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IS THE FOURIER THEORY OF HEAT PROPAGATION PARADOXICAL?

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Il lavoro consiste in un tentativo di difesa della classica teoria di Fourier della propagazione del calore, che è stata accusata di produrre il paradosso secondo il quale il calore si propaga con velocità infinita. Lo scopo è quello di provare che questa accusa, quando la teoria di Fourier venga correttamente interpretata, è infondata.

This paper is concerned with an attempt of defence of the classical theory of Fourier of heat propagation, which has been accused to produce the paradox according to which heat propagates with an infinite speed. The aim is to prove that this accusation, when the Fourier theory is properly interpreted, is unfounded.

The author is particularly happy to dedicate this paper to his dear friend Benedetto Pettineo on the occasion of his 70th anniversary. His *esprit de finesse* will make him to appreciate what is written (...and what is not written) in this paper.

1. The paradox of the Fourier theory.

The Fourier theory of heat conduction [4] has been, during this last half century, subjected to a very serious accusation and alternative theories, starting from 1949 with a celebrated paper by Carlo Cattaneo

[1], have been proposed. The criticism brought to Fourier's theory consists mainly in the remark that, according to this theory, the speed of propagation of heat should be infinite. This is contradicted by the most elementary experiences. An excellent survey of the criticisms to Fourier and of the proposed alternative theories until 1989 can be found in the paper [8] by D.D.Joseph and L. Preziosi.

In order to examine concretely this subject we restrict ourselves to a very simple case. We consider an indefinitely long rectilinear wire which we represent by the real axis such that at the instant $t = 0$ the temperature, in a fixed system of units, is 1 for $x = 0$ and is 0 for $x \neq 0$. We use the following notations:

$u(x, t)$ =temperature of the point x at the instant $t \geq 0$;

q =thermal flux;

χ =coefficient of thermal conductivity;

ρ =density of the wire;

γ =specific heat of the wire.

The Fourier theory hinges on two axioms. The first of them is originated from the experimental observations according to which: i) the heat flows from a point x of the wire to a colder one $x + dx$; ii) the amount of heat which flows from x to $x + dx$ in the unit of time, i.e. *the thermal flux*, is proportional to the difference of the temperature in x and in $x + dx$ and inversely proportional to dx, i.e.

$$q = \chi \frac{u(x, t) - u(x + dx, t)}{dx}.$$

Hence

$$q = -\chi \frac{\partial u}{\partial x}. \tag{1.1}$$

Eq.(1.1) expresses the first of the mentioned axioms.

The second axiom concerns the *conservation of the quantity of heat*. Let us consider two points x and $x + dx$ of the wire. The rate of heat in the interval of time from t to $t + dt$ relative to the piece of wire

$(x, x+dx)$, is given by the increment of the temperature $\frac{\partial u}{\partial t} dt$ times the mass of $(x, x+dx)$, i.e. ρdx, times the specific heat γ of the wire. If the quantity of heat in $(x, x+dx)$ is conserved, the rate of this quantity must be equal to the thermal flux which enters in x diminished by the thermal flux which comes out from $x+dx$, this difference being multiplied by dt:

$$\gamma\rho \frac{\partial u}{\partial t} dx = q(x,t) - q(x+dx,t).$$

Then

$$\gamma\rho \frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{1.2}$$

Eq. (1.2) expresses the *axiom of the conservation of the quantity of heat.*

From (1.1) and (1.2), eliminating q and introducing the coefficient

$$D = \frac{\chi}{\gamma\rho},$$

we get the classical Fourier equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{D}\frac{\partial u}{\partial t} = 0. \tag{1.3}$$

From the theory of this equation and knowing that $u(x,0) = 0$ $(x \neq 0)$, $u(0,0) = 1$, we obtain

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}} e^{-\frac{x^2}{4Dt}} \qquad (t > 0)(^1). \tag{1.4}$$

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(1) It is obvious that for $x \neq 0$

$$\lim_{t \to 0^+} u(x,t) = 0 \tag{*}$$

uniformly in any closed interval which does not contain $x = 0$.
The condition $u(0,0) = 1$ is satisfied in the sense that for any interval (a,b) such that $a < 0, b > 0$

$$\lim_{t \to 0^+} \int_a^b u(x,t) dx = 1. \tag{**}$$

From (1.4) we see that for $|x|$ as large as we wish and for any $t > 0$ we have $u(x,t) > 0$. Hence heat has propagated from $x = 0$ to x in an interval of time no matter how small. This brings to the conclusion that the speed of propagation of heat is infinite.

Cattaneo does not hesitate to call this «*un aspetto paradossale della teoria classica della propagazione del calore*» ([1], p. 83) and reaffirms: «*...mi sembra ben certo che l'immediatezza della propagazione calorifica a distanza sia, almeno in linea di concetto, paradossale*». ([1], p. 86).

M.E. Gurtin and A.C. Pipkin, who, on their turn, have proposed in 1968 an alternative theory to Fourier's, write: «*This equation (i.e.(1.3)), which is parabolic, has a very unpleasant feature: a thermal disturbance at any point of the body is felt instantly at every other point; or in terms more suggestive than precise, the speed of propagation of disturbances is infinite*».. ([7], p. 113).

D.D. Joseph and L. Preziosi write: «*The diffusion equation (i.e. (1.3)) has the unphysical property that if a sudden change of temperature is made at some point on the body, it will be felt instantly everywhere, though with exponentially small amplitudes at distant points. In a loose manner of speaking, we may say that diffusion gives rise to infinite speeds of propagation*». ([8], p. 42).

2. Alternative propositions to the Fourier constitutive laws.

Cattaneo, after summarizing Fourier's theory, writes: «*Se nella teoria che abbiamo riassunto c'è un elemento incerto, mi pare ragionevole ricercarlo nella validità generale ed incondizionata dell'equazione* (1.1). ([1], p. 87).

