(\beta+1)/p}(Q_T)$. Since $(\beta+1)/p > (n+2)/2$, Sobolev's embedding theorem implies that $u(x,t)$ is Hölder continuous in $\bar{\Omega}\times[0,T]$ for any $T>0$. Therefore Schauder's estimate for parabolic equations implies that $u \in C^{2+\alpha,1+\alpha/2}(Q_T)$ for some $\alpha \in (0,1)$. Thus we obtain a global solution. □

We now consider the case that p is equal to the critical number, i.e., $p = n/(n-2)$. The above proof with $q = n/(n-2)$ gives (recalling that $|\Omega| = 1$)

$$
\text{(3.11)}\qquad
\begin{aligned}
&\left(\int_\Omega u^{\beta+1}dx\right)_t + \int_\Omega |\nabla u^{(\beta+1)/2}|^2 dx\\
&\qquad \le C_\beta \frac{\left(\int_\Omega u^{(\beta+1)\frac{n}{n-2}}dx\right)^{\frac{n-2}{n}}}{\left(\int_\Omega u^p dx\right)^{1/p}}\\
&\qquad \le C_\beta \frac{\left(\int_\Omega u^{(\beta+1)\frac{n}{n-2}}dx\right)^{\frac{n-2}{n}}}{t + \int_\Omega u_0(x)dx},
\end{aligned}
$$

where at the final step we have used the inequality (3.7). By Sobolev's embedding theorem (recalling that $(\beta+1)/p > (n+2)/2$), the numerator in the right-hand-side of (3.11) is dominated by

$$C\left(\int_\Omega u^{\beta+1}dx + \int_\Omega |\nabla u^{(\beta+1)/2}|^2 dx\right),$$

where the constant C is the (elliptic) embedding constant and is independent of T. Therefore by choosing $\int_\Omega u_0(x)dx$ large enough, we obtain the same estimate as (3.10). Consequently, we proved

THEOREM 3.2. *Suppose that $p = n/(n-2)$ for $n \geq 3$. Then the solution exists for all $t \in [0, \infty)$, provided that $\int_\Omega u_0(x)dx$ is large enough.* □

4. Blow-up of solutions for $f(u, k(t)) = k(t)u^p$.

In this section we shall construct a solution which blows up in finite time when $p > n/(n-2)$.

THEOREM 4.1. *Suppose that $p > n/(n-2)$. Then there exists a radially symmetric initial datum $u_0(x)$ in $\Omega = B_1(0)$ such that the corresponding solution $u(x,t)$ of (3.1) blows up in finite time at $x = 0$. Furthermore, $x = 0$ is the only blow-up point.*

Proof. Let

$$(4.1) \qquad \varphi(r) = \begin{cases} \dfrac{1}{r^\alpha} & \text{for } \delta < r \leq 1, \\[2ex] \dfrac{1}{\delta^\alpha}\left(1 + \dfrac{\alpha}{2}\right) - \dfrac{\alpha}{2\delta^{\alpha+2}} r^2 & \text{for } 0 \leq r \leq \delta, \end{cases}$$

where we choose

$$\alpha = \frac{2}{p-1}.$$

It is clear that $\varphi \in C^1[0,1] \cap C^\infty([0,\delta) \cup (\delta,1])$, $\varphi'(0) = 0$ and $\varphi'(r) \le 0$. A direct calculation gives

$$\int_0^1 n\omega_n \varphi^p(r) r^{n-1} dr = \int_0^1 n\omega_n r^{n-1-p\alpha} dr + \sigma_1(\delta) \tag{4.2}$$
$$= \frac{n\omega_n}{n-\alpha p} + \sigma_1(\delta),$$

where

$$\sigma_1(\delta) = -\int_0^\delta n\omega_n r^{n-1-p\alpha} dr$$
$$+ \int_0^\delta n\omega_n \left[\frac{1}{\delta^\alpha}\left(1+\frac{\alpha}{2}\right) - \frac{\alpha}{2\delta^{\alpha+2}} r^2\right]^p r^{n-1} dr \tag{4.3}$$
$$= O(\delta^{n-\alpha p}),$$

