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FIXED POINTS OF ASYMPTOTICALLY REGULAR SEMIGROUPS IN BANACH SPACES

JAROSŁAW GÓRNICKI

In this paper we study in Banach spaces the existence of fixed points of (nonlinear) asymptotically regular semigroups. We establish for these semigroups some fixed point theorems in spaces with weak uniform normal structure, in a Hilbert space, in L^p spaces, in Hardy spaces H^p and in Sobolev spaces $W^{r,p}$ for $1 < p < \infty$ and $r \geq 0$, in spaces with Lifshitz's constant greater than one. These results are the generalizations of [8, 10, 16].

Introduction.

Let $(E, \|\cdot\|)$ be a Banach space, C a nonempty closed convex subset of E and G an unbounded subset of $[0, +\infty)$ such that

$t + h \in G$ for all $t, h \in G$,

$t - h \in G$ for all $t, h \in G$ with $t \geq h$

(e.g. $G = [0, +\infty)$ or $G = \mathbb{N}_0$, the set of nonnegative integers).

Let $\mathscr{S} = \{T_t : t \in G\}$ be a family of mappings from C into itself. $\mathscr{S}$ is called an *asymptotically regular semigroup* on C if

1) $T_{s+t}x = T_s T_t x$ for all s, $t \in G$ and $x \in C$,

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2) for each $x \in C$, the mapping $t \mapsto T_t x$ from G into C is continuous when G has the relative topology of $[0, +\infty)$,

3) for each $x \in C$, $h \in G$,

$$\lim_{t \to \infty} \|T_{t+h}x - T_t x\| = 0.$$

The concept of asymptotic regularity is due to Browder and Petryshyn [4]. It is known that if C is bounded and if T is nonexpansive then $T_t := t \cdot I + (1 - t) \cdot T$ is asymptotically regular for each $0 < t < 1$. (There are no restrictions on the geometry of the Banach space; see [5, Th. 2.1]).

1. Fixed Points in Banach spaces with weak uniform normal structure.

To proceed, we establish some preliminaries. Recall that the modulus of convexity of E is the function δ_E defined on $[0, 2]$ by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2} \cdot \|x + y\| : x, y \in B_E \text{ with } \|x - y\| \geq \varepsilon\right\},$$

where B_E is the closed unit ball of E. E is said to be *uniformly convex* if $\delta_E(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$.

Recall that the *normal structure coefficient* $N(E)$ of E is the number defined by

$$N(E) = \inf\left\{\frac{\operatorname{diam} C}{r_C(C)} : \begin{array}{l} C \text{ a bounded convex subset of } E \\ \text{with more than one point} \end{array}\right\},$$

where $r_C(C) = \inf_{x \in C}\left(\sup_{y \in C} \|x - y\|\right)$ is the Chebyshev radius of C relative to itself.

Bynum [6] defined the *weakly convergent sequence coefficient* $WCS(E)$ for a Banach space E which is not Schur (i.e. the weak and strong convergence for sequence in E do not coincide) as the number

$$WCS(E) = \inf \left\{ \frac{D(\{x_n\})}{\inf \left\{ \overline{\lim\limits_{n\to\infty}} \|x_n - y\| : y \in \overline{\text{conv}}\{x_n\} \right\}} \right\}$$

where the first infimum is taken over all weakly (not strongly) convergent sequences $\{x_n\}$ in E and

$$D(\{x_n\}) = \lim_{n\to\infty} \left(\sup\{\|x_i - x_j\| : i, j \geq n\}\right)$$

is the asymptotic diameter of $\{x_n\}$.

A Banach E has *weak uniform normal structure* if $WCS(E) > 1$.

Bynum proved the relation

$$1 \leq N(E) \leq WCS(E) \leq 2$$

in reflexive spaces and computed

$$WCS(\ell^p) = 2^{1/p},\ 1 < p < \infty,\ WCS(\ell^\infty) = 1.$$

Thus

$$WCS(\ell^p) = 2^{1/p} > 2^{(p-1)/p} = N(\ell^p) \text{ for } 1 < p < 2,$$

and ℓ^p is an example of reflexive Banach space such that $N(E)$ and $WCS(E)$ are different. Moreover, if E is a reflexive Banach space with modulus of convexity δ_E, then

(1) $$N(E) \geq (1 - \delta_E(1))^{-1}.$$

Suppose E is uniformly convex Banach space. Then it is easily seen that the equation

(2) $$\alpha^2 \cdot \delta_E^{-1}\left(1 - \frac{1}{\alpha}\right) \cdot \frac{1}{WCS(E)} = 1$$

has a unique solution $\alpha > 1$.

LEMMA 1. *Let E be a uniformly convex Banach space, $\gamma > 1$ be the solution of the equation*

$$\gamma \cdot (1 - \delta_E(\gamma^{-1})) = 1 \tag{3}$$

and $\alpha > 1$ be the solution of the equation (2). Then $\gamma < \alpha$.

Proof. Observe that δ_E is strictly increasing. From (3) we have $\delta_E^{-1}(1 - \gamma^{-1}) = \gamma^{-1}$. This together with the inequality

$$1 < \gamma = \frac{1}{1 - \delta_E(\gamma^{-1})} < \frac{1}{1 - \delta_E(1)} \leq N(E) \leq WCS(E)$$

implies

$$\gamma^2 \cdot \delta_E^{-1}\left(1 - \frac{1}{\gamma}\right) \cdot \frac{1}{WCS(E)} = \frac{\gamma}{WCS(E)} < 1$$

and hence $\gamma < \alpha$. The proof is complete. ■

Recently Doínguez Benavides, López Acedo and Xu [9] proved that

$$WSC(E) = \sup\{M > 0 : M \cdot \overline{\lim_{n\to\infty}} \|x_n - u\| \leq A(\{x_n\})\} \tag{4}$$

where the supremum is taken over all weakly (not strongly) convergent sequences $\{x_n\}$ in E and u is the weak limit of $\{x_n\}$ and

$$A(\{x_n\}) = \overline{\lim_{m\to\infty}} \left(\overline{\lim_{n\to\infty}} \|x_n - x_m\|\right).$$

Recall that a Banach space E satisfies Opial's condition [23] for weach topology if $x_n \rightharpoonup y$ in E implies that

$$\overline{\lim_{n\to\infty}} \|x_n - y\| < \overline{\lim_{n\to\infty}} \|x_n - z\|$$

for all $z \neq y$.

All spaces $l^p (1 < p < +\infty)$ have Opial's property but $L^p[0, 2\pi]$ with $p \in (1, +\infty)\setminus\{2\}$ laks it [23].

THEOREM 1. *Let E be a uniformly convex Banach space with Opial's property, C a nonempty convex weakly compact separable* *subset of E and $\mathscr{S} = \{T_s : s \in G\}$ an asymptotically regular semigroup with*

$$\underline{\lim}_{s\to\infty} \|T_s\| = k < \alpha,$$

where $\alpha > 1$ is the unique solution of the equation (2). Then there exists z in C such that $T_s z = z$ for all $s \in G$.

Proof. The separability of C makes it possible to select a sequence $\{s_\beta\}$ such that

$$\underline{\lim}_{s\to\infty} \|T_s\| = \lim_{\beta\to\infty} \|T_{s_\beta}\| = k < \alpha$$

and

$$\{T_{s_\beta} x\} \text{ converges weakly for every } x \in C.$$

Now we can construct the sequence $\{x_n\} \subset C$ in the following way:

$$\begin{cases} x_0 \in C & \text{arbitrary,} \\ x_{n+1} = \omega - \lim\limits_{\beta\to\infty} T_{s_\beta} x_n, \ n = 0, 1, 2, \dots . \end{cases}$$

Write $r_n = \overline{\lim}_{\beta\to\infty} \|x_{n+1} - T_{s_\beta} x_n\|$. By (4), we have

$$r_n \leq \frac{1}{WCS(E)} \cdot A(\{T_{s_\beta} x_n\})$$

and

$$\begin{aligned} A(\{T_{s_\beta} x_n\}) &= \overline{\lim}_{\beta\to\infty} \left(\overline{\lim}_{\gamma\to\infty} \|T_{s_\beta} x_n - T_{s_\gamma} x_n\| \right) \\ &\leq \overline{\lim}_{\beta\to\infty} \left(\overline{\lim}_{\gamma\to\infty} \left(\|T_{s_\beta} x_n - T_{s_\beta + s_\gamma} x_n\| + \|T_{s_\beta + s_\gamma} x_n - T_{s_\gamma} x_n\| \right) \right) \\ &\leq \overline{\lim}_{\beta\to\infty} \|T_{s_\beta}\| \cdot \overline{\lim}_{\gamma\to\infty} \|x_n - T_{s_\gamma} x_n\|, \end{aligned}$$

that is

$$r_n \leq \frac{k}{WCS(E)} \cdot d(x_n) \tag{5}$$

where

$$d(x_n) = \sup\{\|x_n - T_{s_\gamma} x_n\| : s_\gamma \in G\}.$$

If $d(x_n) = 0$, then $T_{S_\gamma} x_n = x_n$ for $S_\gamma \in G$ and for all $s \in G$, by the asymptotic regularity we have:

$$\|T_s x_n - x_n\| = \|T_S(T_{S_\gamma} x_n) - T_{S_\gamma} x_n\| = \|T_{S+S_\gamma} x_n - T_{S_\gamma} x_n\| \to 0$$

as $\gamma \to \infty$.

