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RANDOM POINTS ASSOCIATED WITH RECTANGLES

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The study of the distribution and moments of the distance between random points within a rectangle or in two coplanar rectangles is required in a wide variety of fields. Formulae for the distributions and arbitrary moments of the distance between two random points associated with one or two rectangles in various situations are given here explicitly. These explicit formulae will be helpful to those who work in various applied areas for the computations required in their problems.

1. Introduction.

Borel (1925) seems to have been the first one to consider the distance between random points in specific elementary geometric figures such as triangles, squares and so on. The theorems on mean values and fixed points of Crofton (1877, 1885) are very general in nature and although they cover particular geometric figures such as one and two rectangles, explicit formulae for these specific situations are not given there. The first papers giving explicit formulae for the density and mean values of the distance between two random points within a rectangle, in adjacent squares and in squares having a common diagonal seem to be those of Ghosh (1943a, 1943b, 1951). Since then several papers have been written on this topic. Expected distance is dealt with by many authors, see for example Christofides and Eilon (1969), Alagar (1976), Daley (1976), Oser (1976), Vaughan (1976) and Hsu (1990).

Some of the practical situations where the expected distance between

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random points in the same or different rectangles is required are the following: For applications in sampling problems and agricultural experimentation see Ghosh (1949), for urban operations see Larson and Odoni (1981), for management and mathematical modelling problems see Eilon et. al (1971), for transportation problems, vehicle routing, dispatching of emergency vehicles in urban settings and random paths across a rectangle see for example Horowitz (1965), Stone (1981) and Vaughan (1981). Horowitz (1965) also mentions some of the applications of random distances in physics such as in measuring the length of the path of a gamma-ray to the wall of a nuclear reactor and the length of a sound ray in a room from one reflection to the next. Kuchel and Vaughan (1981) look at the length of chords in a square which is applicable to modern electronic digitizing pads. Vaughan (1976) examines engineer's "route factor" (that is, the ratio of the average distance by a given route to the average direct distance) for rectangular routing between adjacent squares. Marsaglia et.al (1990) state that the study of distance between random points in a rectangle arise in physical chemistry, chemical physics, material science, operations research and population studies and give a convenient spline function notation for the density which is more suitable for computational purposes and computer graphics. Gaboune et. al (1993) deal with generalized distances such as Manhattan metric and Chebychev metric, besides Euclidean metric. General procedures for tackling expected Euclidean distance in convex bodies with particular reference to polygons may be found in Kendall and Moran (1963), Fairthorne (1965), Coleman (1969, 1973), Ruben (1978), Solomon (1978), and Sheng (1985), among others, from where one can derive the results for special situations but with great effort.

Our aim in the present paper is to give a large number of explicit formulae for the general moments and densities by using elementary methods and without using results from integral geometry, most of which are assumed to be new, (although particular cases are available in the literature) of the distance between two random points where the points could be inside a rectangle, on opposite sides of a rectangle, on adjacent sides of a rectangle, one on a corner and the other inside a rectangle, in two different but similarly oriented rectangles. Euclidean distance, Manhattan distance and Chebychev distance are considered with reference to the densities and general moments. These formulae are expected to come in handy for those who want to apply them to practical situations in various disciplines.

2. Two random points within a rectangle.

When two points are selected at random on a particular side or on the perimeter of a rectangle the problem is equivalent to selecting two random points on a line segment. Here randomness is interpreted as probability being proportional to the respective side or the perimeter as the case may be. The situations of interest are the following: one point is on one side and the second point is on another side (this could be an adjacent side or the opposite side), one point is on a side and one point inside the rectangle, one point at one of the corners and the other inside the rectangle, and both points within the rectangle. When the two points are on two sides the derivation of the distribution of the distance is not difficult as it can be seen from an example to follow. Hence we will start with the case of both the points within the rectangle. Let the sides be of lengths a and b respectively, $a \leq b$. Let the points be $P(x_{11}, x_{12})$ and $Q(x_{21}, x_{22})$. We take the origin at one corner and the axes along two sides.

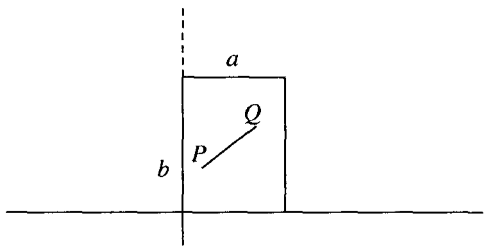

Figure 1

Then the square of the Euclidean distance between P and Q is given by

$$
\begin{aligned}
u &= (x_{21} - x_{11})^2 + (x_{22} - x_{12})^2 \\
&= a^2\left(\frac{x_{21}}{a} - \frac{x_{11}}{a}\right)^2 + b^2\left(\frac{x_{22}}{b} - \frac{x_{12}}{b}\right)^2 \qquad (1) \\
&= a^2(u_1 - u_2)^2 + b^2(u_3 - u_4)^2
\end{aligned}
$$

where u_1, u_2, u_3, u_4 are independently and uniformly distributed over $[0, 1]$. From the uniform distribution the density of $v_1 = a(u_1 - u_2)$, denoted by $f_1(v_1)$, is given by

$$
f_1(v_1) = \begin{cases} \dfrac{1}{a_2}(a + v_1), & -a \leq v_1 \leq 0 \\ \dfrac{1}{a^2}(a - v_1), & 0 \leq v_1 \leq a \\ 0, & \text{elsewhere.} \end{cases}
$$

Hence the density of $w_1 = v_1^2$, denoted by $g_1(w_1)$, is given by

$$g_1(w_1) = \begin{cases} \dfrac{1}{a_2}\left(\dfrac{a}{\sqrt{w_1}} - 1\right), \ 0 < w_1 \leq a^2 \\ 0, \ \text{elsewhere.} \end{cases}$$

The corresponding density for $w_2 = b^2(u_3 - u_4)^2$ is given by

$$g_2(w_2) = \begin{cases} \dfrac{1}{b_2}\left(\dfrac{b}{\sqrt{w_2}} - 1\right), \ 0 < w_2 \leq b^2 \\ 0, \ \text{elsewhere.} \end{cases} \tag{2}$$

Since $u = w_1 + w_2$, the density of u, denoted by $g(u)$, is given by

$$g(u) = \begin{cases} \displaystyle\int_{v=0}^{u} g_1(v)g_2(u-v)dv, \ 0 \leq u \leq a^2 \\ \displaystyle\int_{v=0}^{a^2} g_1(v)g_2(u-v)dv, \ a^2 \leq u \leq b^2 \\ \displaystyle\int_{v=u-b^2}^{a^2} g_1(v)g_2(u-v)dv, \ b^2 \leq u \leq a^2 + b^2 \\ 0, \ \text{elsewhere.} \end{cases}$$

$$= \frac{1}{a^2b^2} \begin{cases} \pi ab - 2(a+b)u^{\frac{1}{2}} + u, \ 0 \leq u < a^2 \\ 2ab\sin^{-1}\left(\dfrac{a}{\sqrt{u}}\right) - a^2 - 2bu^{\frac{1}{2}} \\ +2b(u-a^2)^{\frac{1}{2}}, \ a^2 \leq u < b^2 \\ 2ab\left[\sin^{-1}\left(\dfrac{a}{\sqrt{u}}\right) - \sin^{-1}\sqrt{1 - \dfrac{b^2}{u}}\right] - u \\ -(a^2+b^2) + 2a\sqrt{u-b^2} + 2b\sqrt{u-a^2}, \ b^2 \leq u \leq a^2 + b^2 \\ 0, \ \text{elsewhere.} \end{cases}$$

