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ON THE HYPERGROUPS WITH FOUR PROPER PAIRS AND TWO OR THREE NON-SCALAR ELEMENTS(*)

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In this paper one continues the study of hypergroups with for proper pairs. In particular one considers the case when the scalars group is not empty and the set of non-scalar elements has size two or three.

1. Introduction.

We recall that a *hypergroup* H is a nontempty set equipped with a hyperoperation such that the following two conditions are satisfied:

1.1. $\forall(x, y, z) \in H^3$, $(xy)z = x(yz)$ (associativity);

1.2. $\forall x \in H$, $Hx = xH = H$ (reproducibility).

In this paper, the authors continue the study of the hypergroups H, such that:

1.3. $|P(H)| = |\{(x, y) \in H^2 | \, |xy| \geq 2\}| = 4$.

In [5], [6], [7], one had solved the same problem, when $|P(H)| \leq 3$, while in [3], the authors had examined the previous hypergroups when the scalars group $S(H) = S_l(H) \cap S_r(H)$ is empty, where:

$S_l(H) = \{x \in H | \, |xy| = 1, \ \forall y \in H\}$ (the set of left scalars);

$S_r(H) = \{x \in H | \, |yx| = 1, \ \forall y \in H\}$ (the set of right scalars),

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succeding in determining their structure.

Here we suppose that the scalars group $S(H)$ is not empty and that the set of the non-scalar elements $M(H) = H - S(H)$ has size two or three. We denote by $C(H, S(H), t)$ the class of such hypergroups, where $t = |M(H)| \in \{2, 3\}$.

We observe that $S(H)$ is a closed sub-hypergroup of H (see [5]), thus, we have $S(H)M(H) = S(H)(H - S(H)) = H - S(H) = M(H)$, and the map $\varphi : S(H) \times M(H) \to M(H)$ defined by $\varphi(x, a) = xa$, $\forall(x, a) \in S(H) \times M(H)$ is an action to the left on $M(H)$. In fact, we have $(xy)a = x(ya)$ and $\varepsilon a = a$ (as $\forall x \in S(H)$, $\forall a \in M(H)$, we have $|ax| = 1$, in order to simplify the writing, we shall put $ax = b$ instead of $ax = \{b\}$). In consequence we obtain that $S(H)$ operates to the left on $M(H)$.

Clearly, $S(H)$ also operates to the right on $M(H)$, and we have the following:

PROPOSITION 1.4. *If H is a hypergroup such that $S(H) \neq \emptyset$, $M(H) \neq \emptyset$, then $S(H)$ operates on $M(H)$.*

In the following we shall suppose that $S(H)$ operates on $M(H)$ by the action just defined in proposition (1.4), and $\forall a \in M(H)$, $Stab_L(a)$ (respect. $Stab_R(a)$) will indicate the stabilizer of the element a, under the action to the left (respect. to the right) of $S(H)$ on $M(H)$.

Note that, given a hypergroup $(H, \circ)$, it is possible to consider the hypergroup $(H, \star)$ equipped with the hyperoperation $\star$ such that $\forall(x, y) \in H^2$, $x \star y = y \circ x$. $(H, \star)$ will be called the *symmetric hypergroup* of $(H, \circ)$.

2. The class $C(H, S(H), 2)$.

In this section the set $M(H)$ will be denoted with $\{a, b\}$. We begin with some examples of hypergroups belonging to the class $C(H, S(H), 2)$.

EXAMPLE 2.1. Let G be a group, let g be a subgroup of G of index $[G : g] = 2$ and let a, b be two distinct elements such that

$G \cap \{a, b\} = \emptyset$.

We set $H = G \cup \{a, b\}$ and define on H the following two hyperoperations $\circ_1$, $\circ_2$:

$$H_1 = (H, \circ_1) = \begin{cases} x \circ_1 y = \{xy\} & \text{if } (x, y) \in G^2; \\ x \circ_1 a = \{a\}, x \circ_1 b = \{b\} & \text{if } x \in g; \\ x \circ_1 a = \{b\}, x \circ_1 b = \{a\} & \text{if } x \in G - g; \\ a \circ_1 x = \{a\}, b \circ_1 x = \{b\} & \text{if } x \in G; \\ a \circ_1 a = a \circ_1 b = G \cup \{b\}, b \circ_1 a = b \circ_1 b = G \cup \{a\} & \end{cases}$$

$$H_2 = (H, \circ_2) = \begin{cases} z \circ_2 w = z \circ_1 w & \text{if } (z, w) \in H^2 - \{a, b\}^2; \\ z \circ_2 w = H & \text{if } (z, w) \in \{a, b\}^2 \end{cases}$$

It is easy to show that H_1 and H_2 are hypergroups. In particular, in order to prove the associativity, it is useful to observe that $(G - g)(G - g) = g$.

We denote with H_1' and H_2' respectively the symmetric hypergroups of H_1 and H_2.

EXAMPLE 2.2. In the same hypotheses of 2.1, by setting $K = G \cup \{a, b\}$, we can define the hyperoperations $\diamond_i$ $(i \in \{1, 2, \ldots, 9\})$ in the following way:

$$K_i = (K, \diamond_i) = \begin{cases} x \diamond_i y = \{xy\} & \text{if } (x, y) \in G^2; \\ x \diamond_i a = a \diamond_i x = \{a\}, x \diamond_i b = b \diamond_i x = \{b\} & \text{if } x \in g; \\ x \diamond_i a = a \diamond_i x = \{b\}, x \diamond_i b = b \diamond_i x = \{a\} & \text{if } x \in G - g; \\ a \diamond_i a = b \diamond_i b = X, a \diamond_i b = b \diamond_i a = Y & \end{cases}$$

where:

$X = g$ and $Y = G - g$ under the hyperoperation $\diamond_1$;

$X = G - g$ and $Y = g$ under the hyperoperation $\diamond_2$;

$X = g \cup \{a\}$ and $Y = (G - g) \cup \{b\}$ under the hyperoperation $\diamond_3$;

$X = g \cup \{a, b\}$ and $Y = (G - g) \cup \{a, b\}$ under the hyperoperation $\diamond_4$;

$X = (G - g) \cup \{a, b\}$ and $Y = g \cup \{a, b\}$ under the hyperoperation $\diamond_5$;

$X = (G - g) \cup \{a\}$ and $Y = g \cup \{b\}$ under the hyperoperation $\diamond_6$

$X = Y = G$ under the hyperoperation $\diamond_7$;

$X = G \cup \{a\}$ and $Y = G \cup \{b\}$ under the hyperoperation $\diamond_8$;

$X = Y = K$ under the hyperoperation $\diamond_9$;

For K_1 and K_2 we suppose that G has at least three elements.

We observe that, $X = g \cup \{b\}$ and $Y = (G - g) \cup \{a\}$, or $X = (G-g) \cup \{b\}$ and $Y = g \cup \{a\}$, or $X = G \cup \{b\}$ and $Y = G \cup \{a\}$, then we obtain hyperstructures which are isomorphic, respectively to K_3, K_6 and K_8.

In this case one can also prove that the hyperoperations $\diamond_i$ define on K a structure of hypergroup. We denote by K_i' the symmetric hypergroups of the hypergroups K_i.