Conditions (*), (**) are equivalent to the following

$$\lim_{t\to 0^+} \int_{-h}^{+h} \varphi(x)u(x,t)dx = \varphi(0)$$

for any $\varphi \in C^\infty(\mathbb{R})$ and any $h > 0$.

Actually axiom (1.1) is a result of experimental observations hence it is inevitably conditioned by the physical nature of the material under consideration and by the degree of refinement of the performed measurements.

Eq. (1.1) should be regarded like an *approximation* of the physical reality and for (1.1) a constitutive equation of the following kind should be substituted

$$q = -\chi \frac{\partial u}{\partial x} + R(x, t; u) \tag{2.1}$$

where $R(x, t; u)$ is a suitable *mixed functional* in the sense of Volterra, i.e. R is a functional of u in a given function class for any x and $t (x \in \mathbb{R}, t > 0)$.

Cattaneo firstly assumes

$$R = -\frac{\sigma}{\chi}\frac{\partial q}{\partial t} + \sigma^2 \frac{\partial^2 u}{\partial x \partial t}$$

where σ is a certain positive physical constant depending on the material under consideration. Afterwards he writes: *«approfitteremo della piccolezza del parametro σ per trascurare il termine che contiene il suo quadrato,conservando peraltro il termine in cui σ compare al primo grado»* ([1], p. 93) and he assumes as constitutive equation

$$q = -\chi \frac{\partial u}{\partial x} - \frac{\sigma}{\chi}\frac{\partial q}{\partial t}. \tag{2.2}$$

From (1.2) and (2.2) he gets

$$\sigma\gamma\rho \frac{\partial^2 u}{\partial t^2} - \chi^2 \frac{\partial^2 u}{\partial x^2} + \chi\gamma\rho \frac{\partial u}{\partial t} = 0. \tag{2.3}$$

This is a well known hyperbolic equation, the so called *telegraph equation,* which describes a waves propagation phenomenon where the propagation speed v is finite

$$v = \sqrt{\frac{\chi^2}{\sigma\gamma\rho}}.$$

Different from Fourier's theory, the unknown function $u(x, t)$ is now determined, like a solution of (2.2), *if both the functions* $u(0, x)$

and $u_t(0,x)$ *are known.* Cattaneo in his paper [1] tries to give a physical explanation to this fact.

A theory proposed by M.E. Gurtin and A.C. Pipkin [7] avoids this inconvenience by considering the *past thermal history* of the material system under consideration. Actually these Authors consider the following constitutive equations:

$$q(x,t) = -\int_{-\infty}^{t} a(t-\tau)u_x(x,\tau)d\tau \tag{2.4}$$

$$E(x,t) = b + cu(x,t) + \int_{-\infty}^{t} \beta(t-\tau)u(x,\tau)d\tau \tag{2.5}$$

where b and c are constants and $a(s)$ and $\beta(s)$ $(0 \leq s < +\infty)$ sufficiently smooth functions connected with the past thermal history of the body; c and $a(0)$ are supposed positive.

The functions $E(x,t)$ and $q(x,t)$ are related through the equation

$$\frac{\partial}{\partial t}E(x,t) = -\frac{\partial}{\partial x}q(x,t) + f(x,t) \tag{2.6}$$

where $f(x,t)$ is a known function which measures the interchange of heat between the material system and the external world.

Equations (2.3),(2.4) are obtained from more general non linear constitutive equations, by a *linearization process*, where, assuming

$$\delta = \sup_{\substack{x\in\mathbb{R} \\ -\infty<t<t_0}} \{|u(x,t)-u_0| + |u_x(x,t)|\} \tag{2.7}$$

(u_0 is a suitable constant), it is supposed that any quantity having the order of magnitude of $o(\delta)$ can be considered like *negligible* ([7], p. 124). From (2.5), by differentiating with respect to t and using (2.6), one gets

$$\begin{aligned} &-\frac{\partial}{\partial x}q(x,t) + f(x,t) = \\ &c\frac{\partial}{\partial t}u(x,t) + \beta(0)u(x,t) + \int_{-\infty}^{t} \beta'(t-\tau)u(x,\tau)d\tau. \end{aligned} \tag{2.8}$$

By assuming $\beta(s) \equiv 0$, $c = \gamma\rho$, $f(x,t) \equiv 0$, from (2.8) one gets (1.2). Eq. (2.4) written like (2.1) gives

$$R(x,t;u) = \chi \frac{\partial u}{\partial x} - \int_0^{+\infty} a(s) u_x(t-s) ds.$$

Let us assume $h > 0$ and

$$a(s) \begin{cases} = \dfrac{\chi}{h} e^{-\frac{s^2}{h-s}} & 0 \leq s < h \\ = 0 & s \geq h. \end{cases} \tag{2.9}$$

Then

$$\lim_{h \to 0^+} \left[\chi u(x,t) + \int_0^{+\infty} a(s) u_x(x, t-s) ds \right] =$$

$$\lim_{h \to 0^+} \left[\chi u(x,t) - \frac{\chi}{h} \int_0^{h} e^{-\frac{s^2}{h-s}} u_x(x, t-s) ds \right] =$$

$$\lim_{h \to 0^+} \left[\chi u(x,t) - \chi \int_0^{1} e^{-\frac{h\sigma^2}{1-\sigma}} u_x(x, t-h\sigma) d\sigma \right] = 0.$$

Hence, by assuming $a(s)$ given by (2.9), the Fourier law (1.1) is a limiting case of (2.4). On the other hand, by assuming

$$a(s) = \frac{\chi^2}{\sigma} \exp \frac{-\chi}{\sigma} s,$$

one easily obtains Eq. (2.2) of Cattaneo.