Hence the density of $x = u^{\frac{1}{2}}$ can be written as

$$f(x) = \frac{4x}{a^2b^2}\phi(x) \tag{3}$$

where

$$\phi(x) = \begin{cases} \dfrac{ab\pi}{2} - (a+b)x + \dfrac{x^2}{2}, \ 0 \le x < a \\ ab\sin^{-1}\left(\dfrac{a}{x}\right) - \dfrac{a^2}{2} - b(x^2 - a^2)^{\frac{1}{2}}, \ a \le x < b \\ ab\left[\sin^{-1}\left(\dfrac{a}{x}\right) - \sin^{-1}\sqrt{1 - \dfrac{b^2}{x^2}}\right] - \dfrac{(a^2+b^2)}{2} - \dfrac{x^2}{2} \\ +a\sqrt{x^2-b^2} + b\sqrt{x^2-a^2}, \ b \le x \le \sqrt{a^2+b^2} \\ 0, \text{ elsewhere.} \end{cases}$$

Note that $\sin^{-1}\sqrt{1 - \dfrac{b^2}{x^2}} = \cos^{-1}\left(\dfrac{b}{x}\right)$. Under this substitution, (2) agrees with the result obtained by Ghosh (1943a).

The general h-th moment can be worked out by using (3). That is,

(4)

$$\begin{aligned} E(x^2) = \frac{4}{a^2b^2}&\left\{ab\frac{\pi}{2}\int_0^a x^{h+1}dx - (a+b)\int_0^a x^{h+2}dx\right. \\ &+ \frac{1}{2}\int_0^a x^{h+3}dx - \frac{a^2}{2}\int_a^b x^{h+1}dx - b\int_a^b x^{h+2}dx \\ &- \frac{(a^2+b^2)}{2}\int_b^{\sqrt{a^2+b^2}} x^{h+1}dx - \frac{1}{2}\int_b^{\sqrt{a^2+b^2}} x^{h+3}dx \\ &+ b\int_a^{\sqrt{a^2+b^2}} x^{h+1}\sqrt{x^2-a^2}dx + a\int_b^{\sqrt{a^2+b^2}} x^{h+1}\sqrt{x^2-b^2}dx \\ &\left.+ ab\int_a^{\sqrt{a^2+b^2}} x^{h+1}\sin^{-1}\left(\frac{a}{x}\right)dx - ab\int_b^{\sqrt{a^2+b^2}} x^{h+1}\cos^{-1}\left(\frac{b}{x}\right)dx\right\} \\ = \frac{4}{a^2b^2}&\left\{\frac{1}{2(h+2)}[a^{h+3}(a+\pi b) + b^{h+4} - (a^2+b^2)^{\frac{h}{2}+2}]\right. \\ &- \frac{1}{(h+3)}[a^{h+4} + b^{h+4}] + \frac{1}{2(h+4)}[a^{h+4} + b^{h+4} - (a^2+b^2)^{\frac{h}{2}+2}] \\ &\left.+ bI^{(h)}1 + aI^{(h)}2 + abI_5^{(h)} - abI_6^{(h)}\right\}, \ \Re(h) > -2. \end{aligned}$$

where $I_1^{(h)}$ to $I_6^{(h)}$ are given in the following Lemmas 1 and 2 and $\Re(\cdot)$ denotes the real part of $(\cdot)$. Explicit forms of the moments for $h = 1, 2, 3, 4$ are given by Ghosh (1943a) in terms of hyperbolic functions. Some simpler forms are available from (4) and the following lemmas.

LEMMA 1.

$$I_1^{(h)} = \int_a^{\sqrt{a^2+b^2}} x^{h+1}\sqrt{x^2-a^2}dx$$
$$= \frac{b^3}{3}(a^2+b^2)^{h2}\,{}_2F_1\left(-\frac{h}{2}, 1; \frac{5}{2}; \frac{b^2}{a^2+b^2}\right);$$
$$I_2^{(h)} = \int_b^{\sqrt{a^2+b^2}} x^{h+1}\sqrt{x^2-b^2}dx$$
$$= I_1^{(h)} \text{ with } a \text{ and } b \text{ interchanged;}$$
$$I_3^{(h)} = \int_a^{\sqrt{a^2+b^2}} x^{h+1}(x^2-a^2)^{-\frac{1}{2}}dx$$
$$= b(a^2+b^2)^{\frac{h}{2}}\,{}_2F_1\left(-\frac{h}{2}, 1; \frac{3}{2}; \frac{b^2}{a^2+b^2}\right);$$

and

$$I_4^{(h)} = \int_b^{\sqrt{a^2+b^2}} x^{h+1}(x^2-b^2)^{-\frac{1}{2}}dx$$
$$= I_3^{(h)} \text{ with } a \text{ an d} b \text{ interchaned.}$$

Proof. The proofs for $I_1^{(h)}$ to $I_4^{(h)}$ are similar and hence we consider one case. Take $I_3^{(h)}$. Make the following substitutions on the left side. $x = y\sqrt{a^2+b^2}$, $y^2 = z$, $w = 1-z$, $\dfrac{(a^2+b^2)}{b^2}w = t$. Then we have

$$I_3^{(h)} = (a^2+b^2)^{\frac{h+2}{2}}\int_{\frac{a}{\sqrt{a^2+b^2}}}^{1} y^{h+1}[(a^2+b^2)y^2-a^2]^{-\frac{1}{2}}dy$$
$$= \frac{b}{2}(a^2+b^2)^{\frac{h}{2}}\int_0^1 (1-t)^{-\frac{1}{2}}\left[1-\frac{(a^2+b^2)}{b^2}w\right]^{\frac{h}{2}}dt.$$

One factor can be expanded in the following convergent series.

$$\left[1-\frac{b^2}{a^2+b^2}t\right]^{\frac{h}{2}}=\sum^{\infty} m=0\frac{[-(h/2)]_m}{m!}\left(\frac{b^2}{a^2+b^2}\right)^m t^m.$$

Integrating out t by using a type-1 beta integral we have

$$\int_0^1 t^m(1-t)^{-\frac{1}{2}}dt=\frac{\Gamma(m+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)}=\frac{(1)_m\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)_m}=2\frac{(1)_m}{\left(\frac{3}{2}\right)_m}$$

Substituting back and summing up we have the convergent series in terms of a Gauss' hypergeometric function,

$$b(a^2+b^2)^{\frac{h}{2}}\,{}_2F_1\left(-\frac{h}{2},1;\frac{3}{2};\frac{b^2}{a^2+b^2}\right).$$