EXAMPLE 2.3. Let G be a group, let g_1, g_2 be two distinct subgroups of index $[G : g_1] = [G : g_2] = 2$ and let a, b be two distinct elements such that $G \cap \{a, b\} = \emptyset$. We put $T = G \cup \{a, b\}$ and denote with $\star_1, \star_2$ the following hyperoperations:

$$T_1 = (T, \star_1) =$$
$$\begin{cases} x \star_1 y = \{xy\} & \text{if } (x, y) \in G^2; \\ x \star_1 a = \{a\}, x \star_1 b = \{b\} & \text{if } x \in g_1; \\ x \star_1 a = \{b\}, x \star_1 b = \{a\} & \text{if } x \in G - g_1; \\ a \star_1 x = \{a\}, b \star_1 x = \{b\} & \text{if } x \in g_2; \\ a \star_1 x = \{b\}, b \star_1 x = \{a\} & \text{if } x \in G - g_2; \\ a \star_1 a = a \star_1 b = b \star_1 a = b \star_1 b = G & \end{cases}$$

$$T_2 = (T, \star_2) = \begin{cases} z \star_2 w = z \star_1 w & \text{if } (z, w) \in T^2 - \{a, b\}^2; \\ z \star_2 w = T & \text{if } (z, w) \in \{a, b\}^2 \end{cases}$$

T_1 and T_2 are hypergroups. To verify the associativity, it is useful to observe that $(G - g_1)(G - g_1) = g_1$, $(G - g_2)(G - g_2) = g_2$, $G - (g_1 \cup g_2) \neq \emptyset$ and $[G - (g_1 \cup g_2)][G - (g_1 \cup g_2)] \subset g_1 \cap g_2$.

As usual T_1' and T_2' denote the symmetric hypergroups of T_1 and T_2.

EXAMPLE 2.4. Let G be a group and let a, b be two distinct elements such that $G \cap \{a, b\} = \emptyset$. We put $M = G \cup \{a, b\}$ and define on M, 24 hyperoperations $\otimes_k$, as follows:

$x \otimes_k y = \{xy\}, \forall (x, y) \in G^2, \forall k \in \{1, 2, \ldots, 24\}$;

$x \otimes_k a = a \otimes_k x = \{a\}, x \otimes_k b = b \otimes_k x = \{b\}, \forall x \in G, \forall k \in \{1, 2, \ldots, 24\}$;

$a \otimes_1 a = \{a, b\}, a \otimes_1 b = b \otimes_1 a = G \cup \{b\}, b \otimes_1 b = M$;

$a \otimes_2 a = \{a, b\}, a \otimes_2 b = G \cup \{b\}, b \otimes_2 a = b \otimes_2 b = M$;

$a \otimes_3 a = \{a, b\}, a \otimes_3 b = b \otimes_3 b = M, b \otimes_3 a = G \cup \{b\}$;

$a \otimes_4 a = b \otimes_4 b = \{a, b\}, a \otimes_4 b = b \otimes_4 a = M$;

$a \otimes_5 a = \{a, b\}, a \otimes_5 b = b \otimes_5 a = M, b \otimes_5 b = G \cup \{b\}$;

$a \otimes_6 a = \{a, b\}, a \otimes_6 b = b \otimes_6 a = b \otimes_6 b = M$;

$a \otimes_7 a = G \cup \{a\}, a \otimes_7 b = b \otimes_7 a = M, b \otimes_7 b = \{a, b\}$;

$a \otimes_8 a = G \cup \{a\}, a \otimes_8 b = b \otimes_8 a = M, b \otimes_8 b = G \cup \{b\}$;

$a \otimes_9 a = G \cup \{a\}, a \otimes_9 b = b \otimes_9 a = b \otimes_9 b = M$;

$a \otimes_{10} a = G \cup \{b\}, a \otimes_{10} b = b \otimes_{10} a = \{a, b\}, b \otimes_{10} b = G \cup \{a\}$;

$a \otimes_{11} a = G \cup \{b\}, a \otimes_{11} b = b \otimes_{11} a = \{a, b\}, b \otimes_{11} b = M$;

$a \otimes_{12} a = G \cup \{b\}, a \otimes_{12} b = b \otimes_{12} a = G \cup \{a\}, b \otimes_{12} b = \{a, b\}$;

$a \otimes_{13} a = G \cup \{b\}, a \otimes_{13} b = b \otimes_{13} a = G \cup \{a\}, b \otimes_{13} b = G \cup \{a\}$;

$a \otimes_{14} a = G \cup \{b\}, a \otimes_{14} b = b \otimes_{14} a = G \cup \{a\}, b \otimes_{14} b = M$;

$a \otimes_{15} a = G \cup \{b\}, a \otimes_{15} b = b \otimes_{15} a = M, b \otimes_{15} b = G \cup \{a\}$;

$a \otimes_{16} a = G \cup \{b\}, a \otimes_{16} b = b \otimes_{16} a = b \otimes_{16} b = M;$

$a \otimes_{17} a = a \otimes_{17} b = G \cup \{b\}, b \otimes_{17} a = b \otimes_{17} b = M;$

$a \otimes_{18} a = b \otimes_{18} a = G \cup \{b\}, a \otimes_{18} b = b \otimes_{18} b = M;$

$a \otimes_{19} a = b \otimes_{19} b = M, a \otimes_{19} b = b \otimes_{19} a = \{a, b\};$

$a \otimes_{20} a = b \otimes_{20} b = b \otimes_{20} a = M, a \otimes_{20} b = \{a, b\};$

$a \otimes_{21} a = b \otimes_{21} b = M, a \otimes_{21} b = b \otimes_{21} a = G \cup \{a\};$

$a \otimes_{22} a = b \otimes_{22} b = b \otimes_{22} a = M, a \otimes_{22} b = G \cup \{a\};$

$a \otimes_{23} a = b \otimes_{23} b = b \otimes_{23} a = M, a \otimes_{23} b = G \cup \{b\};$

$a \otimes_{24} a = a \otimes_{24} b = b \otimes_{24} a = b \otimes_{24} b = M.$

Remark that for every $k \in \{1, 2, \dots, 24\}$, $M = (M, \otimes_k)$ is a hypergroup.

We denote by M'_k the symmetric hypergroups of the hypergroups M_k.

We establish the following:

LEMMA 2.5. *if* $H \in C(H, S(H), 2)$ *and* $M(H) = \{a, b\}$, *then* $\forall x \in S(H)$, *we have:*

1) $ax = \{b\} \Leftrightarrow bx = \{a\}$*;*

2) $xa = \{b\} \Leftrightarrow xb = \{a\}$*;*

3) $ax = \{a\} \Leftrightarrow bx = \{b\}$*;*

4) $xa = \{a\} \Leftrightarrow xb = \{b\}$*;*

If there exists $x \in S(H)$ *such that:*

5) $ax = \{b\}$ *and* $xa = \{a\}$, *then* $aa = ba$ *and* $bb = ab$*;*

6) $ax = \{b\} = xa$ *(respect.* $xa = \{b\} = ax$*), then* $aa = bb$ *and* $ab = ba$*;*

7) $ax = \{b\}$ *and* $ax = \{a\}$, *then* $aa = ab$ *and* $ba = bb$.