and ω_n is the volume of the unit ball in R^n. Thus $\sigma_1(\delta) \to 0$ as $\delta \to 0+$, since $p > n/(n-2)$. From our choice of α, we find that for each $\delta \in (0,1)$,

$$\varphi_{rr} + \frac{n-1}{r}\varphi_r = \frac{\alpha}{r^{\alpha+2}}((\alpha+1)-(n-1))$$
$$= -\alpha((n-2)-\alpha)\varphi^p \quad \text{for } \delta \le r < 1,$$

and

$$\varphi_{rr} + \frac{n-1}{r}\varphi_r = -\frac{n\alpha}{\delta^{\alpha+2}} = -n\alpha\varphi^p(\delta)$$
$$\ge -n\alpha\varphi^p(r) \quad \text{for } 0 \le r \le \delta.$$

Since $\varphi \in C^1[0,1]$, the inequality

$$\varphi_{rr} + \frac{n-1}{r}\varphi_r \ge -\beta\varphi^p \quad \text{for } 0 \le r < 1 \tag{4.4}$$

is satisfied in the distribution sense, where

$$\beta = \max[\alpha((n-2)-\alpha), n\alpha].$$

Let $u_0(r) = \mu\varphi(r)$, then

$$(4.5)\quad \begin{aligned} &(u_0)_{rr} + \frac{n-1}{r}(u_0)_r + \frac{1}{2}\left(\frac{1}{\displaystyle\int_0^1 n\omega_n (u_0)^p r^{n-1} dr}\right)(u_0)^p \\ &\geq (-\mu\beta)\left(\frac{u_0}{\mu}\right)^p + \frac{1}{2}\left(\frac{1}{\frac{n\omega_n}{n-\alpha p} + \sigma_1(\delta)}\right)\left(\frac{u_0}{\mu}\right)^p \\ &\geq \frac{1}{4}\frac{n-\alpha p}{n\omega_n}\mu^{-p}(u_0)^p \geq (u_0)^p \quad \text{for } 0 \leq r < 1, \end{aligned}$$

provided we take μ and δ such that $0 < \mu \leq \mu_0$, $0 < \delta \leq \delta_0$ for some sufficiently small μ_0 and δ_0. We now fix $\mu = \mu_0$. Thus $u_0(r)$ will be uniquely determined by the choice of δ.

Next, we want to give an estimate for the modulus of continuity of the integral $\int_{B_1(0)} u^p(x,t)dx$ near $t = 0$ such that the estimate is uniformly valid for small δ. Since we want to prove that the solution blows up in finite time, we will assume for the contrary that it exists for all t; especially, it exists for $0 < t < 1$. Clearly

$$(4.6)\quad \begin{aligned} m &= \int_{B_1(0)} u_0(x)dx \\ &= n\mu\omega_n\left[\frac{1}{n-\alpha} + \left(\frac{2+\alpha}{2n} - \frac{\alpha}{2(n+2)} - \frac{1}{n-\alpha}\right)\delta^{n-\alpha}\right]. \end{aligned}$$

Therefore m is bounded from above and below uniformly for $\delta \leq \delta_0$. As in Section 3, we have

$$(4.7)\quad \int_{B_1(0)} u(x,t)dx = t + m.$$

Notice that $u_0(x) = u_0(r)$ is monotone decreasing in r. By applying the maximum principle to $(r^{n-1}u_r)$, we obtain that

$u_r(r,t) \le 0$. Hence

$$u(r,t)\omega_n r^n = u(r,t)\int_0^r n\omega_n z^{n-1}dz \le \int_0^r u(z,t)n\omega_n z^{n-1}dz$$
$$\le \int_0^1 u(z,t)n\omega_n z^{n-1}dz = t+m,$$

which implies

$$u(r,t) \le \frac{t+m}{\omega_n r^n} \le \frac{1+m}{\omega_n r^n} \quad \text{for } 0<t<1. \tag{4.8}$$