We may assume $d(x_n) > 0$ for all $n \geq 0$. Let $n \geq 0$ be fixed and let $\varepsilon > 0$ be small enough. First choose $s_\lambda \in G$ such that

$$\|T_{s_\lambda} x_{n+1} - x_{n+1}\| \geq d(x_{n+1}) - \varepsilon$$

and the choose $s_0 \in G$ so large that for all $s_\beta \geq s_0$

$$\|T_{s_\beta} x_n - x_{n+1}\| \leq r_n + \varepsilon$$

and

$$\begin{aligned}\|T_{s_\beta} x_n - T_{s_\lambda} x_{n+1}\| &\leq \overline{\lim_{\beta\to\infty}} \|T_{s_\beta} x_n - T_{s_\lambda} x_{n+1}\| \\ &\leq \overline{\lim_{\beta\to\infty}} \left(\|T_{s\beta} x_n - T_{s_\lambda + s_\beta} x_n\| + \|T_{s_\lambda + s_\beta} x_n - T_{s_\lambda} x_{n+1}\|\right) \\ &\leq \|T_{s_\lambda}\| \cdot \overline{\lim_{\beta\to\infty}} \|T_{s_\beta} x_n - x_{n+1}\| \leq k \cdot (r_n + \varepsilon).\end{aligned}$$

It then follows that

$$\left\|T_{s_\beta} x_n - \frac{1}{2}\left(x_{n+1} - T_{s_\lambda} x_{n+1}\right)\right\| \leq k \cdot (r_n + \varepsilon) \cdot \left[1 - \delta_E\left(\frac{d(x_{n+1}) - \varepsilon}{k \cdot (r_n + \varepsilon)}\right)\right]$$

for all $s_\beta \geq s_0$, and hence (by Opial's property)

$$\begin{aligned} r_n &\leq \overline{\lim_{\beta\to\infty}} \left\|T_{s_\beta} x_n - \frac{1}{2}\left(x_{n+1} - T_{s_\lambda} x_{n+1}\right)\right\| \\ &\leq k \cdot (r_n + \varepsilon) \cdot \left[1 - \delta_E\left(\frac{d(x_{n+1}) - \varepsilon}{k \cdot (r_n + \varepsilon)}\right)\right].\end{aligned}$$

Taking the limit as $\varepsilon \downarrow 0$ we obtain

$$r_n \leq k \cdot r_n \cdot \left[1 - \delta_E\left(\frac{d(x_{n+1})}{k \cdot r_n}\right)\right]$$

and

$$d(x_{n+1}) \leq k \cdot r_n \cdot \delta_E^{-1}\left(1 - \frac{1}{k}\right)$$

which together with (5) implies that

$$d(x_{n+1}) \leq \frac{k^2}{WCS(E)} \cdot \delta_E^{-1}\left(1 - \frac{1}{k}\right) \cdot d(x_n).$$

Hence

$$d(x_{n+1}) \leq A \cdot d(x_n) \leq A^n \cdot d(x_0) \tag{6}$$

where $A = \dfrac{k^2}{WCS(E)} \cdot \delta_E^{-1}\left(1 - \dfrac{1}{k}\right) < 1$ by assumption. Noticing

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \overline{\lim_{\beta\to\infty}} \|x_{n+1} - T_{s_\beta} x_n\| + \overline{\lim_{\beta\to\infty}} \|T_{s_\beta} x_n - x_n\| \\ &\leq r_n + d(x_n) \leq \left(\frac{k}{WCS(E)} + 1\right) \cdot d(x_n)\end{aligned}$$

we see from (6) that $\{x_n\}$ is norm Cauchy and hence strongly convergent. Let $z = \lim_{n\to\infty} x_n$. Then we have for each $s_\beta \in G$,

$$\begin{aligned}\|z - T_{s_\beta} z\| &\leq \|z - x_n\| + \|x_n - T_{s_\beta} x_n\| + \|T_{s_\beta} x_n - T_{s_\beta} z\| \\ &\leq (\|T_{s_\beta}\| + 1) \cdot \|z - x_n\| + \|x_n - T_{s_\beta} x_n\|.\end{aligned}$$

Taking the limit superior as $\beta \to \infty$, we obtain

$$\begin{aligned}\overline{\lim_{\beta\to\infty}} \|z - T_{s_\beta} z\| &\leq (1 + k) \cdot \|z - x_n\| + d(x_n) \\ &\leq (1 + k) \cdot \|z - x_n\| + A^n \cdot d(x_0) \to 0\end{aligned}$$

as $n \to \infty$, and, by the asymptotic regularity, we have for $s \in G$

with $\|T_s\| < +\infty$:

$$\begin{aligned}\|T_s z - z\| &\le \|T_s z - T_{s+s_\beta} z\| + \|T_{s+s_\beta} z - T_{s_\beta} z\| + \|T_{s_\beta} z - z\| \\ &\le \|T_s\| \cdot \|z - T_{s_\beta} z\| + \|T_{s+s_\beta} z - T_{s_\beta} z\| + \|T_{s_\beta} z - z\| \\ &= (\|T_s\| + 1) \cdot \|z - T_{s_\beta} z\| + \|T_{s+s_\beta} z - T_{s_\beta} z\| \to 0\end{aligned}$$

as $\beta \to \infty$. Hence, $T_{S_\beta} z = z$ for all $s_\beta \in G$, and for all $s \in G$, by the asymptotic regularity, we have:

$$(*) \qquad \begin{aligned}\|T_S z - z\| &= \|T_S(T_\beta z) - T_{S_\beta} z\| = \\ &= \|T_{S+S_\beta} z - T_{S_\beta} z\| \to 0\end{aligned}$$

This completes the proof. ■

Remark 1. We note that for a Hilbert space $\mathscr{H}$, $\alpha_{\mathscr{H}} = (\sqrt{3} - 1)^{1/2} \approx 1.1687$, and for ℓ^p spaces $(2 \le p < \infty)$,

$$\alpha_p = \frac{1}{2}(2^{p-1} + \sqrt{1 + 2^{3-p}})^{1/p}.$$

We do not know the estimate of the constant α_p for ℓ^p spaces if $1 < p < 2$.

THEOREM 2. *Let E be a Banach space with $WCS(E) > 1$, C a nonempty convex weakly compact and separable subset of E, and $\mathscr{S} = \{T_s : s \in G\}$ be an asymptotically regular semigroup with*

$$\underline{\lim}_{s \to \infty} \|T_s\| = k < \sqrt{WCS(E)}.$$

Then there exists z in C such that $T_s z = z$ for all $s \in G$.