LEMMA 2.

$$\begin{aligned}I_5^{(h)}&=\int_a^{\sqrt{a^2+b^2}}x^{h+1}\sin^{-1}\left(\frac{a}{x}\right)dx\\&=\frac{a^{h+2}}{h+2}\left[-\frac{\pi}{2}+\left(1+\frac{b^2}{a^2}\right)^{\frac{h+2}{2}}\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right.\\&\left.+\left(\frac{b^2}{a^2+b^2}\right)^{\frac{1}{2}}{}_2F_1\left(\frac{h+3}{2},\frac{1}{2};\frac{3}{2};\frac{b^2}{a^2+b^2}\right)\right]\end{aligned}$$

and

$$\begin{aligned}I_6^{(h)}&=\int_b^{\sqrt{a^2+b^2}}x^{h+1}\cos^{-1}\left(\frac{b}{x}\right)dx\\&=\frac{b^{h+2}}{h+2}\left[\left(1+\frac{a^2}{b^2}\right)^{\frac{h+2}{2}}\cos^{-1}\frac{b}{\sqrt{a^2+b^2}}\right.\\&\left.-\left(\frac{a^2}{a^2+b^2}\right)^{\frac{1}{2}}{}_2F_1\left(\frac{h+3}{2},\frac{1}{2};\frac{3}{2};\frac{a^2}{a^2+b^2}\right)\right].\end{aligned}$$

Proof Put $\sin\alpha = \frac{a}{x}$. Then $x^{h+1}dx = -a^{h+2}\frac{\cos\alpha}{(\sin\alpha)^{h+3}}d\alpha$ and

$$I_5^{(h)} = a^{h+2}\int_{\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}}^{\frac{\pi}{2}} \alpha\frac{\cos\alpha}{(\sin\alpha)^{h+3}}d\alpha.$$

Integrating by parts and observing that

$$\int\frac{\cos\alpha}{(\sin\alpha)^{h+3}} = -\frac{1}{(h+2)}\frac{1}{(\sin\alpha)^{h+2}}$$

we have

$$I_5^{(h)} = a^{h+2}\left[-\frac{\pi}{2(h+2)} + \frac{\left(1+\frac{a^2}{a^2}\right)^{\frac{h+2}{2}}}{(h+2)}\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}\right]$$
$$+\left[\frac{a^{h+2}}{(h+2)}\int_{\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}}^{\frac{\pi}{2}}\frac{1}{(\sin\alpha)^{h+2}}d\alpha\right].$$

Making the substitutions $y = \sin\alpha$, and $z = 1-y^2$ we have

$$\int_{\sin^{-1}\frac{a}{\sqrt{a^2+b^2}}}^{\frac{\pi}{2}} d\alpha = \frac{1}{2}\int_{a^2+b^2}^{b^2} z^{-\frac{1}{2}}(1-z)^{-(h+3)}dz$$
$$= \frac{1}{2}\sum_{m=0}^{\infty}\frac{(h+3)_m}{m!}\frac{1}{m+\frac{1}{2}}\left(\frac{b^2}{a^2+b^2}\right)^{m+\frac{1}{2}}$$
$$= \left(\frac{b^2}{a^2+b^2}\right)^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(h+3)_m}{m!}\frac{\left(\frac{1}{2}\right)_m}{\left(\frac{3}{2}\right)_m}\left(\frac{b^2}{a^2+b^2}\right)^m$$
$$= \left(\frac{b^2}{a^2+b^2}\right)^{\frac{1}{2}}\ {}_2F_1\left(h+3,\frac{1}{2};\frac{3}{2};\frac{b^2}{a^2+b^2}\right).$$

Where the ${}_2F_1$ is a hypergeometric function. Hence the result. Observe that a simpler form is available for $h = 1$ by using the formula

$$(5) \qquad \int \frac{1}{\sin^3 ax} dx = -\frac{\cos ax}{2a \sin^2 ax} + \frac{1}{2a} \ln \tan \frac{ax}{2}.$$

Proceeding the same way $I_6^{(h)}$ is evaluated.

For a rectangle of sides a and b, $a \leq b$, it is easy to see from (4) that the expected distance between two random points within the rectangle, $E(x)$, is given by

$$\begin{aligned} 15E(x) = \frac{a^3}{b^2} + \frac{b^3}{a^2} + d\left(3 - \frac{a^2}{b^2} - \frac{b^2}{a^2}\right) \\ + \frac{5}{2}\left(\frac{b^2}{a}\ln\left(\frac{a+d}{b}\right) + \frac{a^2}{b}\ln\left(\frac{b+d}{a}\right)\right) \end{aligned}$$

where

$$d = (a^2 + b^2)^{\frac{1}{2}}.$$

EXAMPLE 1. Two points on opposite sides of a square.

Consider a square of side l. A simple problem one can look at is the expected length of the chord AB if A and B are on opposite sides of the square where A and B are independently and uniformly distributed on the respective sides.

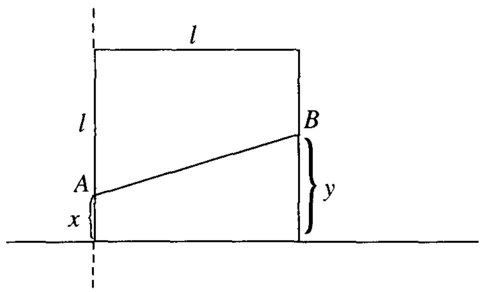

Figure 2

Then

$$|AB| = \sqrt{l^2 + (y-x)^2}.$$

The expected length, denoted by s_1, is given by

$$s_1 = \frac{1}{l^2}\int_{x=0}^{l}\int_{y=0}^{l}[l^2+(y-x)^2]^{\frac{1}{2}}dxdy.$$

Make the transformation $u = x - y$ and $v = x$. Then

$$s_1 = \frac{1}{l^2}\left\{\int_{u=0}^{l}\int_{v=u}^{l}(l^2+u^2)^{\frac{1}{2}}dudv + \int_{u=-l}^{0}\int_{y=0}^{u+l}(l^2 = u^2)^{\frac{1}{2}}dudv\right\}$$

$$(6) \qquad = \frac{2}{l^2}\int_{u=0}^{l}(l-u)(l^2+u^2)^{\frac{1}{2}}du$$

$$= \frac{2}{l^2}\left\{\frac{l^3}{2}[\sqrt{2}+\ln(1+\sqrt{2})] - \frac{l^3}{3}(2\sqrt{2}-1\right\}$$

$$\approx 1.0766l.$$

The h-th moment of the distance $d = |AB|$ is then

$$E(d^h) = \frac{2}{l^2}\int_0^l (l-u)(l^2+u^2)^{\frac{h}{2}}du$$

$$2l^h\left\{\int_{v=0}^{1}(1+v^2)^{\frac{h2}{d}}v - \int_0^1 v(1+v^2)^{\frac{h}{2}}dv\right\}.$$