Thus $u(r,t)$ is uniformly bounded, say, in $[1/2,1]\times[0,1)$. Next, by Hölder's inequality, one can easily derive

$$k(t) = \frac{1}{\int_{B_1(0)} u^p dx} \le \frac{\omega_n^{p-1}}{m^p} \quad \text{for } 0<t<1. \tag{4.9}$$

The function $\omega(r,t) = r^{n-1}u_r$ satisfies the equation

$$\mathscr{L}[w] = 0 \quad \text{for } 0<r<1,\ 0<t<1, \tag{4.10}$$

where $\mathscr{L}[\cdot]$ is defined by

$$\mathscr{L}[\psi] = \psi_t - \psi_{rr} + \frac{n-1}{r}\psi_r - pk(t)u^{p-1}\psi. \tag{4.11}$$

Comparing w with a function v which satisfies the same equation $\mathscr{L}[v]=0$, but equals to 0 on $r=1/2$ and on $r=1$, and equals to $r^{n-1}(u_0)_r(r)$ on $t=0$, we easily derive that

$$u_r(3/4,t) \le -c_0 \quad \text{for } 0<t<1, \tag{4.12}$$

where the constant c_0 is independent of δ because of the estimates (4.8) and (4.9).

We next introduce the auxiliary function as in [10],

$$J = w(r,t) + \varepsilon r^n u^q,$$

where we fix q such that $q < p$, and $2p/(q-1) < n$ (This is possible since $2p/(p-1) < n$). We assume that

$$u(3/4, t) \geq \frac{1}{2} u_0(3/4) \equiv \eta \quad \text{for} \quad O < t < \nu_O \tag{4.13}$$

(we shall justify this assumption later on with ν_0 to be determined). Then by maximum principle,

$$u(r,t) \geq u(3/4,t) \geq \frac{1}{2} u_0(3/4) = \eta \quad \text{for} \quad 0 \leq r \leq 3/4, \ 0 < t < \nu_0.$$

A direct calculation shows that

$$\begin{aligned}
\mathscr{L}[J] &= \varepsilon\{-(p+q)k(t)r^n u^{p+q-1} \\
&\quad - 2nqu^{q-1}r^{n-1}u_r - r^n q(q-1)u^{q-2}(u_r)^2\} \\
&= \varepsilon\{-(p+q)k(t)r^n u^{p+q-1} - 2nqu^{q-1}[J - \varepsilon r^n u^q] \\
&\quad - r^n q(q-1)u^{q-2}(u_r)^2\} \\
&\equiv -2\varepsilon nqu^{q-1}J + \varepsilon c(r,t),
\end{aligned}$$

where the coefficient of J in the above equation is bounded as long as the solution $u(r,t)$ remains bounded. Since $u(r,t) \geq \eta$,

$$\begin{aligned}
c(r,t) &= r^n u^{2q-1}[-(p+q)k(t)u^{p-q} + 2n\varepsilon q] - r^n q(q-1)u^{q-2}(u_r)^2 \\
&\leq r^n u^{2q-1}[-(p+q)k(t)\eta^{p-q} + 2n\varepsilon q] \\
&< 0,
\end{aligned}$$

provided

$$\varepsilon < \frac{p-q}{2nq} k(t)\eta^{p-q}. \tag{4.15}$$

The above inequality is valid for small ε if we have a lower bound for $k(t)$. We assume that (we shall justify this assumption later on)

$$k(t) \geq \frac{1}{2} k(0) \quad \text{for} \quad 0 < t < \nu_O, \tag{4.16}$$

then (4.15) is valid for small ε.