Proof. The separability of C makes it possible to select a sequence $\{s_\beta\}$ such that

$$\underline{\lim}_{s \to \infty} \|T_s\| = \lim_{\beta \to \infty} \|T_{s_\beta}\| = k < \sqrt{WCS(E)}$$

and

$$\{T_{s_\beta} x\} \text{ converges weakly for every } x \in C.$$

Now we can constrct the sequence $\{x_n\} \subset C$ in the following way:

$$\begin{cases} x_0 \in C & \text{arbitrary,} \\ x_{n+1} = \omega - \lim_{\beta\to\infty} T_{s_\beta} x_n, \ n = 0, 1, 2, \dots . \end{cases}$$

Note that the asymptotic regularity on C ensures that

$$x_{n+1} = \omega - \lim_{\beta\to\infty} T_{s_\beta + s_\alpha} x_n, \ s_\alpha \in G.$$

Now we show that $\{x_n\}$ converges strongly. By (4) we have

$$r_n = \overline{\lim_{\beta\to\infty}} \|x_{n+1} - T_{s_\beta} x_n\| \leq \frac{1}{WCS(E)} \cdot A(\{T_{s_\beta} x_n\}).$$

From the asymptotic regularity and the ω − l.s.c. of the norm of E, it follows that

$$A(\{T_{s_\beta} x_n\}) = \overline{\lim_{\beta\to\infty}} \left(\overline{\lim_{\alpha\to\infty}} \|T_{s_\beta} x_n - T_{s_\alpha} x_n\| \right)$$

$$\leq \overline{\lim_{\beta\to\infty}} \left(\overline{\lim_{\alpha\to\infty}} (\|T_{s_\beta} x_n - T_{s_\alpha + s_\beta} x_n\| + \|T_{s_\alpha + s_\beta} x_n - T_{s_\alpha} x_n\|) \right)$$

$$\leq \overline{\lim_{\beta\to\infty}} \|T_{s_\beta}\| \cdot \overline{\lim_{\alpha\to\infty}} \|x_n - T_{s_\alpha} x_n\|$$

$$\leq k \cdot \overline{\lim_{\alpha\to\infty}} \left(\underline{\lim_{\nu\to\infty}} \|T_{s_\nu} x_{n-1} - T_{s_\alpha} x_n\| \right)$$

$$\leq k \cdot \overline{\lim_{\alpha\to\infty}} \left(\overline{\lim_{\nu\to\infty}} \|T_{s_\nu} x_{n-1} - T_{s_\alpha} x_n\| \right)$$

$$\leq k \cdot \overline{\lim_{\alpha\to\infty}} \left(\overline{\lim_{\nu\to\infty}} (\|T_{s_\nu} x_{n-1} - T_{s_\alpha + s_\nu} x_{n-1}\| + \|T_{s_\alpha + s_\nu} x_{n-1} - T_{s_\alpha} x_n\|) \right)$$

$$\leq k^2 \cdot \overline{\lim_{\nu\to\infty}} \|T_{s_\nu} x_{n-1} - x_n\| = k^2 \cdot r_{n-1}.$$

Hence

$$r_n \leq \frac{k^2}{WCS(E)} \cdot r_{n-1} = A \cdot r_{n-1},$$

where $A = \frac{k^2}{WCS(E)} < 1$ by assumption. Now, using the asymptotic regularity and the $\omega-$l.s.c. of the norm of E again, we deduce that

$$\|x_{n+1} - x_n\| \leq \overline{\lim_{\beta\to\infty}} \|x_{n+1} - T_{s_\beta} x_n\| + \overline{\lim_{\beta\to\infty}} \|T_{s_\beta} x_n - x_n\|$$

$$\leq r_n + \overline{\lim_{\beta\to\infty}} \left(\overline{\lim_{\alpha\to\infty}} \|T_{s_\beta} x_n - T_{s_\alpha} x_{n-1}\| \right)$$

$$\leq r_n + \overline{\lim_{\beta\to\infty}} \left(\overline{\lim_{\alpha\to\infty}} (\|T_{s_\beta} x_n - T_{s_\alpha+s_\beta} x_{n-1}\| + \|T_{s_\alpha+s_\beta} x_{n-1} - T_{s_\alpha} x_{n-1}\|) \right)$$

$$\leq r_n + k \cdot r_{n-1}$$

which implies that $\{x_n\}$ is Cauchy. Then $\lim_{n\to\infty} x_n = z \in C$ and $T_s z = z$ for all $s \in G$ (see the proof of Theorem 1). This completes the proof. ■

EXAMPLE 1. Recall that the James quasi-reflexive space J consist of all null sequences $x = \{x^i\} = \sum_{i=1}^{\infty} e_i$ ($\{e_i\}$ is the standard basis in c_0) for which the squared variation

$$\sup_{p_1<\dots<p_m}^{m} \left[\sum_{j=2}^{m} (x^{p_j} - x^{p_{j-1}})^2 \right]^{1/2} \tag{7}$$

is finite. Denote by $\|\cdot\|_1$ the norm of x given by (7). The other two norms $\|\cdot\|_2$ and $\|\cdot\|_3$ are defined by

$$\|x\|_2 = \sup_{p_1<\dots<p_{2m}}^{m} \left[\sum_{j=2}^{m} (x^{p_{2j-1}} - x^{p_{2j}})^2 \right]^{1/2}, \tag{8}$$

$$\|x\|_3 = \sup_{p_1<\dots<p_m}^{m} \left[\sum_{j=2}^{m} (x^{p_j} - x^{p_{j-1}})^2 + (x^{p_m} - x^{p_1})^2 \right]^{1/2}. \tag{9}$$

Recently, Domínguez Benavides, López Acedo and Xu [9] proved that each of James' spaces $(J, \|\cdot\|_j)$, $j = 1, 2, 3$, has weak uniform normal structure and $WCS(J, \|\cdot\|_j) = \sqrt{2}$ for $j = 1, 2$, and

$$WCS(J, \|\cdot\|_3) = \left(\frac{3}{2}\right)^{1/2}. \qquad \blacksquare$$

If Ω is a σ-finite measure space, then for $p \geq 2$, $WCS(L^p(\Omega)) = 2^{1/p}$. If $1 < p < 2$ and Ω satisfies an additional property $\mathscr{R}$ (so that the "Rademacher" sequence of functions can be constructed, for instance $L^p[0, 1]$), then $WCS(L^p(\Omega)) = 2^{(p-1)/p}$, [7].

A general formula for $WCS(E)$ in arbitrary Banach space is not known.

LEMMA 2. *Let E be a uniformly convex Banach space. If $1 < WCS(E) < [1 - \delta_E(1)]^{-2}$ and $\alpha > 1$ is the unique solution of the equation (2), then $\sqrt{WCS(E)} < \alpha$.*

Proof. Let $WCS(E) > 1$ and

$$(\sqrt{WCS(E)})^2 \cdot \delta_E^{-1}\left(1 - \frac{1}{\sqrt{WCS(E)}}\right) \cdot \frac{1}{WCS(E)} < 1,$$

$$\delta_E^{-1}\left(1 - \frac{1}{\sqrt{WCS(E)}}\right) < 1.$$

$\delta_E(\cdot)$ is continuous on $[0, 2)$ and strictly increasing. From this it follows that

$$1 - \frac{1}{\sqrt{WCS(E)}} < \delta_E(1).$$

Hence, $WCS(E) < [1 - \delta_E(1)]^{-2}$, and the proof is complete. $\blacksquare$

2. Fixed points in p-uniformly convex Banach spaces.

It is well known that many problems in a Hilbert space $\mathscr{H}$ are solved by applying the following identity:

$$(10) \quad \|\lambda \cdot x + (1-\lambda) \cdot y\|^2 = \lambda \cdot \|x\|^2 + (1-\lambda) \cdot \|y\|^2 - \lambda \cdot (1-\lambda) \cdot \|x-y\|^2$$

for all $x, y \in \mathscr{H}$ and $0 \leq \lambda \leq 1$. Therefore, one of the natural methods to solve problems in Banach space E is to establish equalities and (usually) inequalities in E analogous to (10).

Let $p > 1$ and denote by λ the number in [0,1] and by $W_p(\lambda)$ the function $\lambda \cdot (1-\lambda)^p + \lambda^p \cdot (1-\lambda)$.

Recall that E is said to be p-uniformly convex if there exists a constant $d > 0$ such that

$$\delta_E(\varepsilon) \geq d \cdot \varepsilon^p \quad \text{for} \quad 0 < \varepsilon \leq 2.$$

We note that a Hilbert space $\mathscr{H}$ is 2-uniformly convex (indeed $\delta_{\mathscr{H}}(\varepsilon) = 1 - \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} \geq \frac{1}{8} \cdot \varepsilon^2$) and L^p space ($1 < p < \infty$) is $\max(2, p)$-uniformly convex.