Proof. (1) Suppose $ax = \{b\}$. If $bx = \{b\}$, then we obtain $H = Hx = (S(H) \cup M(H))x = S(H)x \cup ax \cup bx = S(H) \cup \{b\}$, which is impossible. Therefore $bx = \{a\}$. It is easy to prove the converse.

Analogously, one can prove the (2), (3), (4).

(5) Suppose now $ax = \{b\}$ and $xa = \{a\}$. We have $aa = a(xa) = (ax)a = ba$ and, using (4), $bb = (ax)b = a(xb) = ab$.

In an analogous way, one can prove the (6), (7).

PROPOSITION 2.6. *If H is a hypergroup such that $S(H)\neq\emptyset$, $M(H)\neq\emptyset$, then:*

1) *If $H \in C(H, S(H), 2)$, then $S(H)$ operates transitively to the left on $M(H)$ (respect. to the right on $M(H)$) if and only if $\exists a \in M(H)$ such that $Stab_L(a)\subsetneq S(H)$ (respect. $Stab_R(a)\subsetneq S(H)$);*

2) *If $H \in C(H, S(H), 2)$ and $a \in M(H)$ we have:*
$Stab_L(a)\subsetneq S(H)$ if and only if $[S(H) : Stab_L(a)] = 2$
(respect. $Stab_R(a)\subsetneq S(H)$ if and only if $[S(H) : Stab_R(a)] = 2$).

Proof. (1) Let $M(H) = \{a, b\}$. If $S(H)$ operates transitively to the left on $M(H)$ and $\cdot$ denotes the action, then $S(H).a = M(H)$, whence $\exists x \in S(H)$ such that $x.a = b$. Therefore $x \notin Stab_L(a)$ and $Stab_L(a)\subsetneq S(H)$. On the converse, $\forall x \in S(H) - Stab_L(a)$, $x.a = b$ and $x^{-1}.b = a$. Finally $S(H).a = S(H).b = M(H)$ and the action is transitive.

(2) It results, from (1), because the index of the stabilizer of an element in $S(H)$ is equal to the size of the orbit of the element.

Remark 2.7. Before stating the next theorems, we observe that if $H \in C(H, S(H), 2)$ and $M(H) = \{a, b\}$, then, by lemma (2.5) (4), we have $xa = \{a\} \Leftrightarrow xb = \{b\}$ and so, $Stab_L(a) = Stab_L(b)$.

THEOREM 2.8. *Let $H \in C(H, S(H), 2)$ and $M(H) = \{a, b\}$. If $Stab_R(a) = S(H)$ and $Stab_L(a)\subsetneq S(H)$ (respect. $Stab_L(a) = S(H)$ and $Stab_R(a)\subsetneq S(H)$), then H is isomorphic to one of the hypergroups of example (2.1).*

Proof. By proposition (2.6), we have $[S(H) : Stab_L(a)] = 2$ and $\forall x \in S(H) - Stab_L(a)$, $xa\neq\{a\}$ whence $xa = \{b\}$. Besides

$Stab_R(a) = S(H)$ implies that $ax = \{a\}$, $\forall x \in S(H)$, and so, by lemma (2.5) (7), we have $aa = ab$ and $ba = bb$.

Moreover $S(H) \cup \{b\} \subset H = aH = aS(H) \cup aM(H) = aStab_R(a) \cup aM(H) = \{a\} \cup aa \cup ab$ and thus $S(H) \cup \{b\} \subset aa = ab$. Analogously one can prove the inclusion $S(H) \cup \{a\} \subset ba = bb$.

Besides, if $a \in aa$, then $\{b\} = xa \subset x(aa) = (xa)a = ba$ and conversely, if $b \in ba$, then $\{a\} = x^{-1}b \subset x^{-1}(ba) = (x^{-1}b)a = aa$. Then $a \in aa = ab \Leftrightarrow b \in ba = bb$.

Consequently, only the following two cases are possible:

(i) $aa = ab = S(H) \cup \{b\}$ and $ba = bb = S(H) \cup \{a\}$;

(ii) $aa = ab = ba = bb = H$.

Therefore H is one of the hypergroups presented in example (2.1).

COROLLARY 2.9. *If $H \in C(H, S(H), 2)$, $M(H) = \{a, b\}$ and $\{\varepsilon\} = Stab_L(a) \subsetneq S(H) = Stab_R(a)$, then $|H| = 4$ and H, unless isomorphisms, is one of the following hypergroups:*

o	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	a	ε,x,b	ε,x,b
b	b	b	ε,x,a	ε,x,a

o	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	a	H	H
b	b	b	H	H

Proof. Since $Stab_L(a) = \{\varepsilon\}$ and $[S(H) : Stab_L(a)] = 2$, we have $S(H) \cong Z_2$, and setting $S(H) = \{e, x\}$, the theorem (2.8) completes the proof.

We prove now the:

THEOREM 2.10. *Let $H \in C(H, S(H), 2)$ and $M(H) = \{a, b\}$. If $Stab_L(a) = Stab_R(a) \subsetneq S(H)$, then H is isomorphic to one of the hypergroups, which are constructed in example (2.2).*

Proof. For proposition (2.6), we have $[S(H) : Stab_L(a)] = [S(H) : Stab_R(a)] = 2$, and $\forall x \in S(H) - Stab_L(a) = S(H) -$

$Stab_R(a)$ we have $ax = \{b\} = xa$. By using lemma (2.5) (6), we obtain $aa = bb$ and $ab = ba$. Then, if we denote $Stab_L(a) = Stab_R(a) = N$ and $S(H) = S$, we establish the following properties:

1) $N \cap aa \neq \emptyset \Leftrightarrow N \subset aa$;

2) $S - N \cap aa \neq \emptyset \Leftrightarrow S - N \subset aa$;

3) $N \cap ab \neq \emptyset \Leftrightarrow N \subset ab$;

4) $S - N \cap ab \neq \emptyset \Leftrightarrow S - N \subset ab$;

5) $N \subset aa \Leftrightarrow S - N \subset ab$;

6) $S - N \subset aa \Leftrightarrow N \subset ab$;

7) $a \in aa \Leftrightarrow b \in ab$ and $b \in aa \Leftrightarrow a \in ab$;

8) $aa = N \Leftrightarrow ab = S - N$.

(1) If $x \in N \cap aa$, then $N = xN \subset aaN = aa$.

(2) If $z \in S - N \cap aa$, then $\forall w \in S - N$, we have $\{z^{-1}, w\} \subset S - N$. Besides $[S : N] = 2$, hence $(S - N)(S - N) = N$ and so $wz^{-1} \in N$. Denote $wz^{-1} = \{x\}$, we obtain $\{w\} = xz \subset x(aa) = (xa)a = aa$, whence $S - N \subset aa$.

(3) and (4) can be proved, respectively, as (1) and (2).

(5) If $N \subset aa$, then $S - N = N(S - N) \subset aa(S - N) = ab$. Conversely if $S - N \subset ab$, as $(S - N)(S - N) = N$, we deduce $N \subset ab(S - N) = aa$.