On $\{t=0\}$, for $0<r\le\delta$, (notice that $\alpha p=\alpha+2$),

$$\begin{aligned}(u_0)_r+\varepsilon r(u_0)^p &= -\mu\frac{\alpha}{\delta^{\alpha+2}}r+\varepsilon r\mu^p\left[\frac{1}{\delta^\alpha}\left(1+\frac{\alpha}{2}\right)-\frac{\alpha}{2\delta^{\alpha+2}}r^2\right]^p \\ &\le -\mu\frac{\alpha}{\delta^{\alpha+2}}r+\varepsilon r\mu^p\frac{1}{\delta^{\alpha p}}\left(1+\frac{\alpha}{2}\right)^p \\ &= \frac{\mu}{\delta^{\alpha+2}}\left[-\alpha+\varepsilon\mu^{p-1}\left(1+\frac{\alpha}{2}\right)^p\right] \\ &< 0,\end{aligned}$$

if ε is small enough (independent of δ). For $\delta<r<1$,

$$\begin{aligned}(u_0)_r+\varepsilon r(u_0)^p &= -\mu\frac{\alpha}{\delta^{\alpha+2}}r+\varepsilon r\mu^p\frac{1}{r^{\alpha p}} \\ &= \frac{\mu}{r^{\alpha+1}}-\alpha+\varepsilon\mu^{p-1} \\ &< 0\end{aligned}$$

for small ε. Since $p>q$ and $u_0\ge\eta$ for $0\le r\le 3/4$, it follows that

$$(u_0)_r+\varepsilon r(u_0)^q\le(u_0)_r+\varepsilon\eta^{q-p}r(u_0)^p<0 \quad\text{for}\quad 0\le r\le 3/4,$$

for ε small enough (independent of δ). Thus $J<0$ on $\{t=0, 0\le r\le 3/4\}$. By (4.8) and (4.12), we can choose ε to be small enough (independent of δ) so that $J<0$ on $\{r=3/4, 0<t<\nu_0\}$. We now fix such an ε. Obviously, $J=0$ on $\{r=0, 0\le t<\nu_0\}$. Thus, the maximum principle implies that $J\le 0$ on $\{0\le r\le 3/4, 0\le t<\nu_0\}$, as long as (4.13) and (4.16) remain valid for $0\le t<\nu_0$.

Integrating $-u_r\ge\varepsilon r u^q$, we obtain

$$u^{1-q}(r,t)\ge\frac{q-1}{2}\varepsilon r^2+u^{1-q}(0,t)\ge\frac{q-1}{2}\varepsilon r^2,$$

i.e.,

$$u(r,t) \le \left[\frac{2}{(q-1)\varepsilon}\right]^{q-1} \frac{1}{r^{2/(q-1)}} = \left[\frac{2}{(q-1)\varepsilon}\right]^{q-1} \frac{1}{r^{\gamma}}$$

for $0 < r < 3/4, 0 < t < \nu_0$,

where $\gamma = 2/(q-1) < n/p$. It follows that

$$\sup_{0<\delta\le\delta_0} \int_0^{\lambda} n\omega_n u^p(r,t) r^{n-1} dr \le n\omega_n \left[\frac{2}{(p-1)\varepsilon}\right]^{p(q-1)} \frac{\lambda^{n-\gamma p}}{n-\gamma p}. \tag{4.17}$$

Since $u(r,t)$ is uniformly bounded in the domain $\{\lambda \le r \le 1, 0 \le t \le 1\}$, the standard parabolic estimate implies that, for any $\lambda > 0$,

$$\lim_{t\to 0} \sup_{0<\delta\le\delta_0} \left| \int_\lambda^1 n\omega_n u^p(r,t) r^{n-1} dr - \int_\lambda^1 n\omega_n (u_0)^p r^{n-1} dr \right| = 0. \tag{4.18}$$

The estimates (4.17) and (4.18) imply that

$$\lim_{t\to 0} \sup_{0<\delta\le\delta_0} \left| \int_0^1 n\omega_n u^p(r,t) r^{n-1} dr - \int_0^1 n\omega_n (u_0)^p r^{n-1} dr \right| = 0. \tag{4.19}$$

Therefore, by the continuation argument in t, there exists a $\nu_0 \in (0, 1/2)$ such that (4.13) and (4.16) are valid for $0 < t < \nu_0$, where ν_0 is independent of δ. It follows that all of the above estimates are valid for $0 < t < \nu_0$.