Xu [29] proved that real Banach space E is p-uniformly convex if and only if there exists a positive constant c_p such that for all $\lambda \in [0, 1]$ and $x, y \in E$ the following inequality holds:

$$\|\lambda \cdot x + (1-\lambda) \cdot y\|^p \leq \lambda \cdot \|x\|^p + (1-\lambda) \cdot \|y\|^p - W_p(\lambda) \cdot c_p \cdot \|x-y\|^p. \tag{11}$$

The following Lemma is a crucial tool to prove Theorem 3.

LEMMA 3. ([29, Lemma 2]). *Let $p > 1$ and let E be a p-uniformly convex, C a nonempty closed convex subset of E and $\{x_n\} \subset E$ be a bounded sequence. Then there exists a unique point z in C such that*

$$\overline{\lim_{n\to\infty}} \|x_n - z\|^p \leq \overline{\lim_{n\to\infty}} \|x_n - x\|^p - c_p \cdot \|x - z\|^p$$

for every x in C, where $c_p > 0$ is the constant given in (11). ■

THEOREM 3. *Let $p > 1$ and let E be a p-uniformly convex Banach space, C a nonempty closed convex subset of E and $\mathscr{I} = \{T_s : s \in G\}$ be an asymptotically regular semigroup with*

$$\underline{\lim}_{s\to\infty} \|T_s\| = k < [1 + c_p]^{1/p}.$$

Suppose there is an x_0 in C such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in C such that $T_s z = z$ for all $s \in G$.

Proof. Let $\{s_\beta\}$ be a sequence such that

$$\underline{\lim}_{s\to\infty} \|T_s| = \lim_{\beta\to\infty} \|T_{s_\beta}\| = k < [1 + c_p]^{1/p}.$$

By Lemma 3, we can inductively construct a sequence $\{x_n\}_{n=0}^{\infty}$ in C such that x_n is the unique point such that

$$\overline{\lim_{\alpha\to\infty}}\|T_{s_\alpha}x_{n-1} - x_n\|^p \le \overline{\lim_{\alpha\to\infty}}\|T_{s_\alpha}x_{n-1} - x\|^p - c_p \cdot \|x - x_n\|^p$$

for every x in C, where $c_p > 0$ is the constant given in (11).

Then for each fixed $n \ge 1$ and all s_α, $s_\beta \in G$, by the asymptotic regularity, we have:

$$\begin{aligned}
&c_p \cdot \|x_n - T_{s_\beta}x_n\|^p \\
&\le \overline{\lim_{\alpha\to\infty}}\|T_{s_\beta}x_n - T_{s_\alpha}x_{n-1}\|^p - \overline{\lim_{\alpha\to\infty}}\|x_n - T_{s_\alpha}x_{n-1}\|^p \\
&\le \overline{\lim_{\alpha\to\infty}}\left(\|T_{s_\beta}x_n - T_{s_\alpha+s_\beta}x_{n-1}\| + \|T_{s_\alpha+s_\beta}x_{n-1} - T_{s_\alpha}x_{n-1}\|\right)^p \\
&- \overline{\lim_{\alpha\to\infty}}\|x_n - T_{s_\alpha}x_{n-1}\|^p \le \left(\|T_{s_\beta}\|^p - 1\right) \cdot \overline{\lim_{\alpha\to\infty}}\|x_n - T_{s_\alpha}x_{n-1}\|^p \\
&\le \left(\|T_{s_\beta}\|^p - 1\right) \cdot \overline{\lim_{\alpha\to\infty}}\|x_{n-1} - T_{s_\alpha}x_{n-1}\|^p.
\end{aligned}$$

Taking the limit superior as $\beta \to \infty$ on each side, we get

$$\overline{\lim_{\beta\to\infty}}\|x_n - T_{s_\beta}x_n\|^p \le A \cdot \overline{\lim_{\alpha\to\infty}}\|x_{n-1} - T_{s_\alpha}x_{n-1}\|^p$$

where

$$A = \frac{k^p - 1}{c_p} < 1$$

by assumption of the Theorem. Since

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \|x_{n+1} - T_{s_\beta}x_n\| + \|T_{s_\beta}x_n - x_n\| \\
&\le 2 \cdot \left(\overline{\lim_{\beta\to\infty}}\|x_n - T_{s_\beta}x_n\|^p\right)^{1/p} \\
&\le 2 \cdot \left(A^n \cdot \overline{\lim_{\beta\to\infty}}\|x_0 - T_{s_\beta}x_0\|^p\right)^{1/p} \to 0
\end{aligned}$$

as $n \to \infty$, it follows that $\{x_n\}$ is norm Cauchy. Thus $\{x_n\}$ converges to a point z in C, which is a fixed point of T_s for all $s \in G$ (see the proof of Theorem 1). This completes the proof. ■

By the identity (10) we immediately obtain from Theorem 3 the following result:

COROLLARY 1. *Let C a nonempty closed convex subset of a Hilbert space $\mathscr{H}$ and $\mathscr{I} = \{T_s : s \in G\}$ be an asymptotically regular semigroup with*

$$\varliminf_{s\to\infty} \|T_s\| < \sqrt{2}.$$

Suppose there is an x_0 in C such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in C such that $T_s z = z$ for all $s \in G$. ■

For L^p spaces $(1 < p < \infty)$ we have the following inequalities analogous to (11):

LEMMA 4.
(a) If $1 < p \leq 2$, then we have for all $x, y \in L^p$ and $\lambda \in [0, 1]$,

$$\|\lambda \cdot x + (1-\lambda)\cdot y\|^2 \leq \lambda \cdot \|x\|^2 + (1-\lambda)\cdot \|y\|^2 - \lambda \cdot (1-\lambda)\cdot (p-1)\cdot \|x-y\|^2.$$

(b) Assume $2 < p < \infty$ and t_p is the unique zero of the function $g(x) = -x^{p-1} + (p-1)\cdot x + p - 2$ in the interval $(1, \infty)$. Let

$$c_p = (p-1)\cdot(1+t_p)^{2-p} = \frac{1+(t_p)^{p-1}}{(1+t_p)^{p-1}}$$

and we hav ethe following inequality

$$\|\lambda\cdot x+(1-\lambda)\cdot y\|^p \leq \lambda\cdot\|x\|^p+(1-\lambda)\cdot\|y\|^p - W_p(\lambda)\cdot c_p\cdot\|x-y\|^p$$

for all $x, y \in L^p$ and $\lambda \in [0, 1]$.

■

Remark 2. The inequality (a) is contained in [20, 27], and the inequality (b) in [19, 20]. All constants appearing in the inequalities (e.g. the $(p-1)$ and c_p) are the best possible.

By Lemma 4 we immediately obtain from Theorem 3 the following results:

COROLLARY 2. *Let C a nonempty closed convex subset of L^p $(1 < p \leq 2)$ and $\mathscr{I} = \{T_s : s \in G\}$ be an asymptotically regular semigroup with*

$$\underline{\lim}_{s\to\infty} \|T_s| < \sqrt{p}.$$

Suppose there is an x_0 in C such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in C such that $T_s z = s$ for all $s \in G$. ■

COROLLARY 3. *Let C a nonempty closed convex subset of L^p $(2 < p < \infty)$ and $\mathscr{I} = \{T_s : s \in G\}$ be an asymptotically regular semigroup with*

$$\underline{\lim}_{s\to\infty} \|T_s| < \chi_p = [1 + (p-1)\cdot(1+t_p)^{2-p}]^{1/p}$$

where t_p is the unique zero of the function

$$g(x) = -x^{p-1} + (p-1)\cdot x + p - 2$$

in the interval $(1, \infty)$. Suppose there is an x_0 in C such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in C such that $T_s z = z$ for all $s \in G$. ■

Using the result of Prus and Smarzewski [24, 26] we can obtain from Theorem 3 the fixed point theorem for asymptotically regular semigroups, for example, for Hardy and Sobolev spaces.