(6) If $S - N \subset aa$, then $N = (S - N)(S - N) \subset aa(S - N) = ab$, while if $N \subset ab$, then $S - N = N(S - N) \subset ab(S - N) = aa$.

(7) Taking $x \in S - N$, from $a \in aa$, we obtain $\{b\} = ax \subset (aa)x = a(ax) = ab$. In the same manner, one can prove the other implications.

(8) Suppose $aa = N$. For (7), we have that $ab \cap \{a, b\} = \emptyset$. If $ab \cap N \neq \emptyset$, then taking $x \in ab \cap N$, we obtain, for (5), $\{a, b\} = a(S - N) \cup ax = a((S - N) \cup \{x\}) \subset a(ab) = (aa)b = Nb = \{b\}$, which is absurd and so $ab = S - N$.

On the converse, let $ab = S - N$. For (7), we have $aa \cap \{a, b\} = \emptyset$. If $aa \cap S - N \neq \emptyset$, then using (2), we obtain $S - N \subset aa$. In consequence we have $\{a\} = (S - N)b \subset (aa)b = a(ab) = a(S - N) = \{b\}$, which is a contradiction. Therefore $aa = N$.

Lastly, we observe that (5), (6), (7), (8) allow us to prove easily the following implications:

9) $aa = S - N \Leftrightarrow ab = N$;

10) $aa = N \cup \{a\} \Leftrightarrow ab = (S - N) \cup \{b\}$;

11) $aa = N \cup \{b\} \Leftrightarrow ab = (S - N) \cup \{a\}$;

12) $aa = N \cup \{a, b\} \Leftrightarrow ab = (S - N) \cup \{a, b\}$;

13) $aa = (S - N) \cup \{a\} \Leftrightarrow ab = N \cup \{b\}$;

14) $aa = (S - N) \cup \{b\} \Leftrightarrow ab = N \cup \{a\}$;

15) $aa = (S - N) \cup \{a, b\} \Leftrightarrow ab = N \cup \{a, b\}$;

16) $aa = S \Leftrightarrow ab = S$;

17) $aa = S \cup \{a\} \Leftrightarrow ab = S \cup \{b\}$;

18) $aa = S \cup \{b\} \Leftrightarrow ab = S \cup \{a\}$;

19) $aa = H \Leftrightarrow ab = H$.

COROLLARY 2.11. *If $H \in C(H, S(H), 2)$, $M(H) = \{a, b\}$ and $\{\varepsilon\} = Stab_L(a) = Stab_R(a) \subsetneq S(H)$, then $|H| = 4$ and H, up to isomorphisms, is one of the following hypergroups:*

$\Diamond$	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	b	ε,a	x,b
b	b	a	x,b	ε,a

$\Diamond$	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	b	ε,a,b	x,a,b
b	b	a	x,a,b	ε,a,b

◊	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	b	x,a	ε,b
b	b	a	ε,b	x,a

◊	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	b	x,a,b	ε,a,b
b	b	a	ε,a,b	x,a,b

◊	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	b	ε,x	ε,x
b	b	a	ε,x	ε,x

◊	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	b	ε,x,a	ε,x,b
b	b	a	ε,x,b	ε,x,a

o	ε	x	a	b
ε	ε	x	a	b
x	x	ε	b	a
a	a	b	H	H
b	b	a	H	H

Remark 2.12. If $H \in C(H, S(H), 2)$, $M(H) = \{a, b\}$ and $Stab_L(a) \cup Stab_R(a) \subsetneq S(H)$, then, from proposition (2.6), we obtain $[S(H) : Stab_L(a)] = [S(H) : Stab_R(a)] = 2$. In consequence, $Stab_L(a)$ and $Stab_R(a)$ are maximal subgroups of $S(H)$ and it is impossible that $Stab_L(a) \subsetneq Stab_R(a)$ or $Stab_R(a) \subsetneq Stab_L(a)$.

Therefore, in order to determine the structure of the hypergroups in the class $C(H, S(H), 2)$ we have to examine only the following two cases:

I) $Stab_R(a) \cup Stab_L(a) \subsetneq S(H)$, with $Stab_L(a) - Stab_R(a) \neq \emptyset$ and $Stab_R(a) - Stab_L(a) \neq \emptyset$;

II) $Stab_L(a) = Stab_R(a) = S(H)$.

Now, we prove the

THEOREM 2.13. *If H is a hypergroup in $C(H, S(H), 2)$ and H satisfies the conditions (I), then H is isomorphic to one of the* *hypergroups described in example (2.3).*

Proof. We have that $Stab_L(a) \cup Stab_R(a) \neq S(H)$. Then, taking $x \in S(H) - (Stab_R(a) \cup Stab_L(a))$ and $x' \in Stab_L(a) - Stab_R(a)$, we obtain $ax = \{b\} = xa$, $bx = \{a\} = xb$, $x'a = \{a\}$ and $ax' = \{b\}$.

For lemma (2.5) (5), (7) we also have $aa = ab = ba = bb$. Moreover $S(H) \subset aa$ and $aa \cap \{a, b\} \neq \emptyset \Leftrightarrow \{a, b\} \subset aa$. Consequently, $aa = ab = ba = bb = S(H)$ or $aa = ab = ba = bb = H$ and so H is, unless isomorphisms, one of the hypergroups of example (2.3).

The next lemmas allow us to determine the hypergroups $H \in C(H, S(H), 2)$ such that $|P(H)| = 4$ and $Stab_L(a) = Stab_R(a) = S(H)$. Besides we observe that the cases $|P(H)| = 2$ and $|P(H)| = 3$ have been studied and solved in the papers [7], [5].

LEMMA 2.14. *Let H be a hypergroup and h be a subhypergroup of H such that $H - h = \{a, b\}$, $ah = ha = \{a\}$, $bh = hb = \{b\}$ and $|P(H)| = 4$. Then we have:*

1) $\forall (z, w) \in (H - h)^2$, $zw \cap h \neq \emptyset \Leftrightarrow h \subset zw$;

2) $\forall (z, w) \in (H - h)^2$, $zw \in \{\{a, b\}, h \cup \{a\}, h \cup \{b\}, H\}$;

3) If $aa = \{a, b\}$ *(respect.* $bb = \{a, b\}$*), then* $ab \in \{H, h \cup \{b\}\} \ni ba$ *(respect.* $ab \in \{H, h \cup \{a\}\} \ni ba$*);*

4) If $aa = h \cup \{a\}$ *or* $bb = h \cup \{b\}$, *then* $ab = ba = H$ *and* $bb \neq aa$.

5) If $aa = h \cup \{b\}$ *(respect.* $bb = h \cup \{a\}$*) then:*

I) $b \in ab \Leftrightarrow b \in ba$ *(respect.* $a \in ab \Leftrightarrow a \in ba$*);*

II) $h \subset ab \Leftrightarrow h \subset ba$;

III) $ab = \{a, b\} \Leftrightarrow ba = \{a, b\}$;

IV) $ab = h \cup \{a\} \Leftrightarrow ba = h \cup \{a\}$ *(respect.* $ab = h \cup \{b\} \Leftrightarrow ba = h \cup \{b\}$*);*

V) $ab \in \{aa, H\} \Leftrightarrow ba \in \{aa, H\}$ *(respect.* $ab \in \{bb, H\} \Leftrightarrow ba \in \{bb, H\}$*).*

Proof. (1) If we take $x \in zw \cap h$, then we obtain $h = xh \subset zwh = zw$.