But

$$\int_0^1 v(1+v^2)^{\frac{h}{2}}dv = \frac{1}{(h+2)}[2^{1+\frac{h}{2}}-1],\ \Re(h) > -2$$

and

$$\int_0^1 (1+v^2)^{h2}dv =_2 F_1\left(-\frac{h}{2},\frac{1}{2};\frac{3}{2};-1\right)$$

where the ${}_2F_1$ is a Gauss' hypergeometric function. Substituting back we have

$$(7) \qquad E(d^h) = 2l^h\left[{}_2F_1\left(-\frac{h}{2},\frac{1}{2};\frac{3}{2};-1\right) - \frac{1}{(h+2)}\left(2^{1+\frac{h}{2}}-1\right)\right].$$

When h is a positive even integer this ${}_2F_1$ reduces to a binomial sum. When $h = 1$ the integral can also be evaluated explicitly by using the formula

$$(8) \qquad \int (a^2 + x^2)^{\frac{1}{2}} dx = \frac{x}{2}(x^2 + a^2)^{\frac{1}{2}} + \frac{a^2}{2}\ln(x + \sqrt{x^2 + a^2}).$$

This when evaluated gives the explicit form given for s_1 in (6).

Let s_2 denote the expected length of the chord AB when A and B are on adjacent sides, s_3 that when the points are on the same side, and s the general expected distance when A and B move freely on the perimeter of the square with the points independently and uniformly distributed over the respective sides where the points are found. Then it is easy to see that

$$s_2 = \frac{l}{3}[\sqrt{2} + \ln(1 + \sqrt{2})],$$
$$s_3 = \frac{l}{3}$$

and

$$s = \frac{1}{4}s_1 + \frac{1}{2}s_2 + \frac{1}{4}s_3 \approx 0.7351l.$$

Particular cases of the situation when two points are taken in a rectangle, such as one point on the perimeter and one point inside, two cases to be considered here, and both the points on the perimeter, are of interest in practical problems. Hence these cases will be dealt with as examples to follow.

EXAMPLE 2. Special case 1: one point on the y-axis, or the side parallel to it, and one point inside the rectangle.

In this case $(u_1 - u_2)^2$ of (1) is either u_2^2 or $(1 - u_2)^2$. Note that u_2 and $1 - u_2$ are both uniformly distributed over $[0, 1]$. Then

$$g_1(w_1) = \begin{cases} \dfrac{1}{2a} w_1^{-\frac{1}{2}}, \ 0 < w_1 \leq w^2 \\ 0, \ \text{elsewhere.} \end{cases}$$

Therefore

$$g_1(v)g_2(u - v) = \frac{1}{2ab^2}\{bv^{-\frac{1}{2}}(u - v)^{-\frac{1}{2}}v^{-\frac{1}{2}}\}.$$

Now proceeding as in (2) and (3) the density of $x = u^{\frac{1}{2}}$ is

$$f(x) = \frac{2x}{ab^2}\pi(x)$$

where

$$\phi(x) = \begin{cases} \frac{b\pi}{2} - x,\ 0 < x \le a \\ b\sin^{-1}\left(\frac{a}{x}\right) - a,\ a \le x < b \\ b[\sin^{-1}\left(\frac{a}{x}\right) - a,\ a \le x < b \\ b\left[\sin^{-1}\left(\frac{a}{x}\right) - \cos^{-1}\left(\frac{b}{x}\right)\right] \\ +\sqrt{x^2 - b^2} - a,\ b \le x \le \sqrt{a^2 + bP2} \\ 0,\ \text{elsewhere.} \end{cases}$$

By direct integration the general moment is given by the following:

$$E(x^h) = \frac{2}{ab^2}\left\{\frac{1}{(h+2)}\left[\frac{b\pi}{2}a^{h+2} + a^{h+3} - a(a^2 + b^2)^{\frac{h+2}{2}}\right]\right.$$
$$\left. - \frac{a^{h+3}}{(h+3)} + I_2^{(h)} + bI_5^{(h)} - bI_6^{(h)}\right\},\ \Re(h) > -2$$

where the $I_2^{(h)}$, $I_5^{(h)}$ and $I_6^{(h)}$ are given in Lemmas 1 and 2. Simpler forms for $I_1^{(h)}$ to $I_4^{(h)}$ for $h = 1, 2$ and $I_5^{(h)}$ and $I_6^{(h)}$ for $h = 1$ are available. By using these the first and the second moments can be written down without the help of the hypergeometric functions. These will be illustrated when considering special cases 3 and 4 later on.

EXAMPLE 3. Special case 2: one point on the x-axis or the side parallel to it and one point inside a rectangle.

In this case $g_2(w_2)$ changes to $\frac{1}{2b}w_2^{-\frac{1}{2}}$, $0 < w_2 \le b^2$ and then

$$g_1(v)g_2(u - v) = \frac{1}{2a^2b}[av^{-\frac{1}{2}} - 1][(u - v)^{-\frac{1}{2}}]$$

Now proceeding as in (2) and (3) the density of $x = u^{\frac{1}{2}}$ is

$$f(x) = \frac{2x}{a^2 b}\phi(x)$$

where

$$\phi(x) = \begin{cases} \frac{a\pi}{2} - x,\ 0 < x \le a \\ a \sin^{-1}\left(\frac{a}{x}\right) - x + \sqrt{x^2 - a^2},\ a \le x \le b \\ a\left[\sin^{-1}\left(\frac{a}{x}\right) - \cos^{-1}\left(\frac{b}{x}\right)\right] \\ +\sqrt{x^2 - a^2} - b,\ b \le x \le \sqrt{a^2 + b^2} \\ 0,\ \text{elsewhere.} \end{cases}$$

The general moment is given by

$$E(x^h) = \frac{2}{a^2 b}\left\{\frac{1}{(h+2)}\left[\frac{\pi}{2}a^{h+3} + b^{h+3} - b(a^2+b^2)^{\frac{h+2}{2}}\right]\right.$$
$$\left. - \frac{b^{h+3}}{(h+3)} + I_1^{(h)} + aI_5^{(h)} - aI_6^{(h)}\right\},\ \mathbb{R}(h) > -2$$

where the $l_1^{(h)}$, $I_5^{(h)}$ and $I_6^{(h)}$ are given in Lemmas 1 and 2.

EXAMPLE 4. Special case 3: one point on the y-axis and one on the side parallel to it.