We now compare $u(r,t)$ with the solution $v(r,t)$ of the following problem

$$
\begin{aligned}
&v_t = \Delta v + \frac{1}{2}k(0)v^p && \text{for } 0 \le r < 1,\ t > 0,\\
&\frac{\partial v}{\partial r} = -c^* && \text{for } r = 1, t > 0,\\
&\frac{\partial v}{\partial r} = 0 && \text{for } r = 1, t > 0,\\
&v(r,0) = u_0(r) && \text{for } 0 \le r \le 1,
\end{aligned}
$$

where

$$c^* = -\frac{\partial u_0}{\partial r}\bigg|_{r=1}.$$

By the comparison principle, $u(r,t) \ge v(r,t)$, as long as both solutions exist and $0 \le t < \nu_0$. Let $\psi = v_t - v^p$, then $\dfrac{\partial \psi}{\partial r} > 0$ on $\{r = 1, t > O\}$. By using (4.5), we obtain $\psi \ge 0$ on $\{0 \le r \le 1, t = 0\}$. Thus by using the equation for ψ, we derive that $\psi \ge 0$, as long as the solution exists. Hence

$$v(r,t) \ge \frac{1}{\left(\dfrac{1}{u_0^{p-1}(r)} - (p-1)t\right)^{1/(p-1)}}.$$

Especially,

$$v(0,t) \ge \frac{1}{\left(\dfrac{1}{u_0^{p-1}(0)} - (p-1)t\right)^{1/(p-1)}}.$$

Therefore, the solution $v(r,t)$ blows up at a time $T^* < \nu_0$, if δ is chosen to be small enough. This proves Theorem 4.1. □

Remark 4.1. For a general function $g(t)$, Theorem 4.1 holds if $g(t) \ge g_0 > 0$ and $g'(t) \ge g_0 > 0$ for some positive constant g_0.

5. Nonlinear Boundary Value Problem.

In this section we study the problem with $f(u,k(t))$ appearing in the flux boundary condition. We first consider the case where $f(u,k(t)) = |u|^{p-1}u - k(t)$:

$$u_t = \Delta u \qquad \text{for } x \in \Omega,\ t > 0,$$

$$\frac{\partial u}{\partial \nu} = |u|^{p-1}u - k(t) \qquad \text{for } x \in \partial\Omega, t \geq 0,$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \Omega,$$

where ν is the outward normal on $\partial\Omega$. An additional condition is imposed as follows:

$$\int_\Omega u(x,t)dx = 1 \quad \text{for} \quad t \geq 0. \tag{5.2}$$

Assume that $p > 1$. It is known (cf. [11], etc.) that when $k(t) = 0$, the solution blows up in finite time for any nonnegative $u_0(x)$ which is not identically zero.

It is clear from the standard theory of parabolic equations that the problem (5.1)-(5.2) has a local solution. Will the stabilizing factor $k(t)$ in the boundary condition be able to prevent the blow-up phenomenon? The answer is negative for certain initial values. We shall assume that $u_0(x)$ is smooth, say in $C^3(\bar{\Omega})$, for convenience.

THEOREM 5.1. *The solution of (5.1)-(5.2) blows up in finite time if*

$$J(0) = -\frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \frac{1}{p+1}\int_{\partial\Omega} |u_0|^{p+1} ds$$

is large enough.