(2) Since $|P(H)| = 4$, $\forall (z, w) \in (H - h)^2$, one has $|zw| > 1$. If $zw \cap h = \emptyset$, then one obtains $zw = \{a, b\}$. Otherwise, if $zw \cap h \neq \emptyset$, then, from (1) one has $h \subset zw$. If $zw = h$, then $\{z\} = zh = z(zw) = (zz)w$ and so $\forall u \in zz$, $uw = \{z\}$. Since $w \in \{a, b\}$ and $|P(H)| = 4$, one has $u \in h$, whence $zz \subset h$ and $\{z\} = (zz)w \subset hw = \{w\}$, i.e. $z = w$. If $z = w = a$, then $aa = h$ with $a \notin ab$, because if $a \in ab$, then $h = aa \subset a(ab) = (aa)b = hb = \{b\}$. Besides $H = aH = ah \cup a(H - h) = \{a\} \cup aa \cup ab = \{a\} \cup h \cup ab$ and so $b \in ab$, with $a \notin ab$. Moreover $|ab| > 1$, and thus $ab \cap h \neq \emptyset$ and, from (1), $ab = h \cup \{b\}$. Finally, we obtain $(aa)b \neq a(ab)$ which is impossible.

If $z = w = b$, reasoning in a similar way, we obtain again an absurdity. Therefore one has $h \subset zw$.

(3) Suppose $aa = \{a, b\}$, we have $H = aH = \{a\} \cup aa \cup ab = \{a, b\} \cup ab$ and analogously $H = \{a, b\} \cup ba$. Then $h \subset ab \cap ba$.

If $ab = h \cup \{a\}$, then we obtain $h \subset h \cup bb \subset ab \cup bb = \{a, b\}b = (aa)b = a(ab) = \{a, b\}$, which is a contradiction. Thus, from (2), $ab \in \{h \cup \{b\}, H\}$. In the same way, one proves that $ba \in \{h \cup \{b\}, H\}$.

(4) Let $aa = h \cup \{a\}$. We have $H = aH = h \cup \{a\} \cup ab$ and so $b \in ab$, whence, from (2), we obtain $ab \in \{\{a, b\}, h \cup \{b\}, H\}$.

But if $ab \in \{\{a, b\}, h \cup \{b\}\}$, one has $(aa)b \neq a(ab)$ and thus $ab = H$. In an analogous way, one sees that $ba = H$.

Finally, if $bb = aa = h \cup \{a\}$, then we obtain $(ab)b \neq a(bb)$. Thus $aa \neq bb$. We arrive at the same conclusions, if we suppose $bb = h \cup \{b\}$.

(5) If $aa = h \cup \{b\}$, then we have:

$\{a\} \cup ab = a(h \cup \{b\}) = a(aa) = (aa)a = (h \cup \{b\})a = \{a\} \cup ba$, whence (I) and (II) follow.

From (2), we also obtain (III) and (IV). Finally, (V) is a consequence of (I) and (II).

LEMMA 2.15. *Let H be a hypergroup and h be a subhypergroup of H such that $H - h = \{a, b\}$, $ah = ha = \{a\}$, $bh = hb = \{b\}$, $|P(H)| = 4$. Then we have:*

1) $[aa = \{a, b\}, ab = ba = H] \Rightarrow bb \neq h \cup \{a\}$;

2) $[aa = \{a, b\}, (ab = h \cup \{b\} \text{ or } ba = h \cup \{b\})] \Rightarrow bb = H$;

3) $[aa = h \cup \{b\}, ab = ba = \{a, b\}] \Rightarrow bb \in \{h \cup \{a\}, H\}$;

4) $[aa = h \cup \{b\}, ab = ba = h \cup \{a\}] \Rightarrow bb \in \{\{a, b\}, h \cup \{a\}, H\}$;

5) $[aa = h \cup \{b\}, ab = ba = H] \Rightarrow bb \in \{h \cup \{a\}, H\}$;

6) $[aa = h \cup \{b\}, ab \in \{h \cup \{b\}, H\} \ni ba, ab \neq ba] \Rightarrow bb = H$.

Proof. (1) If $bb = h \cup \{a\}$, then $(ab)b \neq a(bb)$.

(2) Suppose $ab = h \cup \{b\}$. If $bb = h \cup \{b\}$, then $(aa)b \neq a(ab)$, while if $bb \in \{\{a, b\}, h \cup \{a\}\}$, then $(ab)b \neq a(bb)$. So, for the lemma (2.14) (2), obtain that $bb = H$. An analogous reasoning can be done when $ba = h \cup \{b\}$.

(3) Certainly $h \subset bb$ and supposing that $bb = h \cup \{b\}$, one has that $(aa)b \neq a(ab)$, whence, for lemma (2.14) (2), one obtains that $bb \in \{h \cup \{a\}, H\}$.

(4) If $bb = h \cup \{b\}$, then $(aa)b \neq a(ab)$ and so, taking in account the usual lemma (2.14) (2), we obtain that $bb \in \{\{a, b\}, h \cup \{a\}, H\}$.

(5) If $ab = ba = H$ and $bb \in \{\{a, b\}, h \cup \{b\}\}$, one has $(aa)b \neq a(ab)$ and using again lemma (2.14), we deduce that $bb \in \{h \cup \{a\}, H\}$.

(6) Let $ab = h \cup \{b\}$ and $ba = H$ (analogously, one can treat the case $ab = H$ and $ba = h \cup \{b\}$). We obtain $H = Hb = hb \cup (H - h)b = \{b\} \cup ab \cup bb = h \cup \{b\} \cup bb$, whence $a \in bb$, and, for the lemma (2.14) (2), $bb \in \{\{a, b\}, h \cup \{a\}, H\}$. Finally, if $bb = \{\{a, b\}$, then $(aa)b \neq a(ab)$ and if $bb = h \cup \{a\}$, then $(cc)c \neq c(cc)$.

Remark 2.16. The lemma (2.15) can be stated and proved again, exchanging a with b.

In the end of this section we establish:

THEOREM 2.17. *If* $H \in C(H, S(H), 2)$, $M(H) = \{a, b\}$, $Stab_L(a) = Stab_R(a) = S(H)$ *and* $|P(H)| = 4$, *then* H *is isomorphic to one of the hypergroups described in the example (2.4).*

Proof. In virtue of the lemma (2.14) (2), $aa \in \{\{a, b\}, S(H) \cup \{a\}, S(H) \cup \{b\}, H\}$.

If $aa = \{a, b\}$, as a consequence of lemmas (2.14) (3) and (2.15) (1) (2), we can affirm that, up to isomorphims, $H \in \{M_1, \dots, M_6\}$ (see example (4)).

If $aa = S(H) \cup \{a\}$, from lemma (2.14) (4), H is unless isomorphisms one of the following hypergroups M_7, M_8, M_9.

If $aa = S(H) \cup \{b\}$, by using lemmas (2.14) (5) and (2.15) (3), (4), (5), (6), we obtain that H is isomorphic to one of the hypergroups M_i of the example (2.4), for $i \in \{10, \ldots, 18\}$.