Let H^p, $1 < p < \infty$, denote the Hardy space [13] of all functions x analytic in the unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \lim_{r\to 1_-} \left(\frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\Theta})|^p d\Theta \right)^{1/p} < +\infty.$$

Now, let Ω be an open subset of $\mathscr{R}^n$. Denote by $W^{r,p}(\Omega)$, $r \geq 0$, $1 < p < \infty$, the Sobolev space [3] of distributions x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n \leq r$ equipped with the norm

$$\|x\| = \left(\sum_{|\alpha|\leq r} \int_\Omega |D^\alpha x(\omega)|^p d\omega \right)^{1/p}.$$

Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in \Lambda$, be a sequence of positive measure spaces, where the index set Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$, $1 < p < \infty$ and $q = \max(2, p)$, the linear space of all sequences

$$x = \{x_\alpha \in X_\alpha : \alpha \in \Lambda\} \in \ell^q(\Lambda)$$

equipped with the norm

$$\|x\| = \left(\sum_{\alpha \in \Lambda} \|x_\alpha\|_{p,\alpha}^q\right)^{1/q},$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, [22].

Finally, let $L^p = L^p(S_1, \Sigma_1, \mu_1)$ and $L^q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < \infty$, $q = \max(2, p)$ and (S_i, Σ_i, μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach space [12, III.2.10] of all measurable L^p-valued functions x on S_2 such that

$$\|x\| = \left(\int_{S_2} \|x(s)\|_p^q \mu_2(ds)\right)^{1/q}.$$

These spaces are q-uniformly convex with $q = \max(2, p)$ and teh norm in these spaces satisfies

$$\|\lambda \cdot x + (1-\lambda) \cdot y\|^q \leq \lambda \cdot \|x\|^q + (1-\lambda) \cdot \|y\|^q - W_q(\lambda) \cdot d \cdot \|x - y\|^q$$

with the constant

$$d = d_p = \frac{1}{8} \cdot (p-1) \quad \text{for} \quad 1 < p \leq 2 \quad \text{and} \quad d_p = \frac{1}{p \cdot 2^p} \quad \text{for} \quad 2 < p < \infty.$$

Hence the next result follows from Theorem 5.

COROLLARY 4. *Let C be a nonempty closed convex subset of the space E, where $E = H^p$ or $E = W^{r,p}(\Omega)$ or $E = L_{q,p}$ or $E = L_q(L_p)$ and $1 < p < \infty$, $q = \max(2, p)$, $r \geq 0$ and $\mathcal{J} = \{T_s : s \in G\}$ be an asymptotically regular semigroup with*

$$\underline{\lim}_{s \to \infty} \|T_s| < [1 + d]^{1/q}.$$

Suppose there is an x_0 in C such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in C such that $T_s z = z$ for all $s \in G$. ■

3. Fixed points in Banach spaces with Lifshitz's constant $\kappa(E) > 1$.

For a metric space (M, d) we write $B(x, r) = \{y \in M : d(x, y) \leq r\}$ for the closed ball of center x and radius r. We shall use the notion of the *Lifshitz characteristic* $\kappa(M)$ of a metric space M [18], which is defined by

$$\kappa(M) = \sup\left\{ b > 0 : \begin{array}{l} \text{there exists } a > 1 \text{ such that for all} \\ x, y \in M \text{ and all } r > 0,\ d(x, y) > r \\ \text{implies there exists } z \in M \text{ such} \\ \text{that } B(x, br) \cap B(y, ar) \subseteq B(z, r) \end{array} \right\}.$$

Obviously $\kappa(M) \geq 1$, and if M is a nonreflexive Banach space, $\kappa(M) = 1$ [17, Section 37.5]. Define $\kappa_0(E)$ to be the infimum of $\kappa(C)$ where C ranges over all nonempty closed bounded convex subsets of the Banach space E with more than one point. Downing and Turett in [11] proved that for a Banach space E, $\kappa_0(E) > 1$ if and only if $\varepsilon_0(E) < 1$, where $\varepsilon_0(E) = \sup\{\varepsilon : \delta_E(\varepsilon) = 0\}$. In particular, when E is uniformly convex, $\kappa_0(E) > 1$ and for a Hilbert space $\mathcal{H}$, $\kappa_0(\mathcal{H}) = \sqrt{2}$.

The relationships among above geometric coefficients of Banach spaces are the following [1, 28]:

$$1 \leq \kappa_0(E) \leq N(E) \leq WCS(E) \leq 2.$$

THEOREM 4. *Let $(E, \|\cdot\|)$ be a Banach space with $\kappa(E) > 1$ and $\mathcal{I} = \{T_s : s \in G\}$ be an asymptotically regular semigroup defined on E such that*

$$\varliminf_{s\to\infty} \|T_s| = k < \kappa(E).$$

Suppose there is an x_0 in E such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in E such that $T_s z = z$ for all $s \in G$.

Proof. Let $\{s_\beta\}$ be a sequence such that

$$\underline{\lim}_{s\to\infty} \|T_s| = \lim_{\beta\to\infty} \|T_{s_\beta}\| = k < \kappa(E).$$

For any $y \in E$, let

$$r(y) = \inf\left\{R > 0 : \exists_{x\in E} \overline{\lim}_{\beta\to\infty} \|y - T_{s_\beta}x\| \le R\right\}.$$

Observe that $r(y) = 0$ implies $T_s y = y$ for all $s \in G$. Indeed, for $s \in G$ with $\|T_S\| < +\infty$, we have:

$$\|T_s y - y\| \le \|T_s y - T_{s+s_\beta}x\| + \|T_{s+s_\beta}x - T_{s_\beta}x\| + \|T_{s_\beta}x - y\|$$
$$\le (\|T_s\|1+) \cdot \|T_{s_\beta}x - y\| + \|T_{s+s_\beta}x - T_{s_\beta}x\| \to 0$$

as $\beta \to \infty$ by the asymptotic regularity. Hence $T_{s_\beta}y = y$ for all $s_\beta \in G$ and by $(*)$ $T_s y = y$ for alla $s \in G$.

If $\kappa(E) = 1$, then $k < 1$. Without loss of generality we can assume that $\|T_{s_\beta}\| < 1$ for all $s_\beta \in G$. The Banach Contraction Principle implies that for some $s_{\beta_0} \in G$, $T_{s_{\beta_0}}$ has a fixed point in E. Let $z \in E$ and $T_{s_{\beta_0}}z = z$. Assume that for some $s \in G - \{S_{\beta_0}\}$ with $\|T_s\| < +\infty$, $\|T_s z - z\| > 0$. Then

$$\begin{aligned} 0 < \|T_s z - z\| &= \|T_s z - T_{s_{\beta_0}}z\| \\ &\le \|T_s z - T_{s_{\beta_0}}z\| + \|T_{s_{\beta_0}}z - T_{s_{\beta_0}}z\| \\ &\le \|T_s\| \cdot \|z - T_{s_{\beta_0}}z\| + \|T_{s_{\beta_0}}\| \cdot \|T_s z - z\| \\ &\le \|T_{s_{\beta_0}}\| \cdot \|T_s z - z\| \\ &< \|T_s z - z\|, \end{aligned}$$

contradiction. Hence, $T_{s_\beta}z = z$ for all $s_\beta \in G$ and by the asymptotic regularity and by (*), $T_s z = z$ for all $s \in G$.

Assume that $k \ge 1$. For $b \in (k, \kappa(E))$ there exists $a > 1$ such that

$$\forall_{u,v\in E}\forall_{r>0}[\|u - v\| > r \Rightarrow \exists_{w\in E} B(u, br) \cap B(v, ar) \subset B(w, r)].$$

Take $\lambda \in (0, 1)$ such that $\gamma = \min\left(\lambda \cdot a, \frac{\lambda \cdot b}{k}\right) > 1$. We claim that there exists sequence $\{y_n\} \subset E$ having the property:

(12) $\forall_{n \in \mathbb{N}}[r(y_{n+1}) \leq \lambda \cdot r(y_n)$ and $\|y_{n+1} - y_n\| \leq (\lambda + \gamma) \cdot e(y_n)]$.