In this case (1) becomes $a^2 + b^2(u_3 - u_4)^2$ and hence $u = w_2 + a^2$ and the density of $x = u^{\frac{1}{2}}$ is given by

$$f(x) = \begin{cases} \frac{2x}{b^2}[b(x^2 - a^2)^{-\frac{1}{2}} - 1],\ alx \le \sqrt{a^2 + b^2} \\ 0,\ \text{elsewhere.} \end{cases}$$

The general h-th moment is given by

$$E(x^h) = \frac{2}{b^2}\{\frac{1}{(h+2)}[a^{h+2} - (a^2+b^2)^{\frac{h+2}{2}}] + bI_3^{(h)}\},\ \mathbb{R}(h) > -2$$

where $I_3^{(h)}$ is given in Lemma 1. For $h = 1, 2$ some simpler forms are available by using the following formulae.

$$\int x^2(x^2 - a^2)^{-\frac{1}{2}} dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\ln(x + \sqrt{x^2 - a^2}) \tag{9}$$

and

$$\int x^3(x^2 - a^2)^{-\frac{1}{2}} dx = \frac{(x^2 - a^2)^{\frac{3}{2}}}{3} + a^2\sqrt{x^2 - a^2} \tag{10}$$

Then we have

$$E(x) = \frac{2}{3b^2}[a^3 - (a^2 + b^2)^{\frac{3}{2}}] + \sqrt{a^2 + b^2} + \frac{a^2}{b}\ln\left[\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}}\right]$$

and

$$E(x^2) = \frac{a^4}{2b^2} - \frac{(a^2 + b^2)^2}{2b^2} + \frac{2}{3}b^2 + 2a^2.$$

EXAMPLE 5 Special case 4: one point on the x-axis and one on the side parallel to it.

In this case (1) becomes $a^2(u_1 - u_2)^2 + b^2$ and hence $u = w_1 + b^2$. Then the density of $x = u^{\frac{1}{2}}$ is given by

$$f(x) = \begin{cases} \dfrac{2x}{a^2}[a(x^2 - b^2)^{-\frac{1}{2}} - 1], \; b \le x \le \sqrt{a^2 + b^2} \\ \\ 0, \text{ elsewhere.} \end{cases}$$

The general moment is

$$E(x^h) = \frac{2}{a^2}\{\frac{1}{(h+2)}[b^{h+2} - (a^2 + b^2)^{\frac{h+2}{2}}] + aI_4^{(h)}\}, \; \Re(h) > -2$$

where the $I_4^{(h)}$ is given in Lemma 1. Then by using (9) and (10) some explicit forms are the following:

$$E(x^h) = \frac{2}{3a^2}[b^3 - (a^2 + b^2)^{\frac{3}{2}}] + \sqrt{a^2 + b^2}$$
$$+ \frac{b^2}{a}\ln\left[\frac{a}{b} + \left(1 + \frac{a^2}{b^2}\right)^{\frac{1}{2}}\right]$$

and

$$E(x^2) = \frac{1}{2a^2}[b^4 - (a^2 + b^2)^2] + \frac{2}{3}a^2 + 2b^2.$$

EXAMPLE 6 Special case 5: one point on the y-axis and one point on the x-axis.

In this case (1) is of one of the following forms:

$$a^2(0 - u_2)^2 + b^2(0 - u_4)^2 \text{ or } a^2(0 - u_2)^2 + b^2(1 - u_4)^2$$

or

$$a^2(1 - u_2)^2 + b^2(0 - u_4)^2 \text{ or } a^2(1 - u_2)^2 + b^2(1 - u_4)^2.$$

Since u_j and $1 - u_j$ have the same distribution we need to consider only the form $a^2u_2^2 + b^2u_4^2$. Here

$$g_1(w_1) = \frac{1}{2a}w_1^{-\frac{1}{2}}, 0 < w_1 \leq a^2$$

$$g_2(w_2) = \frac{1}{2b}w_2^{-\frac{1}{2}}, 0 < w_2 \leq b^2$$

and then

$$g_1(v)g_2(u - v) = \frac{1}{4ab}v^{-\frac{1}{2}}(u - v)^{-\frac{1}{2}}.$$

Proceeding as before the density of $x = u^{\frac{1}{2}}$ is given by

$$f(x) = \frac{x}{ab}\phi(x)$$

where

$$\phi(x) = \begin{cases} \frac{\pi}{2}, 0 < x < a \\ \sin^{-1}\left(\frac{a}{x}\right), \ a \leq x < b \\ \sin^{-1}\left(\frac{a}{x}\right) - \cos^{-1}\left(\frac{b}{x}\right), \ b \leq x \leq \sqrt{a^2 + b^2} \\ 0, \text{ elsewhere.} \end{cases}$$

The general moment is given by

$$E(x^2) = \frac{1}{ab}\{\frac{\pi a^{h+2}}{2(h+2)} + I_5^{(h)} - I_6^{(h)}\}$$

where the $I_5^{(h)}$ and $I_6^{(h)}$ are given in Lemma 2.

Remark 1. If one point is fixed at one of the corners of the rectangle and the other is a random point within the rectangle then by comparing with (1) observe that the square of the distance between the two points can be expressed as $a^2u_1^2 + b^2u_3^2$ where u_1 and u_3 are independently and uniformly distributed over $[0, 1]$. Hence surprisingly this situation is equivalent to the special case 5 discussed in Example 6. Furthermore, in order to obtain simpler representations for the mean value of the distance the following formula will be helpful.

$$\text{(11)} \qquad \int \frac{1}{\cos^3 ax} dx = \frac{\sin ax}{2a\cos^2 ax} + \frac{1}{2a} \ln\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right).$$

From Example 6 in the light of Remark 1 it is easy to see that the mean distance, $E(x)$, from one of the vertices to a random point inside a rectangle of sides a and b, $a \leq b$, is given by

$$3E(x) = d + \frac{b^2}{2a} \ln\left(\frac{a+d}{b}\right) + \frac{a^2}{2b} \ln\left(\frac{b+d}{a}\right), \quad d = \sqrt{a^2+b^2}.$$

3. Distance between random points in two different rectangles.

Consider two rectangles, one with sides a and b and the other with sides c and d. Let P and Q be random points in these two rectangles respectively, randomness in the sense of probability being proportional to the areas of the respective rectangles. Several possible cases are there. Select an (x, y)-coordinate system with the origin at one of the corners of one of the rectangles, say the one with sides a and b. Let the side of length a be on the x-axis. The second rectangle could be similarly oriented, that is, one side parallel to the x-axis. Even in this case the rectangles could fully or partially overlap or could be nonoverlapping. The second rectangle could be at an angle in the sense one of the sides making an acute angle α with the x-axis, $\alpha \notin 0, \frac{\pi}{2}$. There are several possibilities of the relative lengths of the sides. Both rectangles could be degenerate leaving pieces of line segments, parallel, perpendicular or at an acute angle not equal to 0 or $\frac{\pi}{2}$. In all these cases the points could be entirely within or on one of the sides of the respective rectangles. Thus to exhaust all possible cases will not be an easy task. Here we will consider one case and *a* general procedure for tackling the distributional aspect of the distance between P and Q. Let the two rectangles be similarly oriented as shown in Figure 3.