If $aa = H$ and $bb \in \{\{a, b\}, S(H) \cup \{a\}, S(H) \cup \{b\}\}$, taking in account the remark (2.16), the possible hyperproducts zw, with $(z, w) \in M(H)^2$, can be represented by one of the following ten tables:

⊗	a	b
a	H	$S(H)\cup\{a\}$
b	$S(H)\cup\{a\}$	$\{a,b\}$

⊗	a	b
a	H	H
b	$S(H)\cup\{a\}$	$\{a,b\}$

⊗	a	b
a	H	$S(H)\cup\{a\}$
b	H	$\{a,b\}$

⊗	a	b
a	H	H
b	H	$\{a,b\}$

⊗	a	b
a	H	H
b	H	$S(H)\cup\{b\}$

⊗	a	b
a	H	$\{a,b\}$
b	$\{a,b\}$	$S(H)\cup\{a\}$

⊗	a	b
a	H	$S(H)\cup\{b\}$
b	$S(H)\cup\{b\}$	$S(H)\cup\{a\}$

⊗	a	b
a	H	H
b	H	$S(H)\cup\{a\}$

⊗	a	b
a	H	H
b	$S(H)\cup\{a\}$	$S(H)\cup\{a\}$

⊗	a	b
a	H	$S(H)\cup\{a\}$
b	H	$S(H)\cup\{a\}$

The ten corresponding hypergroups are respectively isomorphic to the hypergroups M_1, M_2, M_3, M_6, M_9, M_{11}, M_{14}, M_{16}, M_{17} and M_{18}.

Finally, supposing that $aa = bb = H$, it is easy to verify that one obtains hypergroups which are isomorphic to the hypergroups M_i for $i \in \{19, \ldots, 24\}$.

COROLLARY 2.18. *The hypergroups of the class* $C(H, S(H), 2)$

such that $|P(H)| = 4$ *are those constructed in the examples (2.1), (2.2), (2.3), (2.4).*

Proof. The hypergroups given in examples (2.1), (2.2), (2.3), (2.4) are hypergroups which belong to the class $C(H, S(H), 2)$, with $|P(H)| = 4$.

On the converse, if $H \in C(H, S(H), 2)$ and $Stab_L(a) \subsetneq S(H)$ or $Stab_R(a) \subsetneq S(H)$, then, from the theorems (2.8), (2.10), (2.13), H is isomorphic to one of the hypergroups described in examples (2.1), (2.2), (2.3). On the other hand, if H is such that $Stab_L(a) = Stab_R(a) = S(H)$ and $|P(H)| = 4$, then H is isomorphic to one of the hypergroups of the example (2.4).

3. The case $C(H, S(H), 3)$.

In this section we shall indicate with $C(H, S(H), 3)$ the class of the hypergroups H, such that $S(H) \neq \emptyset$ and $|M(H)| = 3$.

We begin by giving some examples:

EXAMPLE 3.1. Let G be a group, let g be a subgroup of G of index $[G : g] = 2$ and let a, b, c be three distinct elements such that $G \cap \{a, b, c\} = \emptyset$. If we put $H = G \cup \{a, b, c\}$, we can define on H the following hyperoperation $\circ$:

$(H, \circ) =$

$$\begin{cases} x \circ y = \{xy\} & \text{if } (x, y) \in G^2; \\ x \circ a = a \circ x = \{a\}, x \circ b = b \circ x = \{b\} & \text{if } x \in g; \\ x \circ a = a \circ x = \{b\}, x \circ b = b \circ x = \{a\} & \text{if } x \in G - g; \\ x \circ c = c \circ x = \{c\} & \text{if } x \in G; \\ a \circ a = b \circ b = \{a\}, a \circ b = b \circ a = \{b\} \text{ and} & \\ c \circ c = \{c\}, a \circ c = c \circ a = c \circ b = b \circ c = H. & \end{cases}$$

It is easy to verify that $(H, \circ)$ is a hypergroup. Moroever if $\forall (z, w) \in H^2 - \{(a, a), (a, b), (b, a), (b, b)\}$, we put $z \star w = z \circ w$

and $a \star a = b \star b = \{b\}$, $a \star b = b \star a = \{a\}$, then the hypergroup $(H, \star)$ is isomorphic to $(H, \circ)$.

We denote with H' the symetric hypergroup of H.

EXAMPLE 3.2. In the same hypothesis of example (3.1), we can define on $K = G \cup \{a, b, c\}$ the hyperoperation $\otimes$, in the following way:

$(K, \otimes) =$

$$\begin{cases} x \otimes y = \{xy\} & \text{if } (x, y) \in G^2; \\ a \otimes x = \{a\}, b \otimes x = \{b\} & \text{if } x \in G; \\ x \otimes a = \{a\}, x \otimes b = \{b\} & \text{if } x \in g; \\ x \otimes a = \{b\}, x \otimes b = \{a\} & \text{if } x \in G - g; \\ x \otimes c = c \otimes x = \{c\} & \text{if } x \in G; \\ a \otimes a = a \otimes b = \{a\}, b \otimes a = b \otimes b = \{b\} \text{ and} & \\ c \otimes c = \{c\}, a \otimes c = c \otimes a = c \otimes b = b \otimes c = K. & \end{cases}$$

One can easily prove that $(K, \otimes)$ is a hypergroup.

We denote with K' the symmetric hypergroup of K.

We give some preliminary results, which will be very useful for determining the hypergroups H in the class $C(H, S(H), 3)$, such that $|P(H)| = 4$.

PROPOSITION 3.3. *If H is a hypergroup such that $S(H) \neq \emptyset \neq M(H)$ and $(x, a, b) \in S(H) \times M(H) \times M(H)$ is a tern of elements such that $xa = \{b\}$ (respect. $ax = \{b\}$), then, $\forall z \in M(H)$, one has $|az| = |bz|$ (respect. $|za| = |zb|$).*

Proof. If $a = b$, then the thesis is obvious. Hence we suppose that $a \neq b$ and let $f_x : az \to bz$ and $f_{x^{-1}} : bz \to az$ be the functions such that:

$$f_x(u) = xu, \; f_{x^{-1}}(v) = x^{-1}v, \quad \forall u \in az, \forall v \in bz.$$

The two functions are well defined, because $xu \subset x(az) = (xa)z = bz$ and $x^{-1}v \subset x^{-1}(bz) = (x^{-1}b)z = az$.

Besides we have $(f_{x^{-1}} \circ f_x)(u) = f_{x^{-1}}(f_x(u)) = f_{x^{-1}}(xu) = x^{-1}(xu) = (x^{-1}x)u = \varepsilon u = u$ and analogously $(f_x \circ f_{x^{-1}})(v) = v$. Therefore f_x and $f_{x^{-1}}$ are bijective and $|az| = |bz|$.

We prove now the following:

PROPOSITION 3.4. *If* $H \in C(H, S(H), 3)$ *and* $|P(H)| = 4$, *then* $\forall a \in M(H)$, $|S(H)a| \neq 3$ *and* $|aS(H)| \neq 3$.