Indeed, take y_1 to be an arbitrary point in E and assume $y_1, y_2, \dots, y_n$ are given. We now construct y_{n+1}. If $r(y_n) = 0$ then $y_{n+1} = y_n$. If $r(y_n) > 0$ then there exists $s_{\beta_\alpha} \in G$ such that

$$\|T_{s_{\beta_\alpha}} y_n - y_n\| > \lambda \cdot r(y_n) \quad \text{and} \quad \|T_{s_{\beta_\alpha}}\| \leq k \cdot \gamma.$$

From the definition of $r(y_n)$ there exists $x \in E$ for which

$$\overline{\lim_{\beta \to \infty}} \|y_n - T_{s_\beta} x\| \leq r(y_n) < \gamma \cdot r(\gamma_n).$$

Hence

$$\begin{aligned} \|T_{s_\beta} x - T_{s_{\beta_\alpha}}\| &\leq \|T_{s_\beta} x - T_{s_\beta + s_{\beta_\alpha}} x\| + \|T_{s_\beta + s_{\beta_\alpha}} x - T_{s_{\beta_\alpha}} y_n\| \\ &\leq \|T_{s_\beta} x - T_{s_\beta + s_{\beta_\alpha}} x\| + \|T_{s_{\beta_\alpha}}\| \cdot \|T_{s_\beta} x - y_n\|. \end{aligned}$$

Taking the limit superior as $\beta \to \infty$, by the asymptotic regularity, we get

$$\overline{\lim_{\beta \to \infty}} \|T_{s_\beta} x - T_{s_{\beta_\alpha}} x\| \leq k \cdot \gamma \cdot r(y_n).$$

Since

$$B(y_n, \gamma \cdot r(y_n)) \cap B(T_{s_{\beta_\alpha}} y_n, k \cdot \gamma \cdot r(y_n))$$

$$\subseteq B(y_n, a \cdot \lambda \cdot r)) \cap B(T_{s_{\beta_\alpha}} y_n, b \cdot \lambda \cdot r(y_n)) = D$$

the set D is contained in a closed ball centered at w with radius $\lambda \cdot r(y_n)$. Thus

$$\overline{\lim_{\beta \to \infty}} \|w - T_{s_{\beta_\alpha}} x\| \leq \lambda \cdot r(y_n)).$$

Take $y_{n+1} = w$. It follows from the above that the sequence $\{y_n\}$ satisfies condition (12). Since $\lambda < 1$, $\{y_n\}$ converges to $z \in E$. But since $r(z) = 0$, z is a fixed point of T_s for all $s \in G$. ■

Remark 3. Let E_β be the James spaces, i.e. $E_\beta = (\ell^2, |\cdot|_\beta)$, where $|\cdot|_\beta = \max\{\|\cdot\|_2, \beta\cdot\|\cdot\|_\infty\}$, $1 \le \beta < \infty$. Recently, Domínguez Benavides in the paper [8] proved that for the James spaces E_β

$$\kappa_0(E_\beta) = \sqrt{1+\beta^{-2} - 2\cdot\beta^{-2}\cdot\sqrt{\beta^2-1}} \quad \text{for} \quad 1 \le \beta \le \frac{\sqrt{5}}{2}.$$

If $\beta \ge \dfrac{\sqrt{5}}{2}$ it is known [30, Ex. 2.1.10] that $\kappa_0(E_\beta) = 1$.

Let E be uniformly convex Banach space and

$$\psi_p(\xi) = \xi^p + \phi_p(\xi),\ 0 \le \xi \le 2,\ p \ge 1.$$

where

$$\phi_p(\xi) = \inf\left\{ \frac{\xi\cdot\|x\|^p + (1+\xi)\cdot\|y\|^p - \|\xi\cdot x + 1(1-\xi)\cdot y\|^p}{W_p(\xi)} : \begin{array}{l} x, y \in B(0,1), \\ \|x-y\| \ge \varepsilon, \\ 0 < \xi < 1 \end{array} \right\}.$$

Then ψ_p is continuous and strictly increasing. Observing that $\psi_p(0) = 0$ and $\psi_p(1) > 1$, we have a unique $\xi_p \in (0,1)$ such that $\psi_p(\xi_p) = 1$. Ayerbe and Xu [2] proved that in uniformly convex Banach space E,

$$\kappa_0(E) \ge \sup\left\{ \frac{1}{\xi_p} : 1 \le p < \infty \right\}.$$

In particular, if E is p-uniformly convex for some $p > 1$ and $c_p > 0$ is the constant appearing in (11), then

$$\kappa_0(E) \ge (1+c_p)^{1/p}.$$

For L^p spaces $(1 < p < \infty)$ from Lemma 4 we have

$$\kappa_0(L^p) \ge \left[1 + \frac{1+(t_p)^{p-1}}{(1+t_p)^{p-1}}\right]^{1/p} \quad \text{if } p > 2 \text{ and } \kappa_0(L^p) \ge \sqrt{p} \text{ if } p \le 2.$$

Webb and Zhao [28] obtained the same lower bound for $\kappa(L^p)$ [which is not less than $\kappa_0(L^p)$] for $p > 2$; while their lower bound for $\kappa(L^p)$ for $p < 2$ is implicit; it involves the computation of the maximum of a complicated function.

Recently, Domínguez Benavides and Xu [10] introduced a new geometrical coefficient for Banach spaces.

Let E be a Banach space and C be a nonempty bounded convex subset of E. Suppose τ is a topology on E.

DEFINITION ([10]).

(i) *A number $b \geq 0$ is said to have property (P_τ) with respect to C if there exists some $a > 1$ such that for all $x, y \in C$ and $r > 0$ with $\|x - y\| \geq r$ and each τ-convergent sequence $\{x_n\} \subset C$ for which $\overline{\lim_{n\to\infty}} \|x_n - x\| \leq a \cdot r$ and $\overline{\lim_{n\to\infty}} \|x_n - y\| \leq b \cdot r$, there exists some $z \in C$ such that $\underline{\lim_{n\to\infty}} \|x_n - z\| \leq r$.*

(ii) $\kappa_\tau(C) = \sup\{b > 0 : b$ *has property* (P_τ) *w.r.t.* $C\}$.

(iii) $\kappa_\tau(E) = \inf\{\kappa_\tau(C) : C$ *as above*$\}$.

It is easily seen that $\kappa_\tau(C) \geq \kappa(C)$ for all nonempty bounded convex subsets $C \subset E$. If τ is the weak topology $\sigma(E, E^*)$ of E, then we write $\kappa_\omega(C)$ and $\kappa_\omega(E)$, respectively. If E is a dual space and τ is the weak * topology of E, then we write $\kappa_{\omega^*}(C)$ and $\kappa_{\omega^*}(E)$ for these two coefficients.

THEOREM 5. *Let E be a Banach space and τ a topology on E. Suppose C is a nonempty convex τ-sequentially compact subset of E and $\mathcal{J} = \{T_s : s \in G\}$ be an asymptotically regular semigroup such that*

$$\underline{\lim_{s\to\infty}} \|T_s\| = k < \kappa_\tau(C).$$

Suppose there is an x_0 in C such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in C such that $T_s z = z$ for all $s \in G$

Proof. Let $\{s_\beta\}$ be a sequence such that

$$\underline{\lim_{s\to\infty}} \|T_s\| = \lim_{\beta\to\infty} \|T_{s_\beta}\| = k < \kappa_\tau(C).$$

For any $y \in E$, let

$$r(y) = \inf\{R > 0 : \exists_{x \in E} \overline{\lim_{\beta \to \infty}} \|y - T_{s_\beta} x\| \le R\}.$$

If $r(y) = 0$, then exists a subsequence $\{s_{\beta_\alpha}\}$ such that

$$\overline{\lim_{s \to \infty}} \|y - T_{s_\beta} x\| = \lim_{\alpha \to \infty} \|y - T_{s_{\beta_\alpha}} x\| = 0.$$

Analysis similar to that in the proof of Theorem 4, shows that $T_s y = y$ for all $s \in G$.

If $\kappa_\tau(C) = 1$, there exists $z \in C$ such that $T_s z = z$ for all $s \in G$ (see the proof of Theorem 4).

Assume that $1 \le k < \kappa_\tau(C)$. For $b \in (k, \kappa_\tau(C))$ there exists $a > 1$ satisfy the above definition (i). Take $\nu \in (0,1)$ such that $\mu = \min\left(\nu \cdot a, \dfrac{\nu \cdot b}{k}\right) > 1$. It is readily seen that there for every $x, y \in C$ with $\|x - y\| \ge \nu \cdot r$ and any τ-convergent sequence $\{\xi_\gamma\}$ in C for which $\overline{\lim_{\gamma \to \infty}} \|\xi_\gamma - x\| \le \mu \cdot r$ and $\overline{\lim_{\gamma \to \infty}} \|\xi_\gamma - y\| \le k \cdot \mu \cdot r$, there exists some z in C such that $\overline{\lim_{\gamma \to \infty}} \|\xi_\gamma - z\| \le \nu \cdot r$ Without loss of generality, we may assume that $\|T_{s_\beta}\| \le \mu \cdot k$ for all $s_\beta \in G$. We claim that there exists a sequence $\{y_n\} \subset C$ having the property:

(13) $\forall_{n \in \mathbb{N}} [r(y_{n+1}) \le \nu \cdot r(y_n)$ and $\|y_{n+1} - y_n\| \le (\nu + \mu) \cdot r(y_n)]$.