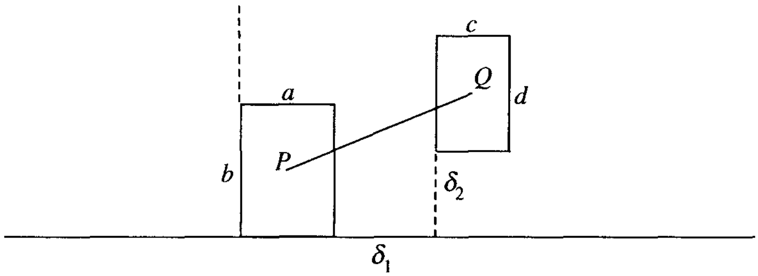

Figure 3

Let $\delta_1 \geq a$ and δ_2 arbitrary. The case of $\delta_2 > 0$ is shown in Figure 3. Denoting $\sim$ to indicate *uniformly distributed over the interval* we have

$$x_{11} \sim [0, a], \; x_{21} \sim [\delta_1, \delta_1 + c]$$

$$x_{12} \sim [0, b], \; x_{22} \sim [\delta_2, \delta_2 + d]$$

which will then imply that

$$u_1 = \frac{x_{21} - \delta_1}{c} \sim [0, 1], \; u_3 = \frac{x_{22} - \delta_2}{d} \sim [0, 1]$$
$$u_2 = \frac{x_{11}}{a} \sim [0, 1], \; u_4 = \frac{x_{12}}{b} \sim [0, 1]$$

where u_1, u_2, u_3, u_4 are independently distributed. Let x be the distance between P and Q and let $u = x^2$.

Then

$$u = (x_{21} - x_{11})^2 + (x_{22} - x_{12})^2$$

$$= [cu_1 + \delta_1 - au_2]^2 + [du_3 + \delta_2 - bu_4]^2.$$

Let us examine the densities of

$$v_1 = cu_1 + \delta_1 - au_2 \text{ and } v_2 = du_3 + \delta_2 - bu_4.$$

Let $z_1 = u_1$ and $z_2 = u_3$. The joint density of v_1 and z_1, denoted by $f(v_1, z_1)$, is given by

$$f(v_1, z_1) = \frac{1}{a} \text{ for } a > 0.$$

Integrating out z_1 the marginal density of v_1 is given by

$$f_1(v_1) = \frac{1}{ac} \begin{cases} v_1 - \delta_1 + a, \delta_1 - a \le v_1 \le \delta_1, & \text{for } c > a, \delta \ge a \\ a, \delta_1 \le v_1 \le c + \delta_1 - a \\ c - v_1 + \delta_1, c + \delta_1 - a \le v_1 \le c + \delta_1 \\ 0, \text{ elsewhere.} \end{cases}$$

If $c < a$ then the intervals will be $[\delta_1 - a, c + \delta_1 - a]$, $[c + \delta_1 - a, \delta_1]$, $[\delta_1, c + \delta_1]$. The density of $w_1 = v_1^2$, denoted by $g_1(w_1)$, is then, for $a > 0$, $c > 0$,

(12)

$$g_1(w_1) = \frac{1}{2ac} \begin{cases} (a - \delta_1)w_1^{-\frac{1}{2}} + 1, (\delta_1 - a)^2 \le w_1 \le \delta_1^2, c > a, \delta_1 \ge a \\ aw_1^{-\frac{1}{2}}, \delta_1^2 \le w_1 \le (c + \delta_1 - a)^2 \\ (c + \delta_1)w_1^{-\frac{1}{2}} - 1, (c + \delta_1 - a)^2 \le w_1 \le (c + \delta_1)^2 \\ 0, \text{ elsewhere.} \end{cases}$$

A similar procedure gives the density of $w_2 = v_2^2$ as

(13)

$$g_2(w_2) = \frac{1}{2bd} \begin{cases} (b - \delta_2)w_2^{-\frac{1}{2}} + 1, (\delta_2 - b)^2 \le w_2 \le \delta_2^2, d > b, \delta_2 \ge b \\ bw_2^{-\frac{1}{2}}, \delta_2^2 \le w_2 \le (d + \delta_2 - b)^2 \\ (d + \delta_2)w_2^{-\frac{1}{2}} - 1, (d + \delta_2 - b)^2 \le w_2 \le (d + \delta_2)^2 \\ 0, \text{ elsewhere.} \end{cases}$$

Hence the density of $x = u^{\frac{1}{2}}$, $u = w_1 + w_2$, is

$$f(x) = 2xg(x^2)$$

where

$$g(u) = \int_v g_1(v)g_2(u - v)dv \tag{14}$$

with $g_1(\cdot)$ and $g_2(\cdot)$ given in (12) and (13) respectively. But for evaluating the integral in (14) we need to consider several regions. Therefore we will not evaluate (14) explicitly for all these regions. Instead, we will look into some special cases which will be listed as examples.

EXAMPLE 7. Two random points on parallel line segments.

Let the random points P and Q be as shown in Figure 4.

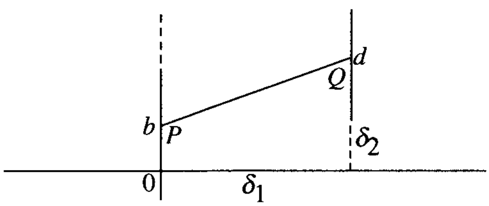

Figure 4

Let the two line segments be of lengths b and d respectively, δ_1 units apart and the piece with length d be displaced by δ_2. We evaluate the density of the distance $x = |PQ|$. Either we could derive the density from first principles or obtain it from $g_2(w_2)$ of (13). Put $a = 0$ and $c = 0$ in (13). Then the square of the distance is $u = w_2 + \delta_1^2$ and the density of u for $d > b$, is given by

$$g(u) = \frac{1}{2bd}\begin{cases} (b-\delta_2)(u-\delta_1^2)^{-\frac{1}{2}}+1, (\delta_2-b)^2+\delta_1^2 \leq u \leq \delta_2^2+\delta_1^2 \\ b(u-\delta_1)^2)^{-\frac{1}{2}}, \delta_1^2+\delta_2^2 \leq u \leq (d+\delta_2-b)^2+\delta_1^2 \\ (d+\delta_2)(u-\delta_1^2)^{-\frac{1}{2}}-1, \\ (d+\delta_2-b)^2+\delta_1^2 \leq u \leq (d+\delta_2)^2+\delta_1^2 \\ 0, \text{ elsewhere.} \end{cases}$$

from which the density of the distance $x = u^{\frac{1}{2}}$ is available. Furthermore, arbitrary moments of $x^2-\delta_1^2$ are easily available from (13). Moments of x can be evaluated by using techniques similar to the ones used in the derivation of the results in Lemma 1.

EXAMPLE 8. Two identical rectangles side by side.

Consider two rectangles of sides a and b, side by side, as shown in Figure 5, and one random point within each of the rectangles.