Proof. If there exists $a \in M(H)$ such that $|S(H)a| = 3$, then $S(H)a = M(H)$ and so there exists $\{x, y\} \subset S(H)$ such that $xa = \{b\}$, $ya = \{c\}$.

Owing to the proposition (3.3), we have $|ba| = |aa| = |ca|$ and if $|aa| = 1$, then $a \in S_R(H) = S(H)$ (see [5]) and this fact is impossible. While, if $|aa| > 1$, since $|P(H)| = 4$, we obtain $b \in S_R(H) = S(H)$ or $c \in S_R(H) = S(H)$ and this is again impossible. Consequently $|aS(H)| \neq 3$.

In the same manner, we can prove that $|aS(H)| \neq 3$.

PROPOSITION 3.5. *Let* $H \in C(H, S(H), 3)$, $M(H) = \{a_1, a_2, a_3\}$ *and* $|P(H)| = 4$. *If* $\exists i \in I_3 = \{1, 2, 3\}$ *such that* $S(H)a_i \neq \{a_i\}$ *then* $\forall \sigma \in S_3$, $\exists r \in I_3$ *such that* $|a_r a_{\sigma(r)}| = 1$.

Proof. Let $x \in S(H)$ such that $xa_i = \{a_k\}$ with $i \neq k$. Assuming that $\{a_j\} = M(H) - \{a_i, a_k\}$, by proposition (3.3), we have $|a_i a_i| = |a_k a_i|$, $|a_i a_k| = |a_k a_k|$, $|a_i a_j| = |a_k a_j|$. If there exists $\sigma \in S_3$ such that $|a_r a_{\sigma(r)}| > 1$, $\forall r \in I_3$, then, supposing that σ is the identity of S_3, we obtain $|a_i a_i| = |a_k a_i| > 1$, $|a_j a_j| > 1$, $|a_k a_k| = |a_i a_k| > 1$. Hence $|P(H)| \geq 5$, which is a contradiction.

We obtain the same conclusions, if we change in all the possible ways the permutation σ in S_3.

As an immediate consequence of the previous proposition, we have the following:

COROLLARY 3.6. *Let* $H \in C(H, S(H), 3)$, $M(H) = \{a_1, a_2, a_3\}$ *and* $|P(H)| = 4$. *If there exists* $\sigma \in S_3$ *such that* $\forall i \in I_3$, $|a_i a_{\sigma(i)}| > 1$, *then* $S(H)a_i = \{a_i\}$, $\forall i \in I_3$.

PROPOSITION 3.7. *Let* $H \in C(H, S(H), 3)$ *and* $|P(H)| = 4$.

If $a \in M(H)$ is such that $|S(H)a| = 2$ (respect. $|aS(H)| = 2$), then $aS(H) = \{a\}$ or $aS(H) = S(H)a$ (respect. $S(H)a = \{a\}$ or $aS(H) = S(H)a$).

Proof. By proposition (3.4), we have $|aS(H)| \neq 3$.

If $|aS(H)| = 1$, then, obviously, $aS(H) = \{a\}$. Then, suppose $|aS(h)| = 2$ and $aS(H) \neq S(H)a$.

By hypothesis, we can suppose that $M(H) = \{a, b, c\}$, $S(H)a = \{a, b\}$ and $aS(H) = \{a, c\}$. We have $S(H)c = \{c\}$ and $bS(H) = b$, so there exists $\{x, y\} \subset S(H)$ such that $xa = \{b\}$, $ay = \{c\}$ and therefore $\{c\} = xc = x(ay) = (xa)y = by = \{b\}$, that is a contradiction. Consequently $aS(H) = S(H)a$.

Remark 3.8. If $H \in C(H, S(H), 3)$, $|P(H)| = 4$ and $M(H) = \{a, b, c\}$ then proposition (3.7) allows us to distinguish the following four cases:

1) $S(H)a = aS(H) = \{a, b\}$ and $S(H)c = cS(H) = \{c\}$;

2) $S(H)a = \{a, b\}$, $S(H)c = \{c\}$ and $zS(H) = \{z\}$, $\forall z \in M(H)$;

3) $aS(H) = \{a, b\}$, $cS(H) = \{c\}$ and $S(H)z = \{z\}$, $\forall z \in M(H)$;

4) $S(H)z = zS(H) = \{z\}$, $\forall z \in M(H)$.

In the following, we shall deal with the cases (1), (2). The case (3) can be treated as the case (2). The authors will study the case (4) in a paper in preparation (see [4]).

We begin this study with two lemmas:

LEMMA 3.9. *Let $H \in C(H, S(H), 3)$ such that $M(H) = \{a, b, c\}$, $|P(H)| = 4$, $S(H)a = \{a, b\}$ (respect. $aS(H) = \{a, b\}$) and $S(H)c = cS(H) = \{c\}$. Therefore we have:*

1) $ca = cb$ (respect. $ac = bc$) and $|aa| = |ba| = |ab| = |bb| = |cc| = 1$, $|ac| = |bc| > 1$, $|ca| = |cb| > 1$;

2) $cc = \{c\}$;

3) $S(H) \cup \{a, b\} \subset ca = cb$ (respect. $S(H) \cup \{a, b\} \subset ac = bc$);

4) $S(H) \cap aa = S(H) \cap ab = S(H) \cap ba = S(H) \cap bb = \emptyset$;

5) $\{c\} \notin \{aa, ab, ba, bb\}$;

6) $ca = cb = H$ *(respect.* $ac = bc = H$*).*

Proof. 1) There exists $x \in S(H)$ such that $xa = \{b\}$ and so $ca = (cx)a = c(xa) = cb$. Moroever, by proposition (3.3), we have $|aa| = |ba|$, $|ab| = |bb|$, $|ac| = |bc|$.

If $|aa| = |ba| > 1$, since $|P(H)| = 4$, we obtain $|ab| = |bb| = |ac| = |bc| = 1$, $|cb| = |cc| > 1$ and $|ca| = 1$, but this is an absurdity, because $ca = cb$.

Analogously, we obtain a contradiction, if we suppose that $|ab| = |bb| > 1$. Therefore, we have $|ac| = |bc| > 1$, $|aa| = |ba| = |ab| = |bb| = |cc| = 1$, $|ca| = |cb| > 1$.

If we assume that $aS(H) = \{a, b\}$, one can show that $ac = bc$ and that the same equalities on the sizes of the hyperproducts are valid.

2) Clearly $[S(H) : Stab_L(a)] = |S(H)a| = 2$ and so $|S(H)| \geq 2$. Now, if $x \in cc \cap S(H)$, then we have $S(H) = S(H)x \subset S(H)cc = cc$ and this is impossible because, by (1), $|cc| = 1$.

Besides, if $cc = \{a\}$, taking $x \in S(H)$ such that $xa = b$ we obtain $(xc)c \neq x(cc)$; while if $cc = \{b\}$, we have $(x^{-1}c)c \neq x^{-1}(cc)$. In both cases, we arrive at a contradiction and consequently, $cc = \{c\}$.

An analogous proof can be done, if $aS(H) = \{a, b\}$.