By differentiating (2.4) once respect to t and once respect to x and differentiating (2.8) with respect to t one gets, eliminating q,

$$\begin{aligned} & c\frac{\partial^2 u}{\partial t^2} - a(0)\frac{\partial^2 u}{\partial x^2} + \beta(0)\frac{\partial u}{\partial t} + \\ & \int_{-\infty}^{t} \beta'(t-\tau) u_t(x,\tau) d\tau - \int_{-\infty}^{t} a'(t-\tau) u_{xx}(x,\tau) d\tau = f_t(x,t). \end{aligned} \tag{2.10}$$

This is an integro-differential equation which assuming $f(x,t) \equiv 0$ contains as particular cases Eq. (1.3) [$c = 0$, $a(0) = \chi$, $\beta(0) = \gamma\rho$,

$\beta'(s) \equiv 0$, $a'(s) \equiv 0$] and Eq.(2.3) $\left[c = \frac{\sigma\gamma\rho}{\chi},\ a(0) = \chi,\ \beta(0) = \gamma\rho,\right.$ $\left.\beta'(s) \equiv 0,\ a'(s) \equiv 0\right]$. If $c > 0$ the speed of propagation of heat is finite like in the case of the *purely* differential equation (2.3).

It is known from the theory of integro-differential equations that in the case when the function

$$\int_{-\infty}^{0} \beta'(t-\tau)u_t(x,\tau)d\tau - \int_{-\infty}^{0} a'(t-\tau)u_{xx}(x,\tau)d\tau$$

is supposed either known or negligible(2), no initial conditions are needed for determining a solution of (2.10). If this hypothesis is not assumed the problem of the existence and of the uniqueness of a solution of (2.10) is still open (see [3]).

The theory of Gurtin and Pipkin is very attractive, however its application to practical problems looks very dubious.(3)

We restrict ourselves to consider here the theories by Cattaneo and by Gurtin and Pipkin. Actually, in our opinion, the theories of Cattaneo and of Gurtin and Pipkin, proposed like alternatives to Fourier's, are the most outstanding. The one by Cattaneo because was the first and because substituted for the Fourier parabolic equation a hyperbolic one. The theory of Gurtin and Pipkin is important for its generality and for imbedding the problem into the context of hereditary phenomena. For other proposed theories we refer the reader to the paper [8] .

3. Proposition for a correct interpretation of Fourier's theory.

When an experimenter measures by his instruments some physical quantity in a fixed system of units, the statement «*the value of this*

(2) This hypothesis was assumed by Volterra in his theory of hereditary elasticity ([12], p. 92).

(3) When the paper [7] by Gurtin and Pipkin appeared I informed about the proposed new heat theory a friend of mine who is an engineer,specialised in projecting heating systems for new buildings, but who has a real interest in new theoretical advances. He looked startled and told me: «*How can I project the heating system for a building taking into account its past thermal history if the building has not yet been constructed?*

quantity is α» means that to the quantity under consideration can be attributed any value of the interval $(\alpha - \varepsilon, \alpha + \varepsilon)$, where ε is some positive constant, which depends on the degree of refinement of the instruments, which have been used for performing the measurement. More small is ε more refined are the instruments. On the other hand, α is expressed by some decimal number where the digits which follow the point are n. The positive integer n can be very large but is bounded and depends either on the computational facilities which the experimenter has at his disposal or on the accuracy he has chosen for his measurements. This implies that ε is not less than 10^{-n}. Hence in Physics the statement «α *is equal to zero*» has a different meaning than in Mathematics, since it expresses only the fact that $|\alpha| < \varepsilon$. The positive number ε must be regarded as a physical constant which should be determined before formulating any mathematical model of the physical phenomenon under consideration. This is practically done when in establishing the mathematical foundations of some physical theory it is stated that some quantity is *negligible*. This means that this quantity either cannot be detected by the instruments or a degree of accuracy has been chosen which is larger than the value of the quantity under consideration. The positive number ε will be denoted as an *upper bound for negligibles.*

With this in mind we believe that the axioms that lay on the basis of Fourier's theory must be understood as follows:

i) *An upper bound* ε *for negligibles has been determined and fixed;*

ii) $q + \chi u_x$ *is negligible*, i.e.

(3.1) $$|q + \chi u_x| < \varepsilon;$$

iii) *Eq. (1.2) holds.*

Of course even Eq. (1.2) could be interpreted in the sense of negligibles, i.e.

(3.2) $$|\gamma \rho u_t + q_x| < \varepsilon.$$

But Eq. (1.2) is considered by Cattaneo, by Gurtin and Pipkin (who imbed it into a more general principle) and by others more like a mathematical principle than like an experimental constitutive law, hence we leave it as an equation. In any case the next developments can be easily adapted to the case when (1.2) is replaced by (3.2).

It is also to be observed that, for any fixed system of units, ε depends on the various physical constants which must be measured. However we are permitted to suppose that the units have been chosen in such a way that a unique ε be considered.

The proposed interpretation of Fourier's theory implies that *it can be applied only to materials such that i), ii), iii) are satisfied.* Of course one should expect that in a theory, where in its constitutive laws the concept of negligible has been taken into account, its results be interpreted by considering the unknown variables, which the theory permits to calculate, determined up to the addition of a negligible quantity.

Unfortunately very often this is not the case. The criticisms to Fourier according to which his theory makes the heat to propagate with infinite speed are a shining example of this attitude.([4])

We shall see in the next Section that, when the results of Fourier's theory are properly interpreted, heat does not propagate with infinite speed, but,if ε is not excessively small, rather slowly. This was known to Maxwell, who, as we shall see later, understood Fourier's theory in its correct meaning.