Indeed, take y_1 to be an arbitrary point in C and assume $y_1, y_2, \dots, y_n$ are given. We now construct y_{n+1}. If $r(y_n) = 0$ then $y_{n+1} = y_n$. If $r(y_n) > 0$ then there exists $s_{\beta_\alpha} \in G$ such that

$$\|T_{s_{\beta_\alpha}} y_n - y_n\| > \nu \cdot r(y_n).$$

From the definition of $r(y_n)$ there exists $x \in E$ for which

$$\overline{\lim_{\beta \to \infty}} \|y_n - T_{s_\beta} x\| \le r(y_n) < \mu \cdot r(y_n).$$

Take a subsequence $\{T_{s_{\beta_\lambda}} x\}$ of $\{T_{s_\beta} x\}$ such that

$$\overline{\lim_{\beta \to \infty}} \|y_n - T_{s_\beta} x\| = \lim_{\lambda \to \infty} \|y_n - T_{s_{\beta_\lambda}} x\|.$$

Since C is τ-sequentially compact, we may assume that $\{T_{s_{\beta_\lambda}}x\}$ τ-converges. Hence

$$\|T_{s_{\beta_\lambda}}x - T_{s_{\beta_\alpha}}y_n\| \leq \|T_{s_{\beta_\lambda}}x - T_{s_{\beta_\lambda}x+s_{\beta_\alpha}}x\| + \|T_{s_{\beta_\lambda}+s_{\beta_\alpha}}x - T_{s_{\beta_\alpha}}y_n\|$$

$$\leq \|T_{s_{\beta_\lambda}}x - T_{s_{\beta_\lambda}+s_{\beta_\alpha}}x\| + \|T_{s_{\beta_\lambda}}\| \cdot \|T_{s_{\beta_\lambda}}x - y_n\|.$$

Taking the limit superior as $\lambda \to \infty$, by the asymptotic regularity, we get

$$\overline{\lim_{\lambda\to\infty}}\|T_{s_{\beta_\lambda}}x - T_{s_{\beta_\alpha}}y_n\| \leq k \cdot \mu \cdot r(y_n).$$

By definition of r it follows that there exists $z \in C$ such that

$$\underline{\lim_{\beta\to\infty}} \|z - T_{s_\beta}x\| \leq \nu \cdot r(y_n).$$

Take $y_{n+1} = z$. If follows from the above that the sequence $\{y_n\}$ satisfies condition (13). Since $\nu < 1$, $\{y_n\}$ is a norm Cauchy sequence and thus strongly convergent to a point $z \in C$. But since $r(z) = 0$, z is a fixed point of T_s for all $s \in G$. ■

Recall that a Banach space E is said to satisfy the *uniform Opial condition* [25] if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \leq \underline{\lim_{n\to\infty}} \|x + x_n\|$$

for each $x \in E$ with $\|x\| \geq c$ and all weakly null sequences $\{x_n\}$ in E such that $\underline{\lim_{n\to\infty}} \|x_n\| \geq 1$. Clearly this condition implies the classical Opial condition.

Opial's modulus r_E of E (or r_τ of E relative to the topology τ) is defined by ([21]):

$$r_E(c) = \inf\{\underline{\lim_{n\to\infty}} \|x + x_n\| - 1\}, \ c \geq 0,$$

where the infimum is taken over all $x \in E$ with $\|x\| \geq c$ and sequences $\{x_n\}$ in E such that $\omega - \lim_{n\to\infty} x_n = 0$ (or τ-converging to 0) and $\underline{\lim_{n\to\infty}} \|x_n\| \geq 1$. It is easy to see that the function r_E is nondecreasing and that E satisfies the uniform Opial condition (τ-uniform Opial condition) if and only if $r_E(c) > 0$ ($r_\tau(c) > 0$) for all $c > 0$. For example, if $E = \ell^p$ $(1 < p < \infty)$ then ([21])

$$r_{\ell^p}(c) = (1 + c^p)^{1/p} - 1, \ c \geq 0.$$

For the space ℓ^1 equipped with the weak * topology

$$r_{\omega^*}(c) = c, \ c \geq 0.$$

In the paper [10] Domínguez Benavides and Xu proved that for a Banach space E with the uniform Opial condition

$$\kappa_\omega(E) = 1 + r_E(1).$$

In particular,

$$\kappa_\omega(\ell^p) = 2^{1/p} \text{ if } 1 < p < \infty \text{ and } \kappa_{\omega^*}(\ell^1) = 2. \tag{14}$$

Applying Theorem 5 and (14) we have the following Corollaries:

COROLLARY 5. *Let C be a nonempty closed convex subset of ℓ^p for $1 < p < \infty$ and $\mathcal{S} = \{T_s : s \in G\}$ be an asymptotically regular semigroup such that*

$$\varliminf_{s\to\infty} \|T_s\| < 2^{1/p}.$$

Suppose there is an x_0 in C such that $\{T_s x_0 : s \in G\}$ is bounded. Then there exists z in C such that $T_s z = z$ for all $s \in G$. ■

COROLLARY 6. *Let C be a nonempty weak * compact convex subset of ℓ^1 and $\mathcal{S} = \{T_s : s \in G\}$ be an asymptotically regular semigroup such that*

$$\varliminf_{s\to\infty} \|T_s\| < 2.$$

Then there exists z in C such that $T_s z = z$ for all $s \in G$. ■

Remark 4. Note that for ℓ^p spaces $(2 < p < \infty)$ the unique solution of (2) (see Remark 1), $\alpha_p < 2^{1/p}$.

Recently, in the paper [8] Domínguez Benavides proved that for reflexive Banach space E,

$$\kappa_\omega(E) \le WCS(E).$$

The following theorem is a generalization of Theorem 2 in [8].

THEOREM 6. *Let E be a reflexive Banach space, C a nonempty bounded closed convex subset of E and $\mathcal{I} = \{T_s : s \in G\}$ be an asymptotically regular semigroup such that*

$$\underline{\lim}_{s\to\infty} \|T_s\| = k < \frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot WCS(E) \cdot (\kappa_\omega(E) - 1)}\right).$$

Then there exists z in C such that $T_s z = z$ for all $s \in G$.

Proof. Let $\{s_\beta\}$ be a sequence such that

$$\begin{aligned}\underline{\lim}_{s\to\infty} \|T_s\| &= \lim_{\beta\to\infty} \|T_{s_\beta}\| \\ &= k < \frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot WCS(E) \cdot (\kappa_\omega(E) - 1)}\right).\end{aligned}$$

For any $x \in E$, let

$$r(x) = \inf\{R > 0 : \exists_{y\in E} \underline{\lim}_{\beta\to\infty} \|x - T_{s_\beta} y\| \le R\}.$$

Observe that $r(x) = 0$ implies $T_s x = x$ for all $s \in G$ (see the proof of Theorem 5). Denote $WCS(E) = W$ and $\kappa_\omega(E) = \kappa$. By Theorem 5 and the following inequality

$$\kappa \le \frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot W \cdot (\kappa - 1)}\right) \le W,$$

we can assume $1 < \kappa \le k < \frac{1}{2}\left(1 + \sqrt{1 + 4 \cdot W \cdot (\kappa - 1)}\right)$.

Since $\frac{k}{W} < \frac{\kappa - 1}{k - 1}$ choose $b < \kappa$ such that $\frac{k}{W} < \frac{b-1}{k-1}$. Let $a > 1$ be the corresponding number to b in the definition of

$\kappa_\omega(E) = \kappa$ such that $\frac{k}{W} < \frac{\frac{b}{a} - 1}{k - 1}$. Next, choose $\varepsilon > 0$ such that $\frac{1 + 2\varepsilon}{a} = \alpha < 1$.