3) From (1), $ca = cb$, and thus we have

$$S(H) \cup \{a, b\} \subset H = cH = c(S(H) \cup M(H)) =$$

$$= cS(H) \cup ca \cup cb \cup cc = \{c\} \cup ca \cup cb \cup \{c\} = \{c\} \cup ca,$$

and finally $S(H) \cup \{a, b\} \subset ca = cb$.

In a similar way, it is possible to prove that if $aS(H) = \{a, b\}$ then $S(H) \cup \{a, b\} \subset ac = bc$.

4) As a consequence of (3), we have $\{a, b\} \subset ca = cb$ and if we suppose that $S(H) \cap aa \neq \emptyset$, taking in account that $|aa| = 1$, we obtain $aa \subset S(H)$, whence $aa \subset (ca)a = c(aa) \subset cS(H) = \{c\}$. Therefore $\{c\} = aa \subset S(H)$, which is impossible.

Thus $S(H) \cap aa = \emptyset$. Analogously, one can prove that $S(H) \cap ab = S(H) \cap ba = S(H) \cap bb = \emptyset$.

5) From (3), $S(H) \cup \{a, b\} \subset ca = cb$ and if we suppose that $aa = \{c\}$, then obtain $\{a, b, c\} = S(H)a \cup aa \subset (S(H) \cup \{a\})a \subset (ca)a = c(aa) = cc$.

If $ab = \{c\}$, then $\{a, b, c\} = S(H)b \cup ab = (S(H) \cup \{a\})b \subset (ca)b = c(ab) = cc$.

If $ba = \{c\}$, then $\{a, b, c\} = S(H)a \cup ba = (S(H) \cup \{b\})a \subset (cb)a = c(ba) = cc$.

If $bb = \{c\}$, then $\{a, b, c\} = S(H)b \cup bb = (S(H) \cup \{b\})b \subset (cb)b = c(bb) = cc$.

In every case, we obtain a contradiction, since $|cc| = 1$.

6) From (5), $\{c\} \cap (S(H)a \cup aa \cup ba) = \emptyset$ and, since $c \in H = Ha = (S(H) \cup M(H))a = S(H)a \cup aa \cup ba \cup ca$, we obtain $c \in ca$ and, by (3), we deduce $ca = cb = H$. Under the hypothesis $aS(H) = \{a, b\}$, one proves that $ac = bc = H$.

We need another lemma:

LEMMA 3.10. *If $H \in C(H, S(H), 3)$, $M(H) = \{a, b, c\}$, $S(H)a = \{a, b\}$ and $S(H)c = \{c\}$ (respect. $aS(H) = \{a, b\}$ and $cS(H) = \{c\}$), then $Stab_L(b) = Stab_L(a)$ (respect. $Stab_R(b) = Stab_R(a)$).* *Proof.* If $x \in Stab_L(b)$, then $xb = b$, $xc = c$, and, since $a \in xH = x(S(H) \cup M(H)) = xS(H) \cup xa \cup xb \cup xc = S(H) \cup xa \cup \{b, c\}$, we have $xa = \{a\}$ and so $x \in Stab_L(a)$. Therefore $Stab_L(b) \subset Stab_L(a)$. In the same way, one can prove the other inclusion and thus $Stab_L(a) = Stab_L(b)$.

If $aS(H) = \{a, b\}$ and $cS(H) = \{c\}$, then we obtain, with similar reasonings, $Stab_R(b) = Stab_R(a)$.

Using the previous lemmas we get:

THEOREM 3.11. *If $H \in C(H, S(H), 3)$, $M(H) = \{a, b, c\}$, $|P(H)| = 4$, $S(H)a = aS(H) = \{a, b\}$ and $S(H)c = cS(H) = \{c\}$, then H is isomorphic to one of the hypergroups of example (3.1).*

Proof. By Lemma (3.9), $aa \in \{\{a\}, \{b\}\}$, $cc = \{c\}$ and $ca = cb = H = ac = bc$.

Let $aa = \{a\}$. There exists $(x, y) \in S(H)^2$ such that $xa = \{b\} = ay$, and so $ab = a(ay) = (aa)y = ay = \{b\}$, $ba = (xa)a = x(aa) = xa = \{b\}$. From lemma (3.10), $y \notin Stab_R(b) = Stab_R(a)$

and thus $bb = (xa)(ay) = (x(aa))y = (xa)y = by = \{a\}$. Besides if $Stab_L(a) - Stab_R(a) \neq \emptyset$, taking $x \in Stab_L(a) - Stab_R(a)$, we obtain $xa = \{a\}$ and $ax = \{b\}$ and so $\{a\} = aa = a(xa) = (ax)a = ba = \{b\}$, which is impossible. In consequence $Stab_L(a) - Stab_R(a) = \emptyset$.

Analogously one can show that $Stab_R(a) - Stab_L(a) = \emptyset$, and so, $Stab_L(a) = Stab_R(a)$. If we put $g = Stab_L(b) = Stab_L(a) = Stab_R(a) = Stab_R(b)$, as $[S(H) : g] = |S(H)a| = 2$, we can affirm that H is isomorphic to one of the hypergroups constructed in example (3.1).

We arrive at the same conclusion if we start from $aa = \{b\}$.

At last we prove

THEOREM 3.12. *If $H \in C(H, S(H), 3)$, $M(H) = \{a, b, c\}$, $|P(H)| = 4$, $S(H)a = \{a, b\}$, $S(H)c = cS(H) = \{c\}$ and $aS(H) = \{a\}$, $bS(H) = \{b\}$, then H is isomorphic to one the hypergroups of example (3.2).*

Proof. We prove in successione the following points:

1) $aa = ab = \{a\}$, $ba = bb = \{b\}$, $cc = \{c\}$;

2) $ca = cb = ac = bc = H$.

(1) From lemma (3.9) (1), (2), (4), (5), we have $|aa| = 1$, $cc = \{c\}$, $S(H) \cap aa = \emptyset$ and $aa \neq \{c\}$. If $aa = \{b\}$, taken $x \in S(H)$ such that $xa = \{b\}$, one has $ab = a(xa) = (ax)a = aa = \{b\}$, and, using the lemma (3.10), it follows that $ba = (xa)a = x(aa) = xb = \{a\}$, whence $(aa)a \neq a(aa)$, which is absurd. Therefore $aa = \{a\}$, and consequently $ab = a(xa) = (ax)a = aa = \{a\}$, $ba = (xa)a = x(aa) = xa = \{b\}$ and $bb = b(xa) = (bx)a = ba = \{b\}$.

(2) By lemma (3.9) (6), we have $ca = cb = H$. Moreover, for (1), $H = aH = a(S(H) \cup M(H)) = aS(H) \cup aa \cup ab \cup ac = \{a\} \cup ac$ and so $S(H) \cup \{b, c\} \subset ac$. In the same way, one can prove that $S(H) \cup \{a, c\} \subset bc$. Besides, if $ac = \{b, c\} \cup S(H)$ and $bc = \{a, c\} \cup S(H)$, we obtain respectively $(aa)c \neq a(ac)$, and $(bb)c \neq b(bc)$. Therefore $ac = bc = H$.

Finally, assuming that $g = Stab_L(a) = Stab_L(b)$, we have $[S(H) : g] = |S(H)a| = 2$ and H is isomorphic to one of the hypergroups of the example (3.2).

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