4. The propagation of heat in a wire.

Let us consider again the example of a wire with a unitary heat

([4]) The Fourier theory of heat propagation is not the only one in Mathematical Physics accused to provide unphysical results. These accusations are a consequence of the fact that approximation assumptions made in establishing the constitutive laws are not maintened in interpreting the final mathematical results furnished by the theory. Another conspicuous example is provided by the mathematical theory of elasticity. Let us recall what A. Clebsch wrote about discrepancies between theoretical results of this theory and physical reality: *«On prend l'habitude d'attribuer ces différences plutôt à une imperfection de la théorie qu'à une irregularité dans son emploi. Serait-ce, par hasard., parce que cette confusion se produit malheureusement trop souvent, que la théorie est si dépreciée dans certain milieux?»* ([2], p. 294).

source at $x = 0$. We wish to calculate how long a point $x \neq 0$ of the wire remains not heated. We assume that the temperature $u(x,t)$ is given by (1.4). Of course this assumption should be justified since being now (1.1) replaced by (3.1), we don't know whether under this new hypothesis $u(x,t)$ could still be represented by (1.4). We shall return on this point in Sect.8.

The problem consists in determining for every $x \neq 0$ some $t(x) > 0$ such that for $0 < t < t(x)$ we have

$$u(x,t) < \varepsilon. \tag{4.1}$$

Since we may (we must!) suppose $t \geq \varepsilon$, (4.1) is satisfied if

$$\frac{1}{2\sqrt{\pi D \varepsilon}} e^{-\frac{x^2}{4Dt}} < \varepsilon,$$

hence

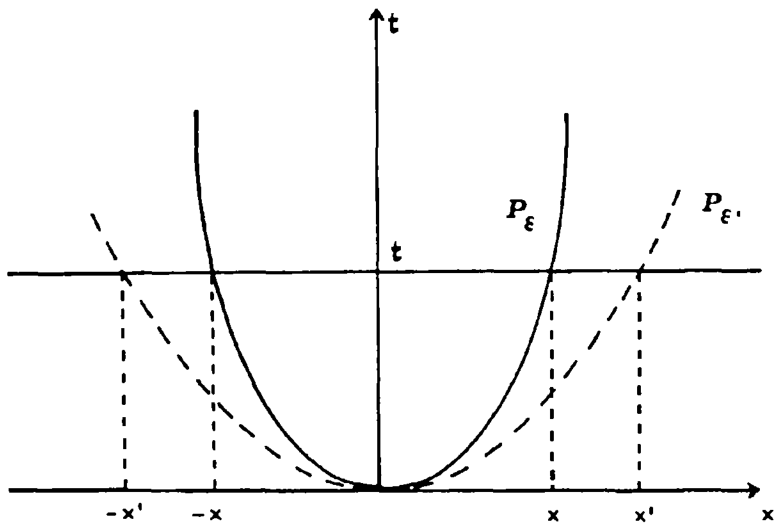
$$-\frac{x^2}{4Dt} < \log(2\varepsilon^{\frac{3}{2}}\sqrt{\pi D}).$$

We can suppose $\varepsilon^3 < (4\pi D)^{-1}$ because of the smallness of ε. It follows that at least until the instant

$$t = \left\{ -\frac{1}{4D} \log(2\varepsilon^{\frac{3}{2}}\sqrt{\pi D}) \right\} x^2 \tag{4.2}$$

the point x is not heated.

Let us consider in the x,t plane the parabola P_ε of Equation (4.2) and draw the parallel to the x-axis at a distance t. From the points where this parallel meets P_ε we draw the perpendiculars to the x-axis. They meet this axis in the points $-x$ and x and the part of the wire exterior to $(-x,x)$ remains not heated at least until the instant t. If we assume $0 < \varepsilon' < \varepsilon$ the parabola $P_{\varepsilon'}$, corresponding to ε' is less concave than P_ε and the corresponding segment $(-x',x')$ contains $(-x,x)$, hence the part of the wire not heated until t decreases with ε. If $\varepsilon \to 0$ the parabola P_ε flattens itself on the x-axis and the propagation of the heat becomes instantaneous. But the limiting case $\varepsilon = 0$ is unrealistic and *cannot correspond to an actual physical situation.*



5. On the compatibility of the criticisms to the Fourier theory with the assumptions made for deriving alternative theories.

We examine now the logical connections between the criticisms to Fourier and the hypotheses assumed by the Authors of alternative theories for deriving their axioms. We restrict ourselves only to the theories proposed by Cattaneo and by Gurtin and Pipkin. However similar arguments can be used in connection with the work of the Authors of other approaches,especially when they derive *linearized theories*(5).

We have seen that Cattaneo in deriving his constitutive equation (2.2) supposes that σ^2 is negligible and retains only a term containing σ. Gurtin & Pipkin in deriving a linear theory from a non linear one assume that any quantity which has an order of magnitude like $o(\delta)$, with δ given by (2.7), is negligible.

Hence, according to Cattaneo, σ^2 is negligible ([1] p. 93) and,

(5) See for instance [11] from p. 265.

assuming in (1.4) $x = 1$, $t = (4D)^{-1}\sigma$,

$$u\left(1, \frac{\sigma}{4D}\right) = \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{1}{\sigma}}$$

is *not* negligible ([1] p. 83, p. 86). For Gurtin and Pipkin $\delta^{1+\alpha}$ $(\alpha > 0)$ is negligible ([7], p. 124) and assuming in (1.4) $x = 1$, $t = (4D)^{-1}\delta$

$$u\left(1, \frac{\delta}{4D}\right) = \frac{1}{\sqrt{\pi\delta}} e^{-\frac{1}{\delta}}$$

is *not* negligible ([7], p. 113).