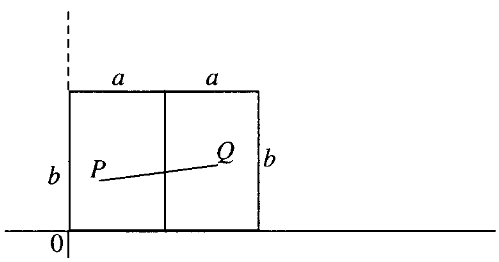

Figure 5

In the notations of (12) and (13) we have $a = c$, $\delta_1 = a$, $\delta_2 = 0$, $b = d$ and

$$g_1(w_1) = \frac{1}{2a^2}\begin{cases} 1, 0 \leq w_1 \leq a^2 \\ 2aw_1^{-\frac{1}{2}}-1, a^2 \leq w_1 \leq (2a)^2 \\ 0, \text{ elsewhere.} \end{cases}$$

Then proceeding as in Example 7 one can see that the density of $u = x^2$, where x is the distance between the points P and Q, is of the following form, for $a \leq b \leq 2a$:

$$
g(u) = \begin{cases}
-u + 2bu^{\frac{1}{2}}, 0 \leq u \leq a^2 \\
2(a^2 + ab\pi) + u + 2(b - 2a)u^{\frac{1}{2}} \\
-4b\sqrt{u - a^2} - 4ab\sin^{-1}\left(\frac{a}{\sqrt{4}}\right), a^2 \leq u \leq b^2 \\
b^2 + 2(a^2 + ab\pi) + 2u - 4au^{\frac{1}{2}} - 4b\sqrt{u - a^2} \\
-4ab\sin^{-1}\left(\frac{a}{\sqrt{u}}\right), b^2 \leq u \leq 4a^2 \\
b^2 - 2a^2 + u - 4b\sqrt{u - a^2} + 2b\sqrt{u - 4a^2} \\
+4ab\sin^{-1}\left(\frac{2a}{\sqrt{u}}\right) - 4ab\sin^{-1}\left(\frac{a}{\sqrt{u}}\right), 4a^2 \leq u \leq a^2 + b^2 \\
-4a^2 - b^2 - u + 2b\sqrt{u - 4a^2} + 4a\sqrt{u - b^2} \\
+4ab\sin^{-1}\left(\frac{2a}{\sqrt{u}}\right) - 4ab\sin^{-1}\sqrt{1 - \frac{b^2}{u}}, a^2 \\
+b^2 \leq u \leq 4a^2 + b^2 \\
0, \text{ elsewhere.}
\end{cases}
$$

The next simple case will be that of two identical rectangles, similarly oriented, but with one corner of one rectangle touching one corner of the other. In this case the density will be slightly more complicated than the one given in Example 8. Since the expression takes up too much space it will not be listed here.

4. Other types of distances.

Our discussion so far was confined to Euclidean distance between two random points. Other generalized distance measures may be of interest in some theoretical and practical investigations. If $P(x_{11}, x_{12})$ and $Q(x_{21}, x_{22})$ are two points on a plane then

$$L_r = \{|x_{21} - x_{11}|^r + |x_{22} - x_{12}|^r\}^{\frac{1}{r}},\ r \geq 1 \tag{15}$$

is a generalized distance between P and Q. We shall investigate the density and moments of L_r for some of the situations discussed so far.

Let the points be within a rectangle of sides a and b, $a \leq b$. Then referring back to (1) we have

$$v_1 = a(u_1 - u_2) \text{ and } v_2 = b(u_3 - u_4).$$

In the present case our interest is in the variables $z_1 = |v_1|^r$ and $z_2 = |v_2|r$. Then the absolute value of v_1 has the density

$$f_{|v_1|}(v_1) = \begin{cases} \frac{2}{a^2}(a - v_1), 0 < v_1 < a \\ 0, \text{ elsewhere.} \end{cases}$$

and $z_1 = |v_1|^r$ has the density

$$f_{|v_1|}(z_1) = \begin{cases} \frac{2}{ra^2} z_1^{\frac{1}{r}-1}(a - z_1^{\frac{1}{r}}), 0 < z_1 < a^r \\ 0, \text{ elsewhere.} \end{cases} \tag{16}$$

A corresponding expression for the density of z_2 is available from $f_{z_1}(z_1)$ by replacing z_1 by z_2 and a by b. Then the density of $u = z_1 + z_2$ is available from (2) where g_1 and g_2 are replaced by $f_{z_1}(z_1)$ and $f_{z_2}(z_2)$, a^2 and b^2 by a^r and b^r respectively. Then

$$\begin{aligned} f_{z_1}(v) f_{z_2}(u - v) = \frac{4}{r^2a^2b^2}[&abv^{\frac{1}{r}-1}(u - v)^{\frac{1}{r}-1} - av^{\frac{1}{r}-1}(u - v)^{2r-1} \\ &- bv^{\frac{2}{r}-1}(u - v)^{1r-1} + v^{2r-1}(u - v)^{\frac{2}{r}-1}]. \end{aligned}$$

The density of $u = z_1 + z_2 = L_r^r$ for $a \leq b$, is available by straight integration with the help of the integrals corresponding to (2). Put $v = uy$ and integrate. The resulting density, denoted by $f(u)$, is the following:

$$f(u) = \frac{4}{r^2a^2b^2} \begin{cases} f_1(u), 0 < u \leq a^r \\ f_2(u), a^r < u \leq b^r \\ f_3(u), b^r < u \leq a^r + b^r \\ 0, \text{ elsewhere.} \end{cases} \tag{17}$$

where

$$
\begin{aligned}
f_1(u) &= abu^{\frac{2}{r}-1}\frac{\Gamma\left(\frac{1}{r}\right)\Gamma\left(\frac{1}{r}\right)}{\Gamma\left(\frac{2}{r}\right)} \\
&\quad -(a+b)u^{\frac{3}{r}-1}\frac{\Gamma\left(\frac{2}{r}\right)\Gamma\left(\frac{1}{r}\right)}{\Gamma\left(\frac{3}{r}\right)}+u^{\frac{4}{r}-1}\frac{\Gamma\left(\frac{2}{r}\right)\Gamma\left(\frac{2}{r}\right)}{\Gamma\left(\frac{4}{r}\right)} \qquad (18)\\
&= abu-\frac{1}{2}(a+b)u^2+\frac{u^3}{6} \quad \text{for } r=1;
\end{aligned}
$$

$$
\begin{aligned}
f_2(u) &= abu^{\frac{2}{r}-1}I\left(\frac{1}{r},\frac{1}{r};\frac{a^r}{u}\right)-au^{3r-1}I\left(\frac{1}{r},\frac{2}{r};\frac{a^r}{u}\right) \\
&\quad -bu^{\frac{3}{r}-1}I\left(\frac{2}{r},\frac{1}{r};\frac{a^r}{u}\right)+u^{\frac{4}{r}-1}I\left(\frac{2}{r},\frac{2}{r};\frac{a^r}{u}\right) \qquad (19)\\
&= \frac{a^2b}{2}+\frac{a^3}{6}-\frac{a^2}{2}u \quad \text{for } r=1;
\end{aligned}
$$