Let x an arbitrary point C and without loss of generality we assume that the sequence $\{T_{s_\beta} x\}$ is weakly convergent. If $r(x) \neq 0$ choose $y \in C$ such that $\lim_{\beta \to \infty} \|T_{s_\beta} y\| < r(x) \cdot (1 + \varepsilon)$. There are two cases:

$$(a) \qquad \overline{\lim_{\beta \to \infty}} \|x - T_{s_\beta} x\| \leq \frac{W \cdot r(x) \cdot (1 + \varepsilon)}{k \cdot a}.$$

Choose arbitrary $\eta > 0$ and γ such that

$$\|x - T_{s_\beta} x\| < \frac{W \cdot r(x) \cdot (1 + \varepsilon)}{k \cdot a} + \eta \quad \text{if} \quad \beta > \gamma.$$

Let $s_\nu > s_\mu > s_\gamma$. If $s_\nu - s_\mu < s_\gamma$ then $\|T_{s_\nu} x - T_{s_\mu} x\| < \eta$ for s_μ large enough. If $s_\nu - s_\mu \geq s_\gamma$ we have

$$\begin{aligned} \|T_{s_\nu} x - T_{s_\mu} x\| &\leq \|T_{s_\nu} x - T_{s_\mu + s_\nu} x\| + \|T_{s_\mu + s_\nu} x - T_{s_\mu} x\| \\ &\leq \|T_{s_\nu} x - T_{s_\mu + s_\nu} x\| + \|T_{s_\mu}\| \cdot \|T_{s_\nu} x - x\| \\ &\leq \|T_{s_\nu} x - T_{s_\mu + s_\nu} x\| + \|T_{s_\mu}\| \cdot \left(\frac{W \cdot r(x) \cdot (1 + \varepsilon)}{k \cdot a} + \eta \right). \end{aligned}$$

Thus the asymptotic diameter of the sequence $\{T_{s_\beta} x\}$

$$\begin{aligned} D(\{T_{s_\beta} x\}) &\leq \lim_{\mu \to \infty} \Big(\|T_{s_\nu} x - T_{s_\mu + s_\nu} x\| + \|T_{s_\mu}\| \cdot \\ &\qquad \cdot \left(\frac{W \cdot r(x) \cdot (1 + \varepsilon)}{k \cdot a} + \eta \right) \Big) \\ &= \frac{W \cdot r(x) \cdot (1 + \varepsilon)}{a} + \eta \cdot k. \end{aligned}$$

Since η is arbitrary we obtain

$$D(\{T_{s_\beta} x\}) \leq \frac{W \cdot r(x) \cdot (1 + \varepsilon)}{a}$$

which, by the definition of the $WCS(E)$ coefficient, implies

$$\inf\left\{\overline{\lim_{\beta\to\infty}}\|T_{s_\beta}x - y\| : y \in \overline{\operatorname{conv}}\{T_{s_\beta}x\}\right\} \le \frac{r(x)\cdot(1+\varepsilon)}{a}.$$

Then there exists z in C such that

$$\underline{\lim}_{\beta\to\infty}\|T_{s_\beta}x - z\| \le \overline{\lim_{\beta\to\infty}}\|T_{s_\beta}x - z\| < \frac{r(x)\cdot(1+2\varepsilon)}{a} < \alpha\cdot r(x).$$

Hence $r(z) \le \alpha\cdot r(x)$ and $\|z - x\| \le \left(1 + \frac{W}{k}\right)\cdot\alpha\cdot r(x)$.

$$(b)\qquad \overline{\lim_{\beta\to\infty}}\|x - T_{s_\beta}x\| > \frac{W\cdot r(x)\cdot(1+\varepsilon)}{k\cdot a}.$$

In this case there exists γ such that

$$\|x - T_{s_\gamma}x\| > \frac{W\cdot r(x)\cdot(1+\varepsilon)}{k\cdot a} \quad\text{and}\quad \frac{k}{W} < \frac{\frac{b}{a}-1}{\|T_{s_\gamma}\|-1}.$$

Choose $y \in C$ that $\underline{\lim}_{\beta\to\infty}\|T_{s_\beta}y - x\| < r(x)\cdot(1+\varepsilon)$ and a sequence $\{T_{s_{\beta_\nu}}y\}$ such that $\underline{\lim}_{\beta\to\infty}\|T_{s_\beta}y - x\| = \lim_{\nu\to\infty}\|T_{s_{\beta_\nu}}y - x\|$. Using the asymptotic regularity of T we obtain

$$\begin{aligned}\overline{\lim_{\nu\to\infty}}\|T_{s_\gamma}x - T_{s_{\beta_\nu}}y\| &\le \overline{\lim_{\nu\to\infty}}\left(\|T_{s_\gamma}x - T_{s_\gamma+s_{\beta_\nu}}y\| + \|T_{s_\gamma+s_{\beta_\nu}}y - T_{s_{\beta_\nu}}y\|\right)\\ &\le \|T_{s_\gamma}\|\cdot\overline{\lim_{\nu\to\infty}}\|x - T_{s_{\beta_\nu}}y\|\\ &= \|T_{s_\gamma}\|\cdot\underline{\lim}_{\beta\to\infty}\|T_{s_\beta}y - x\|\\ &< \|T_{s_\gamma}\|\cdot r(x)\cdot(1+\varepsilon).\end{aligned}$$

Choose λ such that $\dfrac{k}{W} < \lambda < \dfrac{\frac{b}{a} - 1}{\|T_{s_\gamma}\| - 1}$. We have

$$\begin{aligned}
&\overline{\lim_{\nu\to\infty}}\|T_{s_{\beta_\nu}}y - \lambda \cdot T_{s_\gamma}x - (1-\lambda)\cdot x\| \\
&\quad \le \lambda \cdot \overline{\lim_{\nu\to\infty}}\|T_{s_{\beta_\nu}}y - T_{s_\gamma}x\| + (1-\lambda)\cdot \lim_{\nu\to\infty}\|T_{s_{\beta_\nu}}y - x\| \\
&\quad \le \lambda\|T_{s_\gamma}\|\cdot r(x)\cdot(1+\varepsilon) + (1-\lambda)\cdot r(x)\cdot(1+\varepsilon) \\
&\quad < \frac{b\cdot r(x)\cdot(1+\varepsilon)}{a}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\|x - \lambda\cdot T_{s_\gamma}x - (1-\lambda)\cdot x\| &= \lambda\cdot\|T_{s_\gamma}x - x\| \\
&\ge \lambda \cdot \frac{W\cdot r(x)\cdot(1+\varepsilon)}{k\cdot a} > \frac{r(x)\cdot(1+\varepsilon)}{a}.
\end{aligned}$$

By the condition which b satisfies there exists $z \in C$ such that

$$\underline{\lim_{\nu\to\infty}}\|T_{s_{\beta_\nu}}y - z\| \le \frac{r(x)\cdot(1+\varepsilon)}{a} \le \alpha\cdot r(x).$$

Thus $r(z) \le \alpha \cdot r(x)$ and it is easy to check that $\|z - x\| \le r(x)\cdot(1+\varepsilon+\alpha)$.

Define $f(x) = z$, z chosen as in the case (a) and (b). By induction, take $z_0 \in C$ arbitrary and $z_n = f(z_{n-1})$. It is easy to prove that $\{z_n\}$ is norm Cauchy sequence and thus convcergent strongly. Let $\lim_{n\to\infty} z_n = z_\infty$. It is readily seen that $r(z_\infty) = 0$, which implies that z_∞ is a fixed point of T_s for all $s \in G$ (see the proof of Theorem 5). ■

EXAMPLE 2. Domínguez Benavides [8] proved that for the space $E = \ell^2$ renormed by

$$\|\{x_n\}\|_p = \left\{|x_1|^p, \left(\sum_{n=2}^{\infty}|x_n|^2\right)^{p/2}\right\}^{1/p}, \quad 2 < p < +\infty,$$

$\kappa_\omega(E) = 2^{1/p} < WCS(E) = \sqrt{2}$. Thus in this case, for $2 < p < [\log_2(2-2^{-1/4})]^{-1} \approx 4.6946$, the constant which appears in Theorem 6 is stricIty bigger than the constant which appears in Theorem 2.■

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Department of Mathematics
Technical University of Rzeszów
W. Pola 2, P. O. Box 85
35-959 Rzeszów, Poland
e-mail: gornicki@prz.rzeszow.pl