As a numerical example consider $\sigma = \delta = 0.01$, $\alpha = 1$. Then, according to Cattaneo, Gurtin and Pipkin,

$$\sigma^2 = \delta^2 = 10^{-4} \simeq \textit{negligible};$$

$$\frac{1}{\sqrt{\pi\sigma}} e^{-\frac{1}{\sigma}} = \frac{1}{\sqrt{\pi\delta}} e^{-\frac{1}{\delta}} = 2.09882811567 \times 10^{-43}$$

not negligible(6).

6. Maxwell's interpretation of Fourier's theory.

In his book [10] James Clerk Maxwell studies thoroughly Fourier's theory of heat propagation and he exactly perceives in what sense this theory must be understood. It seems that, unfortunately, the analysis of Maxwell of Fourier's theory has been either forgotten or overlooked by the Authors who criticize Fourier. A proof of this is, for instance, what Joseph and Preziosi write in their paper concerning Maxwell:

«His book «Theory of Heat» is based on diffusion and Fourier's law. He **did not note** *that diffusion is associated with infinite speed of propagation»*([8], p. 52).(7)

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(6) *«And why beholdest thou the mote that is in thy brother's eye, but considerest not the beame that is in thine owne eye?»*
(The New Testament: The Gospel according to S. Matthew, Chap.VII: 3).

(7) Bold-face in this and in the next Maxwell statements is ours.

This is not true. Maxwell was very well aware of the fact that propagation of heat, according to Fourier, has theoretically an infinite speed, but he was able to understand and to exactly describe the real meaning to be attached to this theory. Let us follow what Maxwell writes in his book ([10], p. 238-240).

«The discussion of this problem is the subject of the great work of Joseph Fourier... The temperature of every point of the body at a given time is supposed to be known, and it is required to determine the temperature of any given point P after a time t has elapsed... In calculating the temperature of the point P, we must take into account the temperature of every other point Q, however distant, and however short the time may be during which the propagation of heat has been going on. **Hence, in a strict sense, the influence of a heated part of the body extends to the most distant part of the body in an incalculably short time, so that it is impossible to assign to the propagation of heat a definite velocity.** *The velocity of propagation of thermal effects depends entirely on the magnitude of the effect which we are able to recognize; and if there were no limit to the sensibility of our instruments, there would be no limit to the rapidity with which we could detect the influence of heat applied to distant parts of the body.* **But while this influence can be expressed mathematically from the first instant, its numerical value is excessively small... The sensible propagation of heat, so far from being instantaneous, is excessively slow process and the time required to produce... change of temperature... is proportional to the «square» of the «linear dimension».**

These ideas of Maxwell are exactly the ones we have followed in Sections 3 and 4 of this paper.

7. A problem of Analysis: A generalized Cauchy problem for the heat equation.

The mathematical Analysis carried out in this Section will be used in the next one to prove that the classical Fourier solution is *admissible* for describing heat propagation.

Let S_T be the strip of the x, t plane defined by

$$-\infty < x < +\infty, \quad 0 < t < T \quad (T > 0).$$

Assume $\rho \geq 0$ and let $\Im_\rho$ be the function class of real valued function $f(x,t)$ defined in S_T enjoying the following property: for each $f \in \Im_\rho$ two constants c, β exist, such that

$$\begin{aligned} &c > 0 && 0 \leq \beta < 1, \\ &|f(x,t)| \leq ct^{-\beta} && \text{for} |x| \geq \rho. \end{aligned} \tag{7.1}$$

Let $\mathcal{U}_\rho$ be the class of real valued functions $U(x,t)$ defined in S_T enjoying the following properties:

i) $U(x,t) \in C^\infty(S_T)$;

ii) $U_x(x,t)$ belongs to $\Im_\rho$;

iii) $U(x,t)$ and $U_{xx}(x,t) - D^{-1}U_t(x,t)$ belong to $\Im_0$.

Let $a > 0$ be a positive real number arbitrarily given. Let μ (B) be a real valued measure function defined in the σ-ring of the Borel sets of the real axis, such that μ $(B) = 0$ for any $B \subset \mathbb{R}\backslash(-a, a)$.

We consider the following *Generalized Cauchy Problem* for the heat equation:

Given a function $f(x,t) \in \Im_0$, belonging to $C^\infty(S_T)$ and the above considered measure $\mu(B)$, to find a function $U(x,t) \in \mathcal{U}_\rho$ (for any $\rho > a$) such that

$$U_{xx}(x,t) - D^{-1}U_t(x,t) = f(x,t) \text{ in } S_T \tag{7.2}$$

$$\lim_{t \to 0^+} \int_{-h}^{h} \varphi(x)U(x,t)dx = \int_{-h}^{h} \varphi(x)d\mu \tag{7.3}$$

for each $\varphi(x) \in C^\infty(\mathbb{R})$ and each $h > a$.

THEOREM. *The above problem has one and only one solution and* *it is given, for $(x,t) \in S_T$, by*

$$U(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{+\infty} \exp \frac{-(x-\xi)^2}{4Dt} d\mu_\xi \tag{7.4}$$
$$- \frac{D^{\frac{1}{2}}}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} f(\xi,\tau) \exp \frac{-(x-\xi)^2}{4D(t-\tau)} d\xi.$$

By using classical results of the theory of heat equation we see that $U(x,t)$ is a solution of (7.2) which belongs to $C^\infty(S_T)$ (see [6] chap. XXIX; [5]).