Proof. The proof is similar to that of Theorem 2.1. Multiplying the equation (5.1) with u and u_t, respectively, and integrating over

Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega u^2dx\right)+\int_\Omega |\nabla u|^2dx=\int_{\partial\Omega}|u|^{p+1}ds \tag{5.3}$$
$$-\frac{1}{|\partial\Omega|}\int_{\partial\Omega}|u|^{p-1}uds;$$

$$\int_\Omega u_t^2dx+\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla u|^2dx=\frac{d}{dt}\left(\frac{1}{p+1}\int_{\mathsf{l}}^{\partial\Omega} u|^{p+1}ds\right). \tag{5.4}$$

Let

$$J(t)=-\frac{1}{2}\int_\Omega \nabla u^2dx+\frac{1}{p+1}\int_{\partial\Omega}|u|^{p+1}ds,$$

and

$$I(t)=\int_0^t\int_{\partial\Omega}u^2dsdt+A+Bt^2.$$

A similar calculation shows that there exists a constant $\alpha>0$ such that

$$I(t)I''(t)-(1+\alpha)I'(t)^2\geq 0,$$

provided that $J(0)$ is large enough. We shall omit the detail here.□

Next we consider the case with $f(u,k(t))=k(t)u^p$ appearing in the flux boundary condition. In this case, since the embedding inequality is different from the situation considered in Section 3, we have a different inequality about p and n to ensure the existence of a global solution. For simplicity, we again assume that

$$\int_\Omega u(x,t)dx=g(t)\equiv t+m,$$

where $m=\int_\Omega u_0(x)dx$. It is not difficult to see that the result can be carried over for a general bounded smooth function $g(t)$ with $g(t)>0$ and $g'(t)>0$.

The problem is equivalent to:

$$
\begin{aligned}
&u_t = \Delta u && \text{for } x \in \Omega,\ t > 0,\\
&\frac{\partial u}{\partial \nu} = k(t)u^p && \text{for } x \in \partial\Omega, t > 0,\\
&u(x,0) = u_0(x) && \text{for } x \in \Omega,
\end{aligned}
$$

where

$$k(t) = \frac{1}{\int_{\partial\Omega} u^p(x,t)ds}.$$

We assume that $u_0(x) > 0$, $u_0(x) \in C^3(\bar{\Omega})$ and that $u_0(x)$ satisfies the compatibility condition:

$$\frac{\partial u_0(x)}{\partial \nu} = \frac{u_0^p(x)}{\int_{\partial\Omega} u_0^p(x)ds} \quad \text{for } \in \partial\Omega.$$

Again the classical theory of parabolic equations implies that the problem (5.5) has a unique classical solution locally in time. Moreover, the strong maximum principle implies that $u(x,t) > 0$ for $x \in \bar{\Omega}$ as long as it exists.

THEOREM 5.2. *For* $1 < p < (n-1)/(n-2)$, *if* $n \geq 3$, *and* $1 < p < \infty$, *if* $1 \leq n \leq 2$, *the problem (5.5) has a unique solution for all* $t \in [0, \infty)$.

Proof. The proof is similar to that of Theorem 3.1. For any $\beta > 0$, we multiply the equation by u^β, and integrate over Ω to obtain

$$\left(\frac{1}{\beta+1}\int_\Omega u^{\beta+1}dx\right)_t+\frac{4\beta}{4(\beta+1)^2}\int_\Omega|\nabla u^{(\beta+1)/2}|^2dx = \frac{\int_{\partial\Omega}u^{p+\beta}ds}{\int_{\partial\Omega}u^p ds}. \tag{5.6}$$

By Hölder's inequality,

$$\int_{\partial\Omega}u^{p+\beta}ds\le\left(\int_{\partial\Omega}u^{(p+\beta)aq}ds\right)^{1/q}\left(\int_u^{\partial\Omega(p+\beta)(1-a)q^*}ds\right)^{1/q^*}.$$