$$
\begin{aligned}
f_3(u) &= abu^{\frac{2}{r}-1}\left[I\left(\frac{1}{r},\frac{1}{r};\frac{a^r}{u}\right)-I\left(\frac{1}{r},\frac{1}{r};1-\frac{b^r}{u}\right)\right] \\
&\quad -au^{3r-1}\left[I\left(\frac{1}{r},\frac{2}{r};\frac{a^r}{u}\right)-I\left(\frac{1}{r},\frac{2}{r};1-\frac{b^r}{u}\right)\right] \\
&\quad -bu^{3r-1}\left[I\left(\frac{2}{r},\frac{1}{r};\frac{a^r}{u}\right)-I\left(\frac{2}{r},\frac{1}{r};1-\frac{b^r}{u}\right)\right] \qquad (20)\\
&\quad +bu^{4r-1}\left[I\left(\frac{2}{r},\frac{2}{r};\frac{a^r}{u}\right)-I\left(\frac{2}{r},\frac{2}{r};1-\frac{b^r}{u}\right)\right] \\
&= \frac{ab}{2}(a+b)+\frac{1}{6}(a^3+b^3)-\frac{1}{2}(a+b)^2u \\
&\quad +\frac{1}{2}(a+b)u^2-\frac{u^3}{6} \quad \text{for } r=1,
\end{aligned}
$$

with $I(\cdot,\cdot;\cdot)$ being the incomplete beta function, which can also be written as a hypergeometric function ${}_2F_1(\cdot)$, given by

$$
\begin{aligned}
I(\alpha,\beta;z) &= \int_0^z y^{\alpha-1}(1-y)^{\beta-1}dy, \alpha>0, \beta>0, z>0 \\
&= \frac{z^\alpha}{\alpha}\,{}_2F_1(1-\beta,\alpha;\alpha-1;z).
\end{aligned} \tag{21}
$$

For $r=1$ we have the density of L_1 which is listed in (17) to (20). For $r=2$ the density of the Euclidean distance is already given in Section 3. For $r\geq 1$ the density of L_r is available from (17). By examining the individual functions f_1, f_2, f_3 in (17) it is evident that the general h-th moment of L_r or that of u is available from (17) with the help of the following integral which will be stated as a lemma.

LEMMA 3. *For, $\beta\geq\alpha>0$, $\delta>0$, $\epsilon>0$, $\gamma>0$, $\gamma\leq\alpha$, $\gamma\leq\beta$, $0<\Re(h+1)<\delta$,*

$$
\begin{aligned}
&\int_{u=\alpha}^{\beta} u^h \int_0^{\frac{\gamma}{u}} y^{\delta-1}(1-y)^{\epsilon-1}dydu \\
&= \frac{1}{(h+1)}\left\{\beta^{h+1}I\left(\delta,\epsilon;\frac{\gamma}{\beta}\right)-a^{h+1}I\left(\delta,\epsilon;\frac{\gamma}{\alpha}\right)\right. \\
&\left.+\gamma^{h+1}\left[\left(\delta-h-1,\epsilon;\frac{\gamma}{\alpha}\right)-\left(\delta-h-1,\epsilon;\frac{\gamma}{\beta}\right)\right]\right\}.
\end{aligned}
$$

The result follows by changing the order of integration and then using (21).

Remark 2. For obtaining the density and moments of the generalized distance of (15), when the random points are in different rectangles, one can use the above procedures and the densities given in Section 3. For example when $c>a$, $\delta_1\geq a$, $d>b$, $\delta_2\geq b$ the densities of $|v_1|^r$ and $|v_2|^r$ are available from $f_1(v_1)$ and the corresponding $f_2(v_2)$ associated with (12) and (13). Under the above conditions on the parameters $v=|v_1|$ and $v_2=|v_2|$ in this case. Since the explicit forms of the densities and moments for the various cases discussed in Section 3, though not difficult to evaluate, will take up too much space these will not be discussed here for the generalized distance of (15). If only positive integer moments are required then use the fact that $|v_1|$ and $|v_2|$

are independently distributed and hence

$$E[|v_1|^r + |v_2|^r]^m = \sum_{n=0}^{m} \binom{m}{n} [E|v_1|^{r(m-n)}][E|v_2|^{rn}], \tag{22}$$

$$r \geq 1,\ m = 0, 1, \ldots$$

Now the individual moments are available from the densities $f_{z_1}(z_1)$ of (16) and the corresponding $f_{z_2}(z_2)$. For example

$$\begin{aligned} E(z_1^{\gamma}) = E[|v_1|^{r\gamma}] &= \frac{2}{a^2}\int_0^a v_1^{r\gamma}(a - v_1)dv_1 \\ &= \frac{2a^{r\gamma}}{(r\gamma+1)(r\gamma+2)},\ \Re(h) > -\frac{1}{r}. \end{aligned} \tag{23}$$

Substituting back in (22) we have the m-th moment of L_r^r, of (15) as follows:

$$E(L_r^{rm}) = \sum_{n=0}^{m} \binom{m}{n} \frac{a^{r(m-n)}b^{rn}}{(rm - rn + 1)(rm - rn + 2)(rn + 1)(rn + 2)}, \tag{24}$$

$$m = 0, 1, \ldots$$

There are other distance measures based on the largest of $|v_1|$ and $|v_2|$. The distribution problems will be easier to handle in these cases. The simplest procedure would be to look for the distribution function or the cumulative probability function by using the fact that if the largest of $|v_1|$ and $|v_2|$ is less than a fixed number t then both $|v_1|$ and $|v_2|$ must be less than t. Let

$$w = \max\{|v_1|, |v_2|\}. \tag{25}$$

The distribution function of w, denoted by $F_w(w)$, is available from the distribution functions of $|v_1|$ and $|v_2|$. Due to statistical independence

$$F_w(w) = Pr\{w \leq t\} = Pr\{|v_1| \leq t\}Pr\{|v_2| \leq t\}.$$

But

$$Pr\{|v_1| \leq t\} = \frac{2}{a^2}\left(at - \frac{t^2}{2}\right) \text{ and } Pr\{|v_2| \leq t\} = \frac{2}{b^2}\left(bt - \frac{t^2}{2}\right).$$

Observe that if $t > a$, $a \leq b$, then $Pr\{|v_1| \leq t\} = 1$. Hence we have

$$F_w(t) = \begin{cases} 0, t < 0 \\ \frac{4}{a^2b^2}\left(at - \frac{t^2}{2}\right)\left(bt - \frac{t^2}{2}\right), \; 0 \leq t \leq a \\ \frac{2}{b^2}\left(bt - \frac{t^2}{2}\right), \; a < t < b \\ 1, t \geq b \end{cases} \tag{26}$$

The density of w is available by differentiating $F_w(w)$ with respect to t and evaluating at w. That is,

$$f_w(w) = \frac{d}{dt}F_w(t)$$
$$\begin{cases} \frac{2w}{a^2b^2}[4ab - 3(a+b)w + 2w^2], \; 0 \leq w \leq a \\ \frac{2}{b^2}(b - w), \; a < w \leq b \\ 0, \; \text{elsewhere.} \end{cases} \tag{27}$$

From here the h-th moment of w is given by

$$E(w^h) = \frac{2a^{h+1}}{b^2}\left[-\frac{b}{(h+1)} + \frac{(4b+a)}{(h+2)} - 3\frac{(a+b)}{(h+3)} + 2\frac{a}{(h+4)}\right] + 2\frac{b^h}{(h+1)(h+2)}, \; \Re(h) > -1. \tag{28}$$

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