Set

$$u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{+\infty} \exp \frac{-(x-\xi)^2}{4Dt} d\mu_\xi, \tag{7.5}$$

$$v(x,t) = -\frac{D^{\frac{1}{2}}}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} f(\xi,\tau) \exp \frac{-(x-\xi)^2}{4D(t-\tau)} d\xi. \tag{7.6}$$

We have

$$|u(x,t)| \le \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{+\infty} d|\mu|, \tag{7.7}$$

$$|v(x,t)| \le \frac{cD^{\frac{1}{2}}}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{\tau^\beta (t-\tau)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \exp \frac{-(x-\xi)^2}{4D(t-\tau)} d\xi =$$
$$= \frac{cD}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{\tau^\beta} \int_{-\infty}^{+\infty} \exp \frac{-(\xi-x)^2}{4D(t-\tau)} d\frac{\xi - x}{2\sqrt{D(t-\tau)}} = \tag{7.8}$$
$$= \frac{cD}{2} \frac{1}{1-\beta} t^{1-\beta}.$$

(7.9)

$$
\begin{aligned}
|v_x(x,t)| &= \left| \frac{1}{4\sqrt{\pi D}} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} f(\xi,\tau)(x-\xi) \exp \frac{-(x-\xi)^2}{4D(t-\tau)} d\xi \right| \\
&\le \frac{cD^{\frac{1}{2}}}{2\sqrt{\pi}} \int_0^t \frac{d\tau}{\tau^\beta (t-\tau)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \frac{|\xi - x|}{2D(t-\tau)} \exp \frac{-(\xi-x)^2}{4D(t-\tau)} d\xi \\
&\le \frac{cD^{\frac{1}{2}}}{\sqrt{\pi}} \int_0^t \frac{d\tau}{\tau^\beta (t-\tau)^{\frac{1}{2}}} \int_0^{+\infty} \frac{s}{2D(t-\tau)} \exp \frac{-s^2}{4D(t-\tau)} ds = \\
&= \frac{cD^{\frac{1}{2}}}{\sqrt{\pi}} \int_0^t \frac{d\tau}{\tau^\beta (t-\tau)^{\frac{1}{2}}} .
\end{aligned}
$$

Moreover for $\rho > a$ and $|x| \geq \rho$, assuming $\frac{1}{2} < \lambda < 1$,

(7.10)

$$
\begin{aligned}
|u_x(x,t)| &\le \frac{1}{4\sqrt{\pi D^3} t^{\frac{3}{2}-\lambda}} \int_{-\infty}^{+\infty} \frac{1}{|x-\xi|^{2\lambda-1}} \frac{|x-\xi|^{2\lambda}}{t^\lambda} \cdot \\
&\quad \cdot \exp \frac{-(x-\xi)^2}{4t} d|\mu_\xi| \\
&\le \frac{(4D\lambda)^\lambda \exp -\lambda}{4\sqrt{\pi D^3} t^{\frac{3}{2}-\lambda}} \frac{1}{(\rho-a)^{2\lambda-1}} \int_{-\infty}^{+\infty} d|\mu| .
\end{aligned}
$$

From inequalities (7.7), (7.8), (7.9), (7.10) it follows that $U(x,t) \in \mathcal{U}_\rho$ for any $\rho > a$. From (7.8) we deduce

(7.11)
$$\lim_{t \to 0^+} v(x,t) = 0$$

uniformly with respect to x. On the other hand

(7.12)
$$\int_{-h}^{+h} \varphi(x) u(x,t) dx = \int_{-\infty}^{+\infty} \phi(\xi,t) d\mu_\xi$$

where

$$\phi(\xi,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-h}^{+h} \varphi(x) \exp \frac{-(x-\xi)^2}{4Dt} dx .$$

Since

$$\lim_{t \to 0^+} \phi(\xi,t) = \varphi(\xi) \qquad (-h < \xi < h); \; |\phi(\xi,t)| \le \max_{[-h,h]} |\varphi(x)|$$

([6], p. 296-299; [5] p. 24), we have

$$\lim_{t\to 0^+}\int_{-h}^{+h}\varphi(x)u(x,t)dx = \int_{-h}^{+h}\varphi(x)d\mu.$$

From (7.11), (7.12) we deduce (7.3). The proof of the existence is now complete. Let us prove uniqueness. Since the function class where our problem is considered is linear we need to prove that if U is a solution in this class such that $\mu(B)\equiv 0$, $f(x,t)\equiv 0$, then U vanishes identically. It is convenient to set

$$\Gamma(x,t;\xi,\tau)\begin{cases} = \dfrac{D^{\frac{1}{2}}}{2\sqrt{\pi}}\dfrac{1}{(t-\tau)^{\frac{1}{2}}}\exp\dfrac{-(x-\xi)^2}{4D(t-\tau)} & \text{for } t>\tau \\ = 0 & \text{for } t\leq\tau \end{cases}$$

$$LU = U_{xx} - D^{-1}U_t.$$

Let $\psi(x,t)$ be a function of $\overset{\circ}{C}{}^{\infty}(S_T)$ i.e. of class C^∞ in S_T and with a bounded support contained in S_T. Set for $(x,t)\in S_T$

$$v(x,t) = -\int_t^T d\tau\int_{-\infty}^{+\infty}\psi(\xi,\tau)\Gamma(\xi,\tau;x,t)d\xi. \tag{7.13}$$

The function $v(x,t)$ enjoys the following properties ([5], p. 28):

1) $v(x,t)\in C^\infty(\bar{S})$

2) $L^*v \equiv v_{xx} + D^{-1}v_t = \psi$

3) $\lim_{t\to T^-} v(x,t) = 0$

uniformly with respect to $x\in \mathbb{R}$.