We choose, for $n\ge 3$ (in the case $n=1$ or $n=2$ we choose q to be large enough),

$$q>\frac{n-1}{n-2},\ q-\frac{n-1}{n-2}\ll 1,\ (p+\beta)aq=(\beta+1)\frac{n-1}{n-2}.$$

Then $q^*\approx n-1$, and $(p+\beta)(1-a)q^*\approx(p-1)(n-1)$. Since $(p-1)(n-1)<p$, we have

$$(p+\beta)(1-a)q^*<p.$$

Therefore

$$\frac{\int_{\partial\Omega}u^{p+\beta}ds}{\int_{\partial\Omega}u^p ds}\le C\left(\int_{\partial\Omega}u^{(\beta+1)q}ds\right)^{1/q}.$$

If we define $v(x,t)=u(x,t)^{(\beta+1)/2}$, then by (5.6),

$$\left(\frac{1}{\beta+1}\int_\Omega v^2dx\right)_t+\frac{4\beta}{4(\beta+1)^2}\int_\Omega|\nabla v|^2dx \le C\left(\int_{\partial\Omega}v^{\frac{2(n-1)}{n-2}}ds\right)^{1/q}. \tag{5.7}$$

Now we use the following trace inequality (cf. [16]):

$$\|w\|_{L^r(\partial\Omega)} \le C(\|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}).$$

where

$$r = \frac{2(n-1)}{n-2} \quad \text{for } n \ge 3, \ r \text{ can be arbitrary for } n = 1 \text{ or } 2.$$

Since $q > (n-1)/(n-2)$, we obtain

$$\left(\int_\Omega v^2 dx\right)_t + \int_\Omega |\nabla v|^2 dx \le C_\beta \left(\int_\Omega v^2 dx + 1\right).$$

Using Gronwall's inequality, we have

$$\begin{aligned} &\left(\int_\Omega u^{\beta+1} dx\right)_t + \left(\int_{\partial\Omega} u^{(\beta+1)\frac{n-1}{n-2}} dx\right)^{\frac{n-2}{n-1}} \le C_\beta, \\ &\sup_{0\le t\le T} \int_\Omega u^{\beta+1}(x,t) dx + \int_0^T \left(\int_{\partial\Omega} u^{(\beta+1)\frac{n-1}{n-2}} dx\right)^{\frac{n-2}{n-1}} dt \le C_\beta, \end{aligned} \tag{5.8}$$

where the constant C depends only on the known data and T.

Fix β so that $\beta + 1 > n(p-1)$, then by Theorem 1.1 in [12],

$$\sup_{0\le t\le T} \sup_{x\in\bar\Omega} u(x,t) \le C. \tag{5.9}$$

Thus by Theorems 7.1-7.2 in Chapter V of [13] (it is clear the assumptions (7.4)-(7.6) of [13, Chapter V] are satisfied), we immediately obtain that

$$\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\bar Q_T)} \le C,$$

for some $\alpha \in (0,1)$. This estimate implies that the function $k(t)u^p$ is uniformly bounded in the space $C^{1+\alpha,(1+\alpha)/2}$. Consequently, we can use Schauder's estimate to obtain

$$\|u\|_{C^{2+\alpha,1+(\alpha)/2}(\bar Q_T)} \le C.$$

With the above a priori estimate in hand, we can obtain the existence for $t \in [0, T]$, for any $T > 0$. □

Remark 5.1. Similar to Theorem 3.2, the above argument works for the critical number $p = (n-1)/(n-2)$, provided $\inf_{0 \le t \le T} \|u\|_{L^p(\partial\Omega)}(t)$ is large enough, which is the case if we assume that $\min_{x \in \bar{\Omega}} u_0(x)$ is large enough. Global existence is guaranteed in this case.

Remark 5.2. We conjecture that the solution of (5.1)-(5.2) will blow up in finite time for certain initial values if $p > (n-1)/(n-2)$.

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Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556, USA