Let us consider for $h>a$ and $0<\sigma<T$ the domain $R_{h\sigma}: -h<x<h,\ \sigma<t<T$. The Green identity in $R_{h\sigma}$ is the

following:

$$
\begin{aligned}
&\iint_{R_{h\sigma}} UL^* v\,dx\,dt - \iint_{R_{h\sigma}} vLU\,dx\,dt = \\
&\quad = \int_\sigma^T U(h,t)v_x(h,t)dt - \int_\sigma^T U(-h,t)v_x(-h,t)dt \\
&\quad - \int_\sigma^T U_x(h,t)v(h,t)dt + \int_\sigma^T U_x(-h,t)v(-h,t)dt \\
&\quad - \frac{1}{D}\int_{-h}^h U(x,\sigma)v(x,\sigma)dx + \frac{1}{D}\int_{-h}^h U(x,T)v(x,T)dx.
\end{aligned} \tag{7.14}
$$

We have

$$
\begin{aligned}
&\left|\int_{-h}^h U(x,\sigma)v(x,\sigma)dx\right| \le \\
&\left|\int_{-h}^h U(x,\sigma)[v(x,\sigma)-v(x,0)]dx\right| + \left|\int_{-h}^h U(x,\sigma)v(x,0)dx\right| \\
&\le 2ch\sigma^{-\beta} \max_{\substack{x\in[-h,h]\\ t\in[0,T]}} |v_t|\sigma + \left|\int_{-h}^h U(x,\sigma)v(x,0)dx\right|.
\end{aligned}
$$

Hence

$$\lim_{\sigma\to 0^+}\int_{-h}^h U(x,\sigma)v(x,\sigma)dx = 0.$$

From (7.14) we deduce for $\sigma \to 0^+$ and for the assumptions on U and v

$$
\begin{aligned}
&\iint_{R_{h0}} U\psi\,dx\,dt = \\
&\quad = \int_0^T U(h,t)v_x(h,t)dt - \int_0^T U(-h,t)v_x(-h,t)dt \\
&\quad - \int_0^T U_x(h,t)v(h,t)dt + \int_0^T U_x(-h,t)v(-h,t)dt
\end{aligned} \tag{7.15}
$$

with an obvious meaning for R_{h0}.

Let k be such that supp $\psi \subset R_{k0}$. From (7.13) we deduce

$$|v(x,t)| \begin{cases} \leq \dfrac{D^{\frac{1}{2}}}{\sqrt{\pi}} T^{\frac{1}{2}} \max\limits_{R_{k0}} |\psi| \exp \dfrac{-(x-k)^2}{4DT} & \text{for } x > k \\ \leq \dfrac{D^{\frac{1}{2}}}{\sqrt{\pi}} T^{\frac{1}{2}} \max\limits_{R_{k0}} |\psi| \exp \dfrac{-(x+k)^2}{4DT} & \text{for } x < -k \end{cases}$$

$$|v_x(x,t)| \begin{cases} \leq \dfrac{D^{\frac{1}{2}}}{\sqrt{\pi}} T^{\frac{1}{2}} \max\limits_{R_{k0}} |\psi_x| \exp \dfrac{-(x-k)^2}{4DT} & \text{for } x > k \\ \leq \dfrac{D^{\frac{1}{2}}}{\sqrt{\pi}} T^{\frac{1}{2}} \max\limits_{R_{k0}} |\psi_x| \exp \dfrac{-(x+k)^2}{4DT} & \text{for } x < -k. \end{cases}$$

From (7.15) for $h \to +\infty$ we deduce

$$\iint_{S_T} U(x,t)\psi(x,t)dxdt = 0$$

which implies, for the arbitrariness of ψ, $U \equiv 0$ in S_T.

Let us observe that the generalized boundary condition (7.3) reduces to the classical one if $\mu(B)$ is absolutely continuous and its derivative $\mu'(x)$ is a continuous function. In this case U is continuous in $\bar{S}_T$.

8. Admissibility of the Fourier solution for describing the heat propagation.

Let us now consider the propagation of heat in a wire, in the hypothesis of a general distribution of temperature, whose sources are represented by the measure function $\mu(B)$ considered in the previous Section. The particular case of a unique unitary source in the point $x = 0$ corresponds to the case when μ is a Dirac measure with a unit mass concentrated in the point $x = 0$.

Let us denote by $U(x,t)$ the temperature corresponding to this general distribution of temperature for $t = 0$. The existence of $U(x,t)$ is assumed as an axiom of physical evidence. From the analytical point of view we shall assume the very liberal axiom according to which U

belongs to any class $\mathcal{U}_\rho$ for any $\rho > a$. Moreover we suppose that (7.3) is satisfied.

Let us now suppose that an upper bound ε for negligibles has been fixed. We say that a function $u(x,t)$ defined in S_T is *admissible* for describing the heat propagation generated by the given measure μ if we have in S_T

$$|U(x,t) - u(x,t)| < \varepsilon.$$

This means that U and u are indistinguishable the one from the other. We shall call *Fourier's solution* the function $u(x,t)$ given by (7.5). The next theorem proves that if to the material of the wire the Fourier postulates, as expounded in Sect.3, are applicable then the Fourier solution $u(x,t)$ is admissible, i.e. $u(x,t)$ is a solution with an actual physical meaning.

THEOREM. *Let us suppose that*

(8.1) $$|\chi U_x + q| < p\varepsilon \quad (0 \leq p < 1)$$

(8.2) $$\gamma\rho U_t + q_x = 0.$$

If

$$T < \frac{\pi\chi\gamma\rho}{4p^2}$$

the Fourier solution (7.4) is admissible in S_T.

Before proving this theorem we observe, having fixed ε, that under the assumptions (8.1), (8.2) the Fourier theory, as proposed in Sect. 3, is supposed valid. We only remark that (8.1) is, from a strict analytical point of view, more stringent than (3.1). From the physical point of view (3.1) and (8.1) should be considered equivalent since ε and $p\varepsilon$ $(0 \leq p < 1)$ are indistinguishable. The reason why we use (8.1) is purely technical and, on the other hand, it has the advantage to include the classical Fourier theory assuming $p = 0$.

Set

(8.3) $$F(x,t) = \chi U_x(x,t) + q(x,t),$$

we have in S_T

$$|F(x,t)| \leq p\varepsilon. \tag{8.4}$$

From (8.2) and (8.3) we get that U satisfies (7.2) when

$$f(x,t) = \frac{1}{\chi} F_x(x,t). \tag{8.5}$$

From the assumptions on U it follows that $f(x,t)$ is C^∞ in S_T and belongs to $\Im_0$. Hence we have

$$U(x,t) = u(x,t) + v(x,t),$$

where u and v are given by (7.5), (7.6) and $f(x,t)$ by (8.5). Since it is natural to assume that the thermal flux $q(x,t)$ is a function of $C^\infty(S_T)$, we may state that $F(x,t) \in C^\infty(S_T)$.

We have

$$v(x,t) = \lim_{h\to+\infty} \frac{-1}{2\sqrt{\pi\chi\gamma\rho}} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \int_{-h}^{h} F_\xi(\xi,\tau) \exp\frac{-(x-\xi)^2}{4D(t-\tau)} d\xi =$$

$$= \lim_{h\to+\infty} \frac{-1}{2\sqrt{\pi\chi\gamma\rho}} \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \cdot$$

$$\cdot \left[F(h,\tau)\exp\frac{-(x-h)^2}{4D(t-\tau)} - F(-h,\tau)\exp\frac{-(x+h)^2}{4D(t-\tau)} \right] d\tau$$

$$+ \lim_{h\to+\infty} \frac{1}{2\sqrt{\pi\chi\gamma\rho}} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \int_{-h}^{h} F(\xi,\tau)\frac{\partial}{\partial\xi} \exp\frac{-(x-\xi)^2}{4D(t-\tau)} d\xi.$$

From (8.4) it easily follows that the first limit is zero, hence

$$v(x,t) = \frac{1}{4}\sqrt{\frac{\gamma\rho}{\pi\chi^3}} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} F(\xi,\tau)(x-\xi)\exp\frac{-(x-\xi)^2}{4D(t-\tau)} d\xi.$$

Using (8.4) we deduce for $(x,t) \in S_T$

$$|v(x,t)| \leq \frac{p\varepsilon}{\sqrt{\pi\chi\gamma\rho}} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \int_0^{+\infty} \frac{s}{2D(t-\tau)} \exp\frac{-s^2}{4D(t-\tau)} ds =$$

$$= \frac{p\varepsilon}{\sqrt{\pi\chi\gamma\rho}} \int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}}} \leq 2p\varepsilon\sqrt{\frac{T}{\pi\chi\gamma\rho}} < \varepsilon.$$

This proves the theorem.

9. Concluding remarks.

We believe that, as a result of our analysis, the accusation to the Fourier theory of heat propagation of producing the paradox of an infinite speed for this propagation should be considered unfounded. It has been also made clear that the Fourier theory has a limited range which will become more and more restricted with the technical progress of the experimental instruments for measurements and of the computing facilities which, respectively, will permit more refined measures and sharper numerical approximations. On the other hand, for technical problems where neither excessively accurate measurements nor extremely approximated numerical evaluations are needed, the Fourier theory will continue to furnish, as it has furnished in the past, satisfactory services.

The analysis carried out in this paper, concerned with the simplest example of a wire, could be extended, by affording more complicated technical difficulties, to general propagation problems related to two- or three-dimensional material systems. However the conceptual line for rehabilitating Fourier's theory will remain unaltered.

We believe that the best final comment to the «philosophy» of this paper is what Tullio Levi-Civita wrote in a celebrated paper where he succeeded in presenting the Einstein relativity theory like *an evolution* of classical mechanics rather than a *revolution: ...«Nessun ricercatore può essere misoneista, ma molti cultori di scienza possono, direi quasi debbono, essere conservatori per la stessa loro missione di custodire con gelosa cura un certo patrimonio intellettuale ben consolidato, e di vagliare con severo spirito critico tutto ciò che importa variazione od alienazione del patrimonio stesso»...* ([9], p. 10).

REFERENCES

[1] Cattaneo C., *Sulla conduzione del calore,* Atti del Semin. Matem. Univ. di Modena 3 (1948-49), 83-101.

[2] Clebsch A., *Theorie der Elasticität fester Körper,* Teubner, Leipzig 1862. Enlarged French Translation by B.de Saint-Venant and A.Flamant, Dunod, Paris, 1885.

[3] Fichera G., *Avere una memoria tenace crea gravi problemi,* Archive for Rat. Mech. & Anal. **70** (1979), 101-112.

[4] Fourier J., *Théorie analytique de la chaleur,* 1822.

[5] Friedman A., *Partial Differential Equations of Parabolic Type,* Prentice-Hall Inc., Englewood Cliffs N.J., 1964.

[6] Goursat E., *Cours d'Analyse Mathematique,* tome III, 4eme ed. Gauthier-Villars, Paris, 1927.

[7] Gurtin A.C., Pipkin M.E., *A General Theory of Heat Conduction with Finite Wave Speeds,* Archive for Rat. Mech. & Anal. **31** (1968), 113-126.

[8] Joseph L., Preziosi D.D., *Heat Waves,* Rev.Mod. Physics, 1989, 41-73.

[9] Levi Civita T., *Come potrebbe un conservatore giungere alla soglia della nuova meccanica,* Semin. Matem. della R. Univ. di Roma, Rendiconti delle Sedute anno acc. 1918-19, V, (1920), 10-28.

[10] Maxwell J.C., *Theory of Heat,* II ed. Longmans, Green & Co., London,1972.

[11] Müller I.,*Toward Relativistic Thermodynamics,* Archive for Rat. Mech. & Anal. **34** (1969), 259-282.

[12] Volterra V., *Leçons sur les fonctions de lignes,* Collect. de monograph. sur la théorie d. fonctions, Gauthier-Villars, Paris, 1